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THE PRESSURE-VOLUME-TEMPERATURE RELATIONS OF FIFTEEN LIQUIDS.

By P. W. Bridgman.

Investigations on Light and Heat made and published with aid from the Rumford Fund.

THE PRESSURE-VOLUME-TEMPERATURE RELATIONS OF FIFTEEN LIQUIDS.

By P. W. Bridgman.

Presented Dec. 14, 1932. Received Dec. 24, 1932.

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INTRODUCTION.

This paper contains determinations of the volume of a number of liquids between 0° and 95° C and up to 12000 kg/cm^2 by the sylphon method which has been applied and discussed in detail in two preceding papers.[1] The new liquids investigated here fall into three groups. The first group consists of only a single liquid, triethanolamine. The reason for investigating this was that it was hoped that its structural similarity to glycerine would throw some light on the cause of incompressibility in liquids, glycerine being the most incompressible organic liquid yet measured. The second group consists of nine alkyl halides, namely the propyl, butyl, and amyl chlorides, bromides, and iodides. Halogen compounds were chosen for investigation because previously interesting results had been found for mono-clorbenzene and mono-brombenzene as compared with simple benzene, and it seemed that the most promising method of investigating the effect of the halogen was to systematically examine a whole series of compounds. Finally, the third group, five of the isomers of octanol, was investigated in order to find more in detail the effect of systematic structural changes.

The technique of the manipulations and the computations is almost exactly the same as before, particularly in the second of the two papers. The only significant differences are: (1) that readings of volume at atmospheric pressure were made at both 0° and 50°; this is preferable to obtaining the volume at 50° and atmospheric pressure by an extrapolation from readings between 200 and 1000 kg as was done in the previous paper: (2) in the case of the alkyl halides it was possible to make readings at 0° in most cases to 12000 without freezing, thus permitting more complete data than was possible for many of the previous liquids which freeze at lower pressures.

The liquids of the first two groups were obtained from the Eastman Kodak Co. Several of these were further purified, as will be described in the detailed presentation of data. Preliminary examination had shown that there is no important chemical action between these liquids and the solder and brass of which the sylphon is constructed.

For the octanols I am much indebted to Professor E. Emmet Reid of the Johns Hopkins University, who has prepared the complete set of the 22 isomers of octanol, and who placed at my disposal the entire pheric pressure had been determined by Professor Reid. This, together with the freezing pressure at room temperature gave sufficiently good indications of the pressure range to be avoided with the sylphon. In Tables XI to XV the region in which the material is solid is indicated by the blank places; these should not be used, however, to get more than very rough coordinates for the melting curves.

In addition to rough measurements of melting, other measurements were also made on the isomers of octanol at room temperature. Some question as to whether the compounds may have experienced changes between the time of formation and the time of the compressibility measurements arose because of discrepancies between the densities provided me by Professor Reid and those which were demanded by the dimensions of the sylphons. This led to an independent determination of the densities of the five isomers by measurement in a specific gravity bottle containing about 4 gm., the weighings being made to a fraction of a milligram. These specially determined densities are given in the following, and were used as fundamental in reducing measured changes of total volume to fractional changes of volume. As an additional check, Professor Conant was so kind as to have measured in his laboratory the index of refraction for the sodium D line at 25°, from which the molecular refraction can be calculated by combining with my densities. These values are given in the following, as well as the indices and the molecular refractions supplied me by Professor Reid, which I understand are not yet published. The theoretical value for the molecular refraction for all five isomers, disregarding structural differences, is 40.79.

DETAILED DATA.

Triethanolamine. This was most kindly purified under the direction of Professor J. B. Conant of the Chemistry Department by distillation in vacuum over BaO, in order to remove as much of the water as possible, which nevertheless in the opinion of Professor Conant was not completely accomplished. The product of the distillation was a liquid, clear, but of a distinct yellow color. This became browner and cloudy by the end of the filling process. The liquid is extremely viscous and two days were taken in filling the sylphon; the final touches to the filling operation were given at 95° and at 3 mm. pressure. During the filling the liquid was kept under a bell jar or other closed container with $CaCl_2$ present in order to prevent the absorption of water from the air.

Because of the extreme viscosity and unknown liability to freezing the measurements were made in an unusual order: first at 50° to 1000 kg., without return, then immediately to 95° at 1000, and at 95° to 10000 and return with good check, then to 0°, first at atmospheric pressure, and then to 1000, where there was evidence of freezing, which persisted on releasing pressure to atmospheric. The probability is that this, like glycerine, will support great subcooling under ordinary conditions. The normal melting point is given in the Eastman Catalogue as 17° to 19°. Temperature was then raised to 50° with good check in spite of the apparent freezing at 0°, and

TABLE I.

TRIETHANOLAMINE.

Pressure kg/cm^2	Related Volumes		
	0°	50°	95°
0	1.0000	1.0251	
500	.9835	1.0075	
1000	.9699	.9921	1.0130

n-propyl chloride. Runs were made without incident at 0° and 50° to 12000 kg/cm², but at 95° pressure was allowed to get too low, permanently deforming the sylphon and probably developing a small leak, which, however, was not discovered until the completion of the run. A repetition was therefore attempted with a second sylphon, but this proved defective and leaked. A third sylphon was therefore used for the run at 95°; this gave a perfect check on releasing pressure, and also on returning at 95° after excursions to 0° and 50°. Connection was established with the previous runs at 0° and 50° by measurements at 1000 and 5000 at both these temperatures. The average deviation from smooth curves of a single one of the 38 points obtained at 0° and 50° with the first sylphon was 0.06% of the maximum effect; the results with the third sylphon were equally as smooth.

TABLE II.

N-PROPYL CHLORIDE.

Pressure kg/cm²	Relative Volumes		
	0°	50°	95°
0	1.0000	1.0757	
500	.9555	1.0046	
1000	.9261	.9645	1.0084
1500	.9024	.9357	.9734
2000	.8833	.9134	.9463
3000	.8541	.8790	.9051
4000	.8319	.8531	.8766
5000	.8138	.8324	.8539
6000	.7987	.8155	.8350
7000	.7852	.8003	.8192
8000	.7732	.7875	.8052
9000	.7625	.7759	.7929
10000	.7526	.7658	.7817
11000	.7440	.7563	.7723
12000	.7366	.7476	.7632

The numerical results are given in Table II. As explained, the relative volumes at 0° and 50° were obtained with one sylphon and those at 95° with another. The volumes obtained at 1000 and 5000 at 0° and 50° with the latter sylphon by way of check were not as good as could be desired, and were on the average 0.0030 higher than those obtained with the first sylphon. It is probable, therefore, that the 95° results for this liquid are not as good as usual, and less weight should be attached to them.

The volumes of the table at 0° and 50° are for 0.9160 gm. The measured expansion of 1 gm at atmospheric pressure between 0° and 50° was 0.0823 cm^3, and the measured density at atmospheric pressure at 24° was 0.8839. I. C. T. gives for the density at 20° 0.890.

n-propyl bromide. Two sylphons were used for this liquid. Runs were made in the regular order at 0°, 50°, and 95°. There were irregularities at the maximum pressure at 0° with the first sylphon, which, however, seemed to have no effect on the later readings, checks being obtained between the readings with increasing and decreasing pressure. A check was not obtained, however, after the run at 95° at 0° at atmospheric pressure, and examination of the sylphon disclosed a leak. The complete series of runs was therefore repeated with a second sylphon. On working up the results, much greater irregularity was disclosed by the high pressure measurements at 0°,

TABLE III.

the irregularity being so great that the results of the first run are doubtless to be preferred. It seems now that there can be little doubt that freezing took place at the highest pressures at 0° and that both sylphons were weakened at this time, but that the actual leak in the first sylphon developed only after release of pressure to nearly atmospheric at 95°. The following table is based on the results obtained with the first sylphon. The average deviation from smooth curves of a single one of the 61 readings was 0.11% of the maximum effect.

The volumes given in Table III are for 1.3777 gm. The measured density at 25° was 1.3337, and the expansion of 1 gm between 0° and 50°, 0.0478 cm^3. I. C. T. gives for the density at 20° 1.353.

n-propyl iodide. This was prepared under the direction of Professor Conant by shaking with "hypo" and $CaCl_2$ and then fractional distillation. It was kept in the dark after purification to avoid decomposition.

TABLE IV.

N-PROPYL IODIDE.

Pressure kg/cm²	Relative Volumes		
	0°	50°	95°
0	1.0000	1.0509	
500	.9656	1.0085	
1000	.9412	.9775	1.0106
1500	.9214	.9528	.9813
2000	.9055	.9336	.9595
3000	.8795	.9027	.9238
4000	.8581	.8792	.8973
5000	.8409	.8594	.8755
6000	.8262	.8435	.8577
7000	.8128	.8289	.8422
8000	.8012	.8163	.8288
9000	.7906	.8049	.8169
10000	.7810	.7945	.8060
11000	.7715	.7850	.7961
12000	.7625	.7761	.7864

Measurements on this were made without incident; the conventional order was followed, 0°, 50°, 95°, and then the return check reading at 0°. The average deviation from smooth curves of a single one of the 51 readings was 0.04%.

The relative volumes are given in Table IV, which is constructed for 1.778 gm. The observed density at atmospheric pressure at 24.5° was 1.7303, and the measured expansion of 1 gm. between 0° and 50°, 0.0319 cm^3. I. C. T. gives for the density at 20° 1.747.

n-butyl chloride. Runs were made on this without incident except for a leak at the gauge plug, which was repaired without displacing the sylphon. The conventional runs at 0°, 50°, 95°, and return check reading at 0° were made. At 0° the maximum pressure reached

TABLE V.

N-BUTYL CHLORIDE.

Pressure kg/cm^2	Relative Volumes		
	0°	50°	95°
0	1.0000	1.0614	
500	.9598	1.0066	1.0526
1000	.9310	.9695	1.0062
1500	.9098	.9436	.9728
2000	.8928	.9234	.9491

n-butyl bromide. The measurements on this substance went without accident or incident. The material was used without purification directly as received from Eastman. It was especially noticed on emptying the sylphon at the end of the runs that the liquid remained clear and colorless. Runs were made as usual at 0°, 50°, and 95°, to 12000 at each temperature, and then the return check reading at 0°. The average deviation from smooth curves of a single one of the 60 readings was 0.016% of the maximum effect.

TABLE VI.

N-BUTYL BROMIDE.

Pressure kg/cm^2	Relative Volumes		
	0°	50°	95°
0	1.0000	1.0613	
500	.9641	1.0101	1.0512
1000	.9384	.9759	1.0102
1500	.9191	.9499	.9788
2000	.9025	.9297	.9554
3000	.8743	.8993	.9202
4000	.8531	.8759	.8942
5000	.8363	.8565	.8717
6000	.8214	.8392	.8530
7000	.8084	.8243	.8377
8000	.7968	.8115	.8244
9000	.7862	.8005	.8123
10000	.7767	.7902	.8014
11000	.7683	.7811	.7919
12000	.7609	.7726	.7836

Relative volumes are given in Table VI, which is constructed for 1.3063 gm. At atmospheric pressure the measured density at 25° was 1.2685, and the expansion of 1 gm between 0° and 50°, 0.0452 cm^3. I. C. T. gives for the density at 20° 1.275.

n-butyl iodide. This was purified under the direction of Professor Conant in the same way as propyl iodide, and it was similarly protected from the action of light before use. The conventional runs at 0°, 50°, and 95° and the return check reading at 0° were made without incident. When the sylphon was emptied at the end of the run a perceptible but small amount of reddish sediment was found, which doubtless was stannic iodide, from the solder of the sylphon. The iodides always seem to have some chemical activity; they are always perceptibly colored by dissociated iodine, which doubtless is the active agent. However, the amount of such activity was very slight because the return check reading at 0° after the high pressure runs was as good or better than usual. The average deviation from smooth curves of a single one of the 53 readings was 0.08% of the maximum effect.

TABLE VII.

N-BUTYL IODIDE.

Pressure kg/cm²	Relative Volumes		
	0°	50°	95°
0	1.0000	1.0508	
500	.9687	1.0094	1.0458
1000	.9453	.9785	1.0104
1500	.9258	.9545	.9821
2000	.9104	.9357	.9606

lar than usual. This was ascribed to an improper configuration of the flexible leads. After the first run at 0°, when a maximum of only 8000 kg was attained, the sylphon was removed from the press and the difficulty with the leads apparently remedied, as proved by the perfectly satisfactory subsequent readings at 50° and 95°. On repeating the 0° run however, after a perfect check at atmospheric pressure, there was again trouble of some sort, for the readings were again irregular, although there was no difficulty with the contact as before. The maximum pressure reached on the repetition at 0° was 10100. It is not unlikely that the irregularities were due to partial freezing. The average deviation from smooth curves of a single one of the 33 readings at 50° and 95° was 0.05% of the maximum effect, while at 0° the corresponding deviation for 25 readings was 0.27%. At 0° practically all of the irregularity occurred below 6000.

TABLE VIII.

N-AMYL CHLORIDE.

Pressure kg/cm^2	Relative Volumes		
	0°	50°	95°
0	1.0000	1.0617	
500	.9632	1.0080	
1000	.9369	.9739	1.0064
1500	.9156	.9481	.9766
2000	.8982	.9277	.9526
3000	.8712	.8956	.9165
4000	.8491	.8716	.8891
5000	.8319	.8515	.8678
6000	.8180	.8350	.8498
7000	.8059	.8206	.8346
8000	.7948	.8082	.8213
9000	.7845	.7971	.8093
10000	.7748	.7866	.7987
11000	.7660	.7777	.7883
12000		.7696	.7792

Relative volumes are shown in Table VIII, which is constructed for 0.9143 gm. At atmospheric pressure the measured density at 25° was 0.8872, and the measured expansion of 1 gm between 0° and 50°, 0.0670 cm³. I. C. T. gives for the density at 20° 0.883, less than my value at 25°, and indicative of a real discrepancy, although a slight one.

n-amyl bromide. This was purified under the direction of Professor Conant in the same way as the iodides. Two complete runs were made with this material with two sylphons. At 0° on the first run pressure was pushed to 11700, which apparently froze the liquid. After the termination of the 50° and the 95° runs the sylphon was found to have lost considerable weight, leak evidently having been produced by damage during freezing. The readings were therefore discarded, and the runs repeated with another sylphon. There was no special incident during the runs with the second sylphon, except that on the initial application of pressure up to about 5000 at 0° a leak developed in the connecting pipe which made it necessary to release the pressure to zero, renew the washers on the pipe, and repeat the run. The maximum pressure at 0° was now restricted to 9000, at which there was no trace of any freezing phenomenon. Runs were

TABLE IX.

N-AMYL BROMIDE.

Pressure kg/cm²	Relative Volumes		
	0°	50°	95°
0	1.0000	1.0573	
500	.9680	1.0090	
1000	.9428	.9770	1.0101
1500	.9233	.9528	.9803
2000	.9072	.9338	.9588
3000	.8807	.9031	.9241
4000	.8600	.8795	.8975
5000	.8429	.8599	.8766
6000	.8284	.8439	.8590
7000	.8153	.8299	.8438
8000	.8036	.8172	.8301
9000	.7930	.8063	.8182
10000		.7971	.8081
11000		.7879	.7990
12000		.7790	.7905

made at 50° and 95° to the full 12000 and return without incident, and then the return check reading was made at 0°. The check was not as good as usual, so that it seemed desirable to repeat the entire 0° run. This was done with a satisfactory check at all the points except atmospheric pressure. On taking down the apparatus, weighing the sylphon indicated no trace of leak. No certain explanation could be found for the unusually large discrepancies at 0° at atmospheric pressure; it is possible that unusually high friction in the internal guides of the sylphon may have been responsible. However, in view of the checks at all other pressures, there is no reason to fear important error. The volume assumed at 0° at atmospheric pressure was the mean of the two volumes on the first application, which differed by only 0.1% from the mean. The mean of the two zeroes on the second application also agreed with the first mean. The mean deviation from smooth curves of a single one of the 50 readings (discarding the atmospheric readings at 0°) was 0.07%.

Relative volumes are given in Table IX, which is constructed for 1.2505 gm. At atmospheric pressure the measured density at 24.5° was 1.2616; I. C. T. gives 1.223 at 20°. The expansion of 1 gm at atmospheric pressure between 0° and 50° was not obtained by direct measurement, but at 50° an extrapolation to atmospheric pressure was made from 100 kg; the range of extrapolation is so slight that there can be no important error. The value thus found for the expansion was 0.0456 cm^3.

n-amyl iodide. This was purified under the direction of Professor Conant in the same way as the other iodides and butyl bromide. The sylphon was filled in a darkened room to avoid decomposition. Regular runs were made without incident at 0° and 50°. At 0° the full 12000 was applied with no signs of freezing, although when the results were plotted on a large scale the irregularities beyond 9000 at 0° seemed somewhat larger than usual. One would have been prepared for freezing in view of the known freezing of amyl chloride and bromide; it may well be that the liquid subcooled and the irregularity of the high pressure points at 0° may have been due to the great viscosity of the subcooled liquid. On completing the run at 50° and passing to 95°, a short circuit developed in the leads which made it necessary to take the apparatus apart, without, however, touching the sylphon. The apparatus was then reassembled, and the 95° run, up and down, made with satisfactory results, but on trying for the lowest reading at 95° trouble of some sort again developed in the connections, making the reading impossible. Neither was it possible to get readings at 50° and 0°, so that the usual check could not be made. On hunting for the trouble after dismounting the apparatus the sylphon was found to have developed a leak, with a total loss of weight of 0.2 gm. It was therefore necessary to repeat the measurements with a new sylphon, but in view of the regularity of the readings with the first sylphon up to the very end, the simple check was first made of determining the difference between the 0°, 50°, and 95° isotherms at atmospheric and 5000 kg/cm² with the new sylphon.

TABLE X.

N-AMYL IODIDE.

Pressure kg/cm²	Relative Volumes		
	0°	50°	95°
0	1.0000	1.0529	
500	.9685	1.0100	
1000	.9442	.9793	1.0093
1500	.9250	.9559	.9806
2000	.9094	.9376	.9585
3000	.8831	.9080	.9247
4000	.8624	.8835	.8994
5000	.8451	.8640	.8783
6000	.8304	.8476	.8607
7000	.8173	.8337	.8453
8000	.8064	.8211	.8312
9000	.7965	.8105	.8192
10000	.7873	.8006	.8085
11000	.7786	.7911	.7987
12000	.7706	.7816	.7896

These were found to check perfectly, so that further repetition was dispensed with. It is evident that the leak in the first sylphon must have developed after the last reading at 95°.

The average deviation from smooth curves of a single one of the 55 readings with the first sylphon was 0.10% of the maximum effect.

Relative volumes are given in Table X, which is constructed for 1.5683 gm. The measured density at atmospheric pressure at 24.5° was 1.5287; I. C. T. gives 1.517 at 20°, less than my value at 24.5°, indicating some discrepancy. It would seem probable that any ordinary impurity would make the liquid lighter rather than heavier. The atmospheric volume at 50° was obtained, as for the preceding liquid, by an extrapolation from 100 kg. The expansion of 1 gm between 0° and 50° was found to be 0.0331 cm^3.

Octanol-3. $C - C - \overset{\overset{\displaystyle CH_3}{|}}{C} - C - C - C - C - C$. The initial attempt to get a run at 95° failed because of a ground on one of the connections. This was repaired without disturbing the sylphon, and three successful runs obtained in the order 50°, 0°, 95°. The final points at 95° agreed with the initial ones before the short circuit developed, so that the usual check at 0° was not necessary. The average deviation from smooth curves of a single one of the 48 readings was 0.10% of the maximum effect.

TABLE XI.

OCTANOL-3

Pressure kg/cm^2	Relative Volumes		
	0°	50°	95°
0	1.0000	1.0480	
500	.9697	1.0090	
1000	.9479	.9800	1.0105
1500	.9301	.9583	.9849
2000	.9156	.9407	.9646
3000	.8916	.9124	.9332
4000	.8722	.8910	.9086
5000		.8737	.8891
6000		.8584	.8728
7000		.8452	.8589
8000		.8338	.8465
9000			.8349
10000			.8250
11000			.8158
12000			.8073

Relative volumes are given in Table XI, which is constructed for 0.8370 gm. At atmospheric pressure the measured density at 22.2° was 0.8177, and the thermal expansion of 1 gm between 0° and 50°

0.0574 cm³. Professor Conant's index of refraction was 1.4251, which combined with my density gives 40.74 for the refraction constant. Professor Reid's values for index and refraction constant are 1.4209 and 40.41 respectively. All these are at 25°.

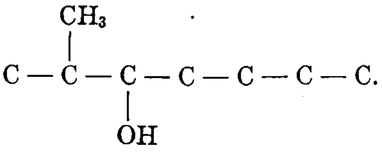
2-methyl heptanol-3. 
$\mathrm{C-\overset{\overset{\displaystyle CH_3}{|}}{C}-\underset{\underset{\displaystyle OH}{|}}{C}-C-C-C-C}$. Runs were made without incident at 50°, 0°, 95°, and the return check reading at 50°. The average deviation from smooth curves of a single one of the 51 readings was 0.08% of the maximum effect.

TABLE XII.

2-METHYL HEPTANOL-3.

Pressure kg/cm²	Relative Volumes		
	0°	50°	95°
0	1.0000	1.0505	
500	.9690	1.0078	
1000	.9459	.9789	1.0121
1500	.9283	.9565	.9851
2000	.9140	.9394	.9647
3000	.8897	.9216	.9428
4000	.8707	.8891	.9080
5000	.8544	.8714	.8888
6000		.8568	.8719
7000		.8435	.8569
8000		.8318	.8441
9000		.8213	.8331
10000		.8117	.8232
11000			.8139
12000			.8049

Relative volumes are given in Table XII, which is constructed for 0.8392 gm. At atmospheric pressure the measured density at 22.2° was 0.8211, and the thermal expansion of 1 gm between 0° and 50°, 0.0603 cm³. Professor Conant's index of refraction was 1.4255, which combined with my density gives 40.68 for the refraction constant. Professor Reid's values for index and refraction constant are 1.4246 and 40.31 respectively, all for 25°.

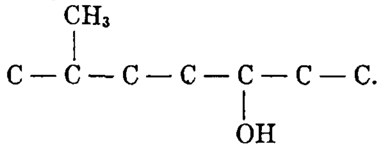
2-methyl heptanol-5. 
$C-\overset{\overset{\displaystyle CH_3}{|}}{C}-C-C-\underset{\underset{\displaystyle OH}{|}}{C}-C-C$. Runs were made without incident at 50°, 0°, 95°, and the return check reading at 50°. The average deviation from smooth curves of a single one of the 49 readings was 0.05%.

TABLE XIII.

2-METHYL HEPTANOL-5

Pressure kg/cm²	Relative Volumes		
	0°	50°	95°
0	1.0000	1.0477	
500	.9679	1.0071	
1000	.9449	.9774	1.0085
1500	.9269	.9549	.9815
2000	.9122	.9375	.9612
3000	.8882	.9099	.9297
4000	.8698	.8887	.9054
5000	.8538	.8716	.8855
6000		.8561	.8696
7000		.8425	.8556
8000		.8305	.8433
9000		.8198	.8318
10000		.8102	.8220
11000		.8023	.8127
12000			.8038

Relative volumes are given in Table XIII, which is constructed for 0.8337 gm. At atmospheric pressure the measured density at 24.4° was 0.8147, and the thermal expansion of 1 gm between 0° and 50°, 0.0571 cm³. Unfortunately there was not enough of the material to permit a measurement of its index. Professor Reid's value for index and refraction constant are 1.4112 and 41.65 respectively.

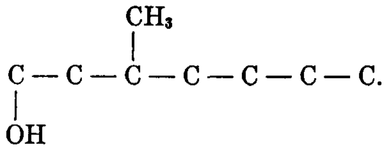
3-methyl heptanol-1. 
$\underset{\text{OH}}{\underset{|}{\text{C}}} — \text{C} — \overset{\text{CH}_3}{\overset{|}{\text{C}}} — \text{C} — \text{C} — \text{C} — \text{C}$. Compressibility measurements were made without incident at 50°, 0°, 95°, and the return check reading at 50°. The average deviation from smooth curves of a single one of the 50 readings was 0.03%.

TABLE XIV.

3-METHYL HEPTANOL-1.

Pressure kg/cm²	Relative Volumes		
	0°	50°	95°
0	1.0000	1.0446	
500	.9714	1.0065	
1000	.9499	.9791	1.0090
1500	.9323	.9585	.9834
2000	.9181	.9422	.9642
3000	.8946	.9147	.9342
4000	.8760	.8939	.9105
5000	.8605	.8764	.8912
6000	.8468	.8616	.8748
7000	.8341	.8482	.8611
8000		.8366	.8487
9000		.8260	.8371
10000		.8167	.8270
11000		.8083	.8179
12000			.8098

Relative volumes are given in Table XIV, which is constructed for 0.8426 gm. The density measured at atmospheric pressure at 22.2° is 0.8265, and the expansion of 1 gm between 0° and 50°, 0.0530. Professor Conant's index of refraction was 1.4304, which combined with my density gives for the refraction constant 40.82. Professor Reid's values for index and refraction constant are respectively 1.4225 and 42.21.

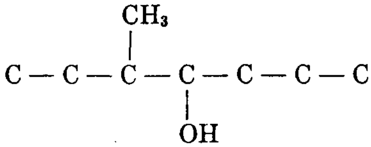
3-methyl heptanol-4.  $C-C-\overset{CH_3}{\overset{|}{C}}-\underset{\underset{OH}{|}}{C}-C-C-C$. Compressibility runs were made without incident at 50°, 95°, 0°, and the return check reading at 50°. The average deviation from smooth curves of a single one of the 54 readings was 0.09%.

TABLE XV.

3-METHYL HEPTANOL-4.

Pressure kg/cm²	Relative Volumes		
	0°	50°	95°
0	1.0000	1.0506	
500	.9697	1.0091	
1000	.9476	.9814	1.0125
1500	.9305	.9599	.9864
2000	.9162	.9421	.9664
3000	.8929	.9146	.9349
4000	.8739	.8925	.9101
5000	.8591	.8762	.8917
6000	.8453	.8614	.8754
7000		.8490	.8614
8000		.8368	.8487
9000		.8263	.8381
10000		.8167	.8283
11000		.8081	.8193
12000			.8109

Relative volumes are shown in Table XV, which is constructed for 0.8455 gm. The density measured at atmospheric pressure at 22.2° is 0.8272, and the expansion of 1 gm between 0° and 50°, 0.0599. Professor Conant's index of refraction was 1.4286, which combined with my density gives for the refraction constant 40.92. Professor Reid's values for index and refraction constant are respectively 1.4211 and 39.61.

DISCUSSION.

This discussion will not be concerned with general features of the pressure-volume-temperature relations which are common to all liquids, for these have been sufficiently considered in previous papers, but we shall be interested only in features characteristic of these particular liquids.

Triethanolamine ($N(C_2H_5O)_3$) was tried, as already stated, because experience with glycerine and similar compounds had suggested that the three OH groups might give it unusual incompressibility. As a matter of fact the compressibility is about 50 per cent greater than that of glycerine, which thus retains its position as the most incompressible organic compound, and is almost exactly the same as that of ethylene glycol. It is to be considered, however, that the rest of the molecule of triethanolamine is much more complicated than in the case of either glycerine or ethylene glycol, so that the OH groups occupy a proportionally smaller part of the total structure. The fact that the compressibility is so low in spite of the greater complication of the rest of the molecule is again evidence of the great effect of the OH group in consolidating the entire molecular structure and imparting to it relative incompressibility.

Consider next the halogen compounds. The most outstanding effect of the addition of a halogen is that the substance is carried away from the gaseous condition toward incompressibility and liquidity. Propane and butane are, of course, under ordinary conditions gaseous, while pentane is one of the most compressible of liquids. The addition of the halogen lowers the vapor pressure and decreases the compressibility by a marked degree. At 0° the volume compression under 10000 kg/cm^2 of the amyl halogens is roughly about 0.8 as much as that of normal pentane. The compression of amyl alcohol is about the same as that of the halogen compounds, so that roughly the effect of a halogen is about the same as that of an OH group in decreasing the compressibility.

If the relative volumes at 50° of the nine halogen compounds are plotted against pressure in the same diagram, certain regularities stand out. In the groups containing a common halogen the compressibility decreases on passing from propyl to butyl to amyl groups, that is, in the direction of increasing molecular weight, and in the groups containing a common alkyl the compressibility decreases as the atomic weight of the halogen increases. An exception to this general trend are butyl and amyl iodides, which have almost exactly the same compressibility.

If the volume increment on increasing temperature at constant pressure is plotted against pressure, curves will be obtained which drop rapidly with increasing pressure, as has been universally found before. The decrease of thermal expansion with increasing pressure is, however, not so large as has been previously found for other classes of compounds. As a rough average the thermal expansion of these alkyl halides at 12000 kg/cm^2 is between one third and one fourth of its value at atmospheric pressure. Among other substances instances are not uncommon in which the drop is twice as rapid. In smaller detail, the curves of thermal expansion against pressure are not entirely smooth, but exhibit various episodes, as has been previously found for other liquids. Among the most striking of such episodes shown by the alkyl halides is a stretch from 2000 to 6000 kg for butyl bromide, for which the thermal expansion between 0° and 50° is higher than is to be expected from the rest of the curve, and the last 6000 kg of the curve for butyl iodide, which shows a nearly constant expansion between 0° and 50°.

Without doubt the most significant method of analysis for the nine alkyl halides is to compare the molecular volumes instead of the relative volumes as given in the tables. The molecular volumes are obtained at once from the data of the tables by dividing the figures there given by the number of grams for which the tables are constructed (that is, by the density at 0° C at atmospheric pressure), and multiplying by the molecular weight. The volumes thus obtained may be compared either by comparing the volume increments on changing the alkyl group, keeping the halogen fixed, or by changing the halogen, keeping the alkyl group fixed. The differences of molecular volume obtained in this way are given in Tables XVI and XVII.

Table XVI gives the increments of volume on adding a CH_2 group to the molecule. It is in the first place evident that this volume is a strong function of pressure, that is, the additional space required on forcing a CH_2 group into the molecule becomes less as pressure increases. This is naturally to be described as due to the compression of the CH_2 group itself. The absolute value of the volume of the CH_2 group varies considerably, depending on the kind of molecule to which it is attached, as may be seen by an inspection of the table, the range being from 17.09 to 15.36 at atmospheric pressure, and from 13.69 to 11.18 at 12000. The proportional loss of volume with pressure, as shown by the decrement in any of the columns of Table XVI, is more nearly constant. Disregarding the initial entry in the first column, which is abnormal and is connected with the abnormally large volume of propyl chloride due to the nearness of the critical point, the volume of the CH_2 group at 12000 varies from 0.72 to 0.79 of its volume at atmospheric pressure. The ratio of the total volume of the compound at 12000 to that at atmospheric pressure varies from 0.72 to 0.75. The average compressibility of the CH_2 is therefore approximately the same as that of the entire compound, not a particularly surprising result.

The increment of volume on adding a CH_2 group to propyl chloride or iodide to form the butyl compound is less than the increment on adding CH_2 to butyl chloride or iodide to form amyl chloride or iodide. This means that as the molecule becomes more complex the increment of volume due to the addition of successive CH_2 groups becomes less, as seems not unnatural. This sequence does not hold for the bromides, however. Examination shows that the sequence would be restored if the volumes of butyl bromide were increased somewhat. Table XVII, as will be shown in more detail presently, indicates the

TABLE XVI.

Differences of Molecular Volumes at 50°.

Pressure kg/cm²	Propyl chloride to butyl chloride	Butyl chloride to amyl chloride	Propyl bromide to butyl bromide	Butyl bromide to amyl bromide	Propyl iodide to butyl iodide	Butyl iodide to amyl iodide
0	16.15	15.36	16.28	16.37	17.09	15.38
500	16.63	14.75	15.62	15.92	16.53	14.57
1000	16.29	14.56	15.31	15.62	16.01	14.18
1500	16.13	14.13	14.91	15.42	15.68	13.96
2000	15.98	13.85	14.57	15.27	15.47	13.67
3000	15.46	13.55	14.32	14.73	15.13	13.20
4000	15.20	13.25	14.16	14.35	14.70	12.78
5000	14.98	12.85	13.98	14.01	14.48	12.45
6000	14.67	12.72	13.65	13.85	14.17	12.21
7000	14.53	12.49	13.41	13.74	13.97	12.05
8000	14.28	12.40	13.17	13.57	13.76	11.86
9000	14.14	12.27	13.06	13.39	13.64	11.76
10000	13.94	12.08	12.92	13.40	13.48	11.62
11000	13.82	11.98	12.87	13.19	13.37	11.46
12000	13.69	11.91	12.70	13.04	13.30	11.18

same sort of thing, namely that butyl bromide has at all pressures a volume somewhat too small to fit in regularly with that of the other compounds.

Consider next Table XVII, which shows the differences of volume between chlorine and bromine and between bromine and iodine in the various compounds. The state of affairs is entirely different from that of Table XVI; here the columns at first increase with increasing pressure and then fall off somewhat. The total range of variation is less than in Table XVI. A possible explanation is obviously that the volumes of chlorine, bromine, and iodine are not the same sort of functions of pressure, but the compressibility of chlorine varies rapidly with pressure, dropping off rapidly with increasing pressure at the low pressure end of the range, the compressibility of bromine is intermediate in behavior, while the compressibility of iodine remains more nearly constant over the entire pressure range. A simple graphical construction will show at once the plausibility of

TABLE XVII.

DIFFERENCES OF MOLECULAR VOLUMES AT 50°.

Pressure kg/cm²	Propyl chloride to propyl bromide	Propyl bromide to propyl iodide	Butyl chloride to butyl bromide	Butyl bromide to butyl iodide	Amyl chloride to amyl bromide	Amyl bromide to amyl iodide
0	2.80	5.46	2.93	6.27	3.94	5.28
500	4.19	6.11	3.18	7.02	4.35	5.67
1000	4.36	6.42	3.38	7.12	4.44	5.68
1500	4.51	6.37	3.29	7.14	4.48	5.68
2000	4.63	6.30	3.32	7.20	4.64	5.60
3000	4.64	6.32	3.50	7.13	4.68	5.60
4000	4.56	6.38	3.52	6.92	4.62	5.35
5000	4.50	6.31	3.50	6.81	4.66	5.25
6000	4.45	6.28	3.53	6.80	4.56	5.16
7000	4.44	6.20	3.32	6.76	4.57	5.07
8000	4.43	6.11	3.32	6.70	4.49	4.99
9000	4.40	6.04	3.32	6.62	4.44	4.99
10000	4.31	6.02	3.29	6.58	4.61	4.80
11000	4.24	6.01	3.29	6.51	4.50	4.78
12000	4.23	5.90	3.24	6.50	4.37	4.64

such an explanation, which is, moreover, consistent with what other lines of evidence would lead us to expect.

In all cases the difference of volume betweeen bromine and chlorine is less than the difference between iodine and bromine. This difference is much accentuated in the butyl compounds. The butyl compounds would fall more nearly into line with the others if the volumes of butyl bromide were somewhat increased, showing, as already suggested, that the volume of butyl bromide is anomalously low.

TABLE XVIII.

VOLUMES AT 50° OF 0.8392 GRAMS OF VARIOUS OCTANOLS.

Pressure kg/cm^2	Octanol-3	2-methyl heptanol-3	2-methyl heptanol-5	3-methyl heptanol-1	3-methyl heptanol-4
0	1.0508	1.0505	1.0546	1.0404	1.0428
500	1.0116	1.0078	1.0137	1.0024	1.0016
1000	.9826	.9789	.9838	.9752	.9741
1500	.9608	.9565	.9612	.9546	.9527
2000	.9432	.9394	.9437	.9384	.9351
3000	.9148	.9216	.9159	.9110	.9078
4000	.8933	.8891	.8946	.8903	.8858
5000	.8760	.8714	.8773	.8729	.8697
6000	.8606	.8568	.8618	.8581	.8550
7000	.8474	.8435	.8481	.8448	.8427
8000	.8360	.8318	.8360	.8322	.8306
9000		.8213	.8252	.8227	.8201
10000		.8117	.8156	.8134	.8106
11000			.8076	.8050	.8021

The difference between the volumes of the halogens in these compounds is several fold greater than one would calculate from the atomic radii in solid compounds as given by X-ray analysis.

Finally, consider the volumes of the isomers of octyl alcohol. The comparison here should be of the volumes of equal masses. In Table XVIII are shown the volumes at 50° in cm^3 of 0.8392 gm, 0.8392 being very nearly the average density of the five isomers at 0°. The most striking feature is doubtless the approximate equality of the volumes of all five isomers at all pressures. The range of variation becomes less as pressure increases, the extreme variation at the maximum pressure being somewhat less than half of the variation at atmospheric pressure, but the effect is not striking because the variation is everywhere so small. Furthermore, the volume of every *pair* of substances does not tend to become more nearly equal at high pressure, as can be seen at once from the table in the case of octanol-3 and 2-methyl heptanol-3. 2-methyl heptanol-1 and 3-methyl heptanol-4 are a pair whose volume curves cross. Further study of the details of Table XVIII does not appear profitable at present. It was a considerable disappointment to find the situation so featureless. This, in addition to previous experience, has convinced me that the behavior of isomers under high pressure is so similar that it will not pay to examine the question experimentally further until there is some theory which will indicate just what the crucial effects may be.

One would expect that the average compressibility of the octanols would be less than that of octane, according to the general rule that the addition of an OH group decreases the compressibility. This is in fact the case; for example, the compression of n-octane at 50° under a pressure of 7000 kg/cm^2 (the highest pressure at which n-octane is liquid at that temperature) is 1.16 times as great as the compression of octanol-3 under the same conditions.

I am indebted to my assistant Mr. L. H. Abbot for making the readings. I am also indebted for financial assistance to the Rumford Fund of the American Academy of Arts and Sciences for the purchase of supplies, and to the Milton Fund of Harvard University for the salary of my assistant.

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References.

[1] P. W. Bridgman, Proc. Amer. Acad. 66, 185–233, 1931; 67, 1–27, 1932.

[2] P. W. Bridgman, Proc. Amer. Acad. 47, 347–438, 1912; 47, 441–558, 1912; 51, 55–124, 1915; 52, 91–187, 1916. Phys. Rev. 3, 126–141, 153–203, 1914; 6, 1–33, 94–112, 1915.

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COMPRESSIBILITIES AND PRESSURE COEFFICIENTS OF RESISTANCE OF ELEMENTS, COMPOUNDS, AND ALLOYS, MANY OF THEM ANOMALOUS.

By P. W. Bridgman.

Investigations on Light and Heat made and published with aid from the Rumford Fund.

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INTRODUCTION.

This paper represents the accumulation of a year and a half or two years' measurements of compressibility and pressure coefficient of resistance by methods already extensively used and described.[1]

The particular feature of this work is the discovery of a number of highly anomalous effects. Some of the anomalies are large scale and striking, as in the case of Ag_2O, already published elsewhere,[2] or of the element chromium. But most of the anomalies are small scale in character, and demand careful use of the apparatus in order to be sure that they exist. In fact some of these effects might previously have escaped notice, and have been dismissed as merely due to accidental irregularities. My assistant, Mr. L. H. Abbot, has developed a great deal of skill in the use of the lever piezometer, and has furthermore had the patience to make the large number of closely spaced readings necessary to decide whether an irregularity in the data is due to a real anomaly characteristic of the substance, or is only accidental. Every precaution has been taken to be sure that these anomalies are real and not in the apparatus. The apparatus itself has been carefully studied and recalibrated. The fact that all the readings of compressibility given in the following have been obtained with the same apparatus, and the fact that anomalies of different substances are so entirely different in character is itself guarantee of the genuineness of the effects. The explanation of the effects is not yet obvious; there is no reason to think that a single type of explanation will apply to all the phenomena. It is to be remarked that similar anomalies are beginning to appear in many places, the anomalies of NH_4Cl first discovered by Simon[3] being perhaps the best known. Whatever the explanation, a new field is here opening before us, just on the threshold of the last decimal place, which will occupy us experimentally for a long time.

DETAILED DATA.

Elements.

Columbium. This material was obtained from the Fansteel Co. of Chicago, and was a product of their Research Department. The measurements were made in 1929, and have been waiting since for a suitable opportunity for publication. The compressibility sample was in the form of a rod 2.6 cm. long and 3 mm. in diameter. The compressibility measurements were made as usual at 30° and 75° without incident. At 30° the average arithmetical deviation from a straight line of a single one of the 11 readings was 0.13% of the total effect, and at 75° the corresponding figure was 0.25%. The results follow:

$$\text{At } 30°,\ -\Delta V/V_0 = 5.700 \times 10^{-7}p - 2.22 \times 10^{-12}p^2,$$

$$\text{At } 75°,\ -\Delta V/V_0 = 5.778 \times 10^{-7}p - 2.12 \times 10^{-12}p^2,\ p \text{ in kg/cm}^2.$$

It is thus slightly less compressible than iron.

The resistance specimen was in the form of wire 10 cm. long and 0.050 cm. in diameter. The resistance under pressure was measured on the potentiometer, employing four terminals in the conventional way, the connections being made by spot welding. The specific resistance at 30° was 23.3×10^{-6} ohms per cm^3., somewhat high for a metal of this character. The average temperature coefficient of resistance between 0° and 100°, obtained by linear extrapolation of the readings at 30° and 75° was 0.00228. This is considerably lower than is to be expected for a pure metal, and is probable indication of not very high purity. The effect of impurity is known in practically all cases to be greater on the electrical properties than on the compressibility. The effect of pressure on resistance was measured at 30° and 75°. At each temperature the resistance is sensibly linear; at 30° the average arithmetical deviation of a single one of the 11 readings from a straight line was 0.10%, and at 75° the corresponding figure for the same number of readings was 0.27% of the maximum effect. The coefficients found were as follows:

At 30°, average coefficient 0–10000 kg, -1.21×10^{-6},

At 75°, average coefficient 0–10000 kg, -1.25×10^{-6}.

The coefficient is thus normal in sign and close in numerical value to that of cobalt, which also has about the same compressibility. Columbium thus appears to be an unexciting, prosaic substance.

Rhodium. Measurements had been made previously on this material;[4] the point in repeating them was that I had now obtained material of considerably higher purity from Johnson Matthey and Co. of London.

The compressibility measurements were made on a rod 2.5 cm. long and 2 mm. in diameter. The measurements went without incident, at 30° and 75° as usual. At 30° the average arithmetical departure from a straight line of a single one of the 17 readings was 2.9% of the maximum effect, and at 75° the corresponding figure for 14 readings was 0.7%. The results are:

$$\text{At } 30°,\ -\Delta V/V_0 = 3.606 \times 10^{-7}p - 2.73 \times 10^{-12}p^2,$$

$$\text{At } 75°,\ -\Delta V/V_0 = 3.702 \times 10^{-7}p - 2.75 \times 10^{-12}p^2.$$

The thermal expansion was measured in a specially constructed apparatus used previously in measuring the expansion of single crystals. The apparatus is described in my paper dealing with single crystals.[5] The mean linear differential expansion between Rh and Fe in the temperature range between 12° and 30° was found to be 3.49×10^{-6}, which, assuming 12.0×10^{-6} for Fe, gives 8.5×10^{-6} for Rh. International Critical Tables (I. C. T.) gives 8.4×10^{-6}.

The electrical resistance was measured on a swaged wire about 10 cm. long and 0.08 cm. in diameter. The potentiometer method with four terminals was used, the terminals being attached with soft solder. Two runs were made at 30°, agreeing within the limits of error, and a single run at 75°. At each temperature the relation between pressure and resistance was sensibly linear. The results were:

At 30°, average coefficient between 0 and 12000 kg, -1.764×10^{-6},

At 75°, average coefficient between 0 and 12000 kg, -1.708×10^{-6}.

The specific resistance, measured at 30°, was 4.95×10^{-6}, which, corrected by the temperature coefficient, gives 4.35×10^{-6} at 0° C. I. C. T. gives 4.69.

The average temperature coefficient of resistance between 0° and 100°, obtained by linear extrapolation of measurements at 30° and 75° was 0.00462. This is much higher than my previous value, 0.00399, and is evidence of the much higher purity of this material; the same is also shown by the low specific resistance.

The previous relation between pressure and resistance at 30° was: $\Delta R/R_0 = -1.738 \times 10^{-6}p + 9.7 \times 10^{-12}p^2$, which gives for the average coefficient between 0 and 12000 -1.62×10^{-6}, against -1.76 above. The difference is in the direction always found without exception to be the effect of impurity. The compressibility of the former sample at 30°, however, was: $-\Delta V/V_0 = 3.72 \times 10^{-7}p - 2.67 \times 10^{-12}p^2$, somewhat higher than the value found for this new specimen. This is somewhat unusual, but there is no universal rule here as for the resistance. It would indicate an impurity of higher compressibility in the previous sample, such, for example, as Pd.

Ruthenium. This has not been previously measured. It belongs to the hexagonal system, so that the following results can only be a mean of some sort for different directions. The material was obtained from Johnson and Matthey at the same time as the rhodium. The compressibility specimen was in the form of a swaged rod 2.7 cm. long and 3 mm. in diameter. It was not perfectly regular in appearance, but appeared as if there might be seams in it. On the first application of pressure there was a large change of zero. This may have been due to seams, or to the fact that the crystal system is not cubic; large effects of this nature are to be expected for non-cubic materials. Two more applications of 12000 were made before the routine measurements were attempted. Also at 75° an additional seasoning application of 12000 was made before the measurements. This seasoning seemed effective, for the change of zero during the measurements, both at 30° and 75°, was not more than the irregularity of the other points. At 30° the average arithmetical departure from a straight line of a single one of the 13 readings (after making one discard) was 1.3% of the maximum effect, and at 75° the corresponding figure for 15 readings was 0.64%. As in the case of Rh, the relative irregularity is greater than usual because of the smallness of the compressibility; the absolute irregularity is not greater, and in fact is even less than usual. There may be some significance in the fact that the discarded point at 30° was at the maximum pressure. This was off the curve by one third of the total effect; there is perhaps a possibility that this is a genuine anomaly like that of many of the other substances of this paper, but it would have been very difficult to establish, because it was at the end of the pressure range, and the matter was not investigated further.

The results for volume compressibility, assuming equal linear compressibility in all directions, are:

$$\text{At } 30^\circ, -\Delta V/V_0 = 3.42 \times 10^{-7}p - 2.13 \times 10^{-12}p^2,$$

$$\text{At } 75^\circ, -\Delta V/V_0 = 3.45 \times 10^{-7}p - 2.13 \times 10^{-12}p^2.$$

The average compressibility is therefore slightly less than that of Rh, and fits in with what might be expected from the periodic table.

The linear thermal expansion at atmospheric pressure was measured in the same special apparatus as used for Rh. The mean value between 32° and 13° was 6.75×10^{-6}. I. C. T. gives 9.1; this very marked divergence is perhaps to be ascribed to the non-cubic symmetry of the crystal, for it is now pretty well established that measurements by different observers on different specimens of non-cubic metals are likely to give different results. The principal value of such measurements on multi-crystalline aggregates is in establishing upper and lower limits for the constants corresponding to different directions in the crystal.

The resistance was measured on the compressibility sample, the technical difficulties having proved too great to permit swaging it to a more slender rod, as had been possible with Rh. The current was led in through copper terminals making soldered connection over the entire flat ends. In this way the lines of current flow are straight from the beginning, and end effects are avoided. The current, about one half amp., was made as large as possible without introducing troublesome thermal effects. It was supplied from a storage battery, through heavy manganin resistances in an oil bath. The resistance was very low, only 0.00022 ohms, so that the measurements could not be as accurate as usual, in spite of the large current. Pressure measurements were made at 0° and 95°. Within the limits of error the relation between pressure and resistance was linear. At 0°, the average arithmetical departure of a single one of the 16 readings from a straight line was 2.2% of the pressure effect, and the corresponding figure at 95° for 13 readings was also 2.2%. The results were:

At 0°, average coefficient 0–12000, -2.48×10^{-6},

At 95°, average coefficient 0–12000, -3.20×10^{-6}.

So large an effect of temperature is not usual.

Other electrical data were also obtained. At 0° the specific resistance is 7.64×10^{-6}. I. C. T. gives 10×10^{-6}, to only one significant figure. The temperature coefficient of resistance at atmospheric pressure was determined from readings at 0°, 24° 50°, and 95°. The relation between resistance and temperature is of the second degree in the temperature, namely:

$$R = R_0 [1 + .00494t - 3.64 \times 10^{-6}t^2].$$

The negative sign for the second term is not usual for most metals, but seems to be of frequent occurrence in the metals of the platinum groups. This formula gives for the average coefficient between 0° and 100° 0.00458. There seems to be no previous value for comparison, but this is high for a metal of this character, considerably higher than for pure platinum, for example, and is in so far evidence of high purity.

Chromium. The compressibility of this element has been previously measured,[6] but the purity was not high, and repetition was desirable. The previous sample was highly brittle, and in appearance was like the metal which has been available for many years made by the Goldschmidt process. Its compressibility was not notable in any way, being 5.19×10^{-7} at 30°, somewhat less than for Fe. For the new measurements I was fortunate to obtain from Mr. P. H. Brace of the Research Department of the Westinghouse Electric and Manufacturing Co. a swaged rod of Cr. Mechanically, this was soft enough to be readily filed; the hardness of commercial chromium plating is said to be due to occluded hydrogen. This specimen had been made in connection with a systematic study of the properties of the pure metal, and every effort had been made to obtain metal of high purity; the fact that it was malleable enough to be swaged is evidence that it was much purer than the previous sample. In order to further test the purity, Dr. Martin Grabau was kind enough to make a spectroscopic examination under the direction of Professor F. A. Saunders, and it was found to be of quite an exceptional degree of purity, the only detected impurity being a doubtful trace of magnesium.

Only one piece of this chromium was available, a rod 4 cm. long and 2.5 mm. in diameter. The resistance was first measured, and then the length of the rod was cut to 2.7 cm. for the compressibility measurements. The resistance measurements suggested most unusual compressibility effects, which were indeed found. Readings of compressibility were made at the usual 30° and 75°, and also at − 40°, since the resistance measurements had suggested the possibility of new effects at low temperatures. The temperature bath at − 40° consisted of about 5 gallons of alcohol, rapidly stirred, into which small pieces of solid CO_2 were dropped when required to maintain the temperature constant as indicated on a pentane thermometer. The regulation was done by hand, one observer continually watching the temperature bath, while the other made the compressibility readings. The results were highly unusual, as shown in Figure 1. In this figure the actual displacements of the slider of the potentiometer are recorded, multiplied by such a constant as to reduce them approximately to changes of length relative to the steel of which the piezometer is made. At 30° and 75° Cr is on the average less compressible than Fe, but the difference is far from linear, and behaves in the complicated way shown. Except for one point which may have involved a blunder in reading, although this is not likely, all the points at these two temperatures lie close to a single curve, that is, there is no difference between increasing and decreasing pressure. In fact, the direction of alteration of pressure was changed several times in order to be sure that the curve is really characteristic of the metal, and does not involve its past history. At − 40°, on the other hand, the character of the phenomena is entirely altered; the compressibility is now greater than that of Fe, the sinuosities in the curve of change of

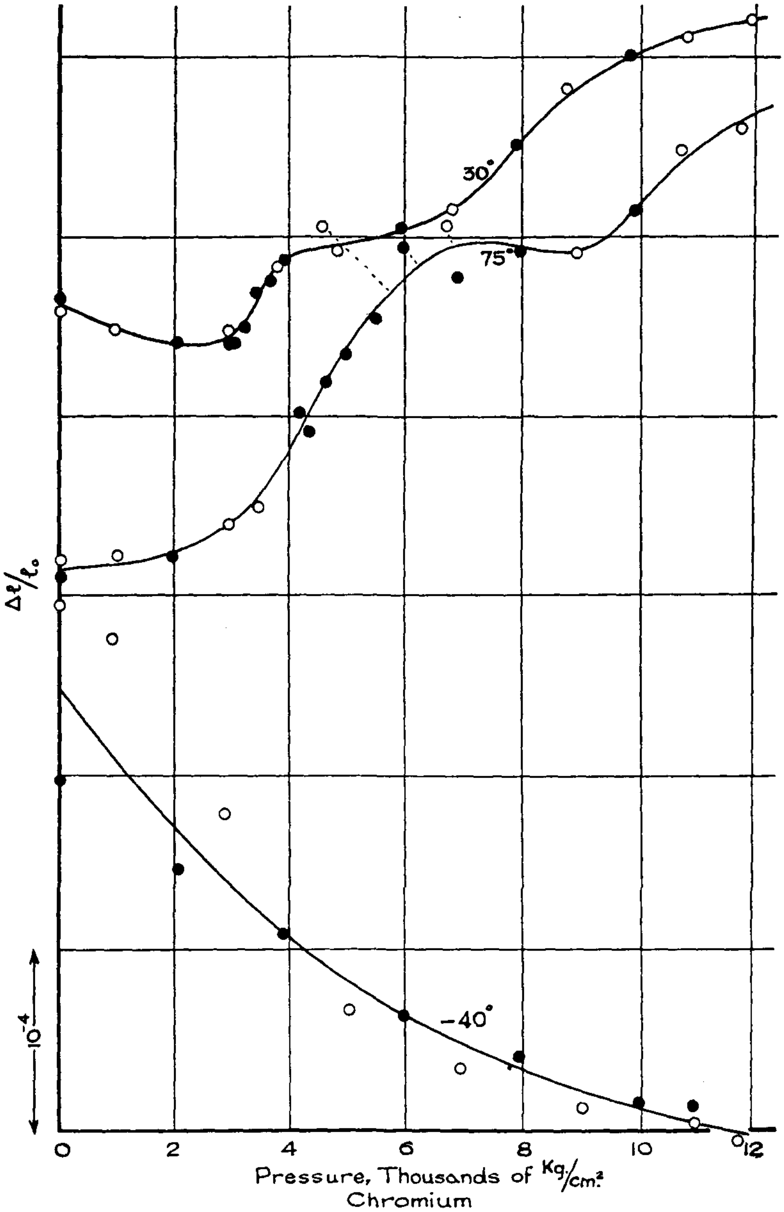

FIGURE 1. Deviations from linearity of the linear compression of chromium. The open circles are for increasing pressure and the solid circles for decreasing pressure.

length have now disappeared, and there is a distinct hysteresis between the readings with increasing and decreasing pressure. It is possible that the hysteresis would have disappeared if the pressure run had been repeated, but because of the inconvenience of maintaining temperature at − 40°, this was not attempted, but the mean result, indicated by the curve, was used in the final calculations. The final results for change of volume are shown in Table I. Chromium is cubic, so that the volume changes can be accurately computed from the changes of length in a single direction. In making the calculations at − 40° the compressibility of iron was extrapolated linearly from the measured values at 30° and 75°.

TABLE I.

RELATIVE VOLUME CHANGES OF CHROMIUM.

Pressure kg/cm²		$\Delta V/V_0$	
	− 40°	+ 30°	+ 75°
1000		000606	000552
2000	.001320	001167	001089
3000		1611	1605
4000	2577	2109	2049
5000		2622	2430
6000	3717	3129	2826
7000		3618	3336
8000	4869	4044	3876
9000		4473	4410
10000	5940	4938	4878
11000		5427	5310
12000	6981	5910	5772

The resistance was measured, as already stated, on the same rod as that from which the compressibility sample was cut. It proved not to be possible to soft solder this, and spot welding on so large a diameter was not practical with the equipment on hand. Current connections at the two ends were made by massive sleeves of copper driven tightly over the ends. Under pressure these become even tighter because of differential compressibility. The design of the current connections is thus such as to start the lines of current flow approximately straight, avoiding end effects. The potential terminals were fine springs. It was soon found that the behavior of resistance is so complicated as to make it desirable to extend the study of resistance over a wider temperature range than usual. In addition to the conventional measurements at 30° and 75°, runs were also made at 0°, − 20°, − 40°, and − 78.5°, the latter being the normal sublimation point of solid CO_2. The temperatures below 0° were obtained in an alcohol bath, cooled with solid CO_2 and regulated manually by a second observer, as already explained. The greatest difficulty to be anticipated at low temperatures is the freezing of the pressure transmitting liquid. Of all the substances liquid under ordinary atmospheric conditions, and therefore capable of being used without the introduction of complicated precompressors, as would be necessary in handling a gas, iso-pentane is that with the lowest freezing point, and this was accordingly tried. It proved gratifyingly successful. It allowed readings at − 78.5° to a maximum pressure of 8000 kg/cm^2 before its viscosity became too great (or before it froze, which, was not determined), and at − 40° and higher, readings to the full 12000.

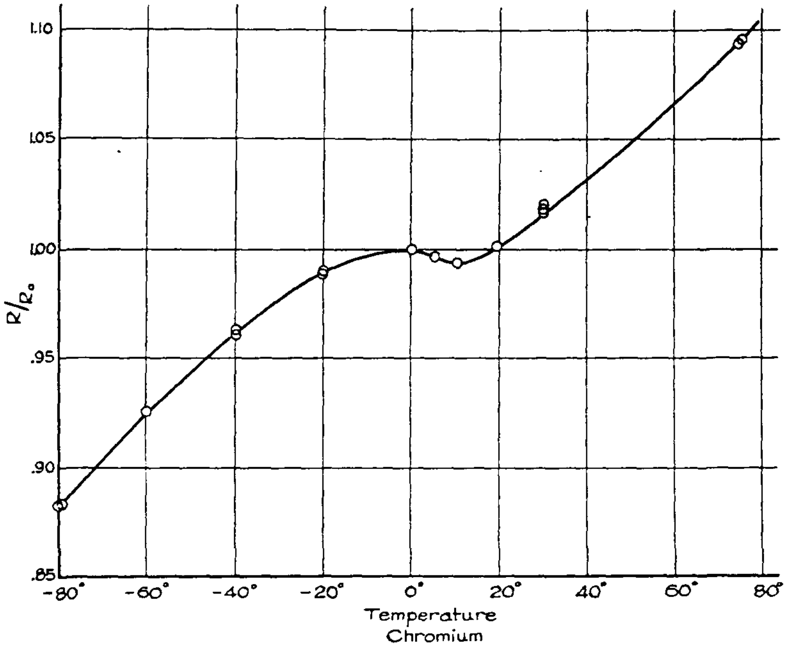

FIGURE 2. The relative resistance of chromium at atmospheric pressure as a function of temperature.

Not only is the resistance abnormal when measured as a function of pressure at constant temperature, but it is also abnormal at atmospheric pressure as a function of temperature, an effect apparently not observed before because pure enough metal was not available. At 0° at atmospheric pressure the specific resistance is 18.9×10^{-6}. The only figure given by I. C. T. is 2.6 for the pure metal on the authority of Shukov. This cannot possibly be correct, but would indicate a misprint somewhere. The relative resistance as a function of temperature at atmospheric pressure is reproduced in Figure 2. The curve is very similar in character to the curve for the volume of liquid water, with an abnormality in the neighborhood of 0° C. of much the same kind. It is to be noticed that there is no hysteresis between resistance and temperature, but within the limits of error the relation is single valued. The order of the readings was as follows: first, 3 points at 30°, made before and after the pressure run, then at 75°, 30° again, 0°, − 78.5°, − 60°, − 40°, − 20°, + 20°, + 30°, + 10°, and + 5°.

The relative resistance as a function of pressure is reproduced in Figure 3, where all the experimental points are shown. The scale of the figure is not large enough to show any departure of these points from smooth curves. In practically all cases the points alternated between increasing and decreasing pressure, readings being made on the odd thousands with increasing pressure, and on the even thousands with decreasing pressure. There is no appreciable hysteresis, even at − 40°, although the compressibility measurements, made after the resistance measurements, did show hysteresis. The crossing of the curves means that there are regions, varying with the pressure, in which the temperature coefficient of resistance is negative. This, of course, is to be expected because such a phenomenon has already been found at atmospheric pressure.

The curves for the resistance and the compressibility must speak for themselves, and I shall not attempt to make any comment except to emphasize that whatever internal change it is which is responsible for the anomalies occurs for the most part without hysteresis, and to remark that manganese, which occurs next to chromium in the periodic table, is known to have polymorphic transitions at atmospheric pressure and to have a highly complicated crystal structure with 54 atoms in the unit cell. It is also to be noted that internal changes like these of pure chromium may be suppressed by impurities, the previous measurements of compressibility having given no hint of the complications now found.

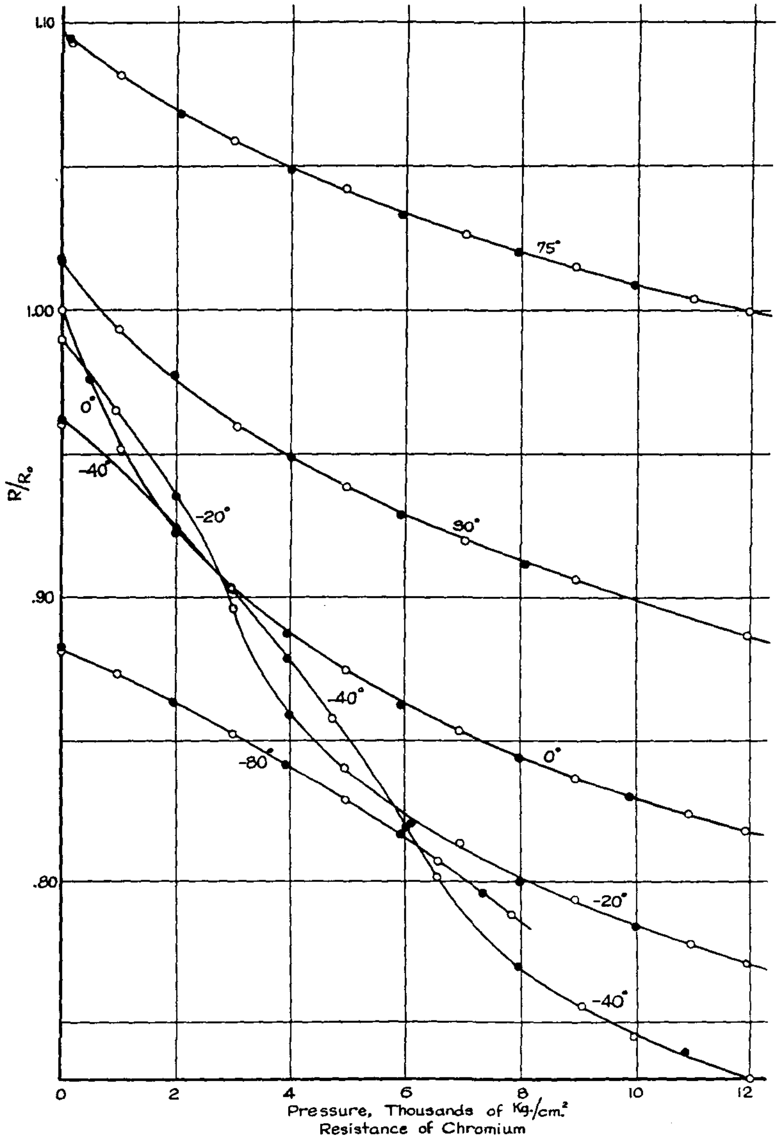

FIGURE 3. The relative resistance of chromium as a function of pressure at various temperatures. The open circles are for increasing pressure and the solid circles for decreasing pressure.

Arsenic. Measurements have already been made of polycrystalline arsenic on two different occasions.[7] The second measurements were made on samples which were approximately single grains, in that there was a strongly preferred direction of orientation. Surprising results were found; it appeared that there must be a polymorphic transition in the neighborhood of 5500 kg., and it was evident that there were very large differences in different directions. However, the results were not consistent, and in particular there were obscure effects on the first application of pressure for which no parallel in the behavior of other substances was known. It seemed evident that satisfactory results could not be expected until truly single crystals were available.

Measurements have now been made on a number of single crystals, without even yet, however, leading to a completely satisfactory cleaning up of the situation. The preparation of single crystals proved an unexpectedly complicated and difficult matter, and Dr. W. E. Danforth, as my assistant, spent a good part of an academic year working out the technique and producing the crystals. The original material was of the "Kahlbaum" grade. This proved to have a very appreciable fraction of non-volatile impurity, probably carbon or silicon. The metal was first purified by a vacuum distillation in pyrex at 500° C, removing the non-volatile impurities, and also separating out the volatile As_2O_3, the metallic arsenic condensing in the intermediate part of the tube. The final crystallization was in heavy quartz tubes. The powdered arsenic from the first purification was introduced into a bulb at one end of the tube, which was bent into several straight lengths to give different orientations in the final crystal. During the exposure to the atmosphere incident to getting into the quartz, more As_2O_3 was formed, which had to be eliminated by a second distillation. This second distillation was at 300°, which drove the As_2O_3 into an end of the tube left projecting from the furnace. This end was then sealed off from the rest of the tube, thus removing the As_2O_3. The quartz was then lowered at about 30 cm. per hour from a furnace maintained at 950°, well above the triple point temperature. The pressure at the triple point is 36 atmospheres. The quartz is amply strong to hold this pressure if perfect seals are made. The casting obtained after lowering out of the furnace was expected to be a single grain, but experience proved that it usually was not. Accordingly, after the first lowering from the furnace the now empty bulb was sealed off, leaving merely the bent tube fairly well filled with arsenic. This was now run through the furnace again in the opposite direction, and this second crystallization usually produced a single grain.

There is a very strong tendency for the principal hexagonal axis to lie in a horizontal plane (cleavage plane vertical), and in fact this proved to be practically invariable, so that the only way of getting other orientations was to bend the tube. Successful specimens of high orientations are extremely difficult to get, however, because the thermal contraction across the cleavage planes is so high that if the cleavage plane makes much of an angle with the length the rod will break spontaneously into short pieces a few mm. long on cooling. The highest orientation that was obtained was 50° between cleavage plane and length, and this was exceptional, angles greater than 10° being rare.

Crystals were made in two sizes: 3 mm. in diameter for measurements of compressibility and about 1 mm. for measurements of resistance. Resistance measurements were made on two samples of 90° between axis and length (cleavage plane parallel to length), one of 83°, and one of 73°. Compressibility measurements were made on two 90° orientations, one 79°, one 56.5°, and one 40°. The results were highly abnormal; it is difficult to decide just how much should be reproduced here, since the abnormalities are in some respects not reproducible. In general characterization, it appears that the direction perpendicular to the axis, in the cleavage plane, is abnormal, but along the axis, the behavior is quite smooth and canonical. One might at first be inclined to expect the opposite behavior, looking perhaps for irregularities associated with the closing together of the atoms in the direction of greatest separation, that is, across the cleavage plane. That the abnormalities are in the opposite direction indicates interpenetration effects in the direction of closest packing, the fields of force being more symmetrical at larger distances of atomic separation. The crystal arrangement of arsenic is much like that of bismuth and antimony, which are also abnormal in many particulars, but not to as extreme a degree as arsenic. One may suspect an unusual asymmetry in the arsenic atom.

The two compressibility measurements on the two 90° orientations agreed in showing several ranges of pressure in which the change of length is approximately a linear function of pressure, the different ranges running into each other continuously with a break only in the derivative, so that there is no evidence for a polymorphic transition of the ordinary kind, as was suspected from the previous measurements. But on the other hand, there is no exact correspondence between the results for the different samples. The first sample showed at 30° linear ranges from 0 to 4060, from 4060 to 6300, and from 6300 to 12000 kg/cm². This specimen showed no hysteresis at all on the very first application of pressure. Similarly at 75°, there were three ranges, from 0 to 3400, from 3400 to 7600, and from 7600 to 12000, and there was also no hysteresis except at the two lowest points, 700 kg. and atmospheric pressure. The second 90° orientation, on the other hand, showed at 30° on the first application of pressure very

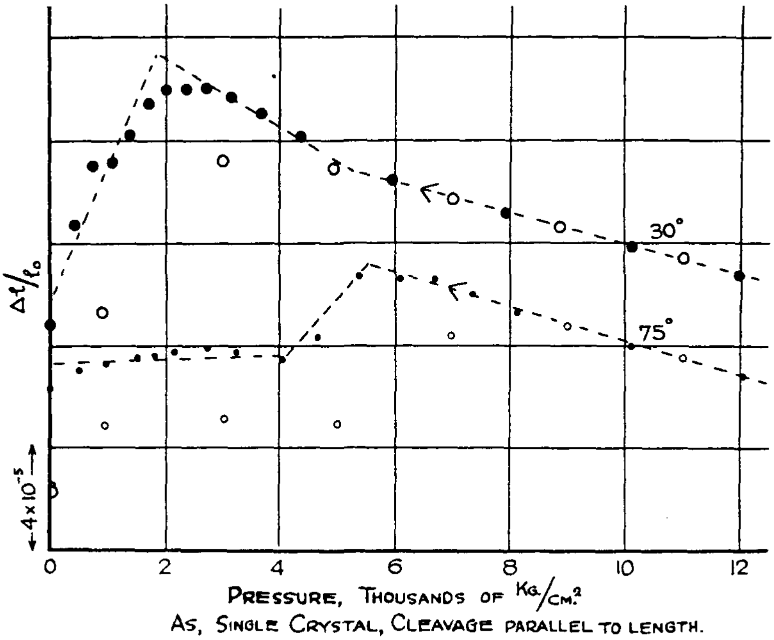

FIGURE 4. Differential linear compression of single crystal arsenic, cleavage plane parallel to length. The larger circles are for 30° and the smaller for 75°; open circles with increasing pressure, and solid circles with decreasing pressure.

marked hysteresis up to 5200, but from here on the same perfectly linear hebavior up to 12000. Similarly at 75°, there was hysteresis, this time up to 8000, but from here on linear behavior. The low pressure episodes on this second sample were not linear like those of the first; they are reproduced in figure 4. The straight lines indicated in this figure are obviously only a rough approximation, but they are sufficiently good considering the failure of reproducibility, and it is the results given by the straight lines which are reproduced below. The 79° orientation gave essentially a linear relation at both temperatures over the entire pressure range, except for slight curvature at the low pressure end. There was, however, marked hysteresis at 30°. The 56.5° orientation gave results of considerable irregularity, with hysteresis. They can be fairly well reproduced by assuming at each temperature two linear ranges; at 30° from 0 to 7100 and from 7100 to 12000, and at 75° from 0 to 4800 and from 4800 to 12000. It is not certain, however, whether these effects are legitimate, because of difficulties in the preparation of the specimen. Two pieces of this orientation were used, each only 2.3 mm. long, piled together with

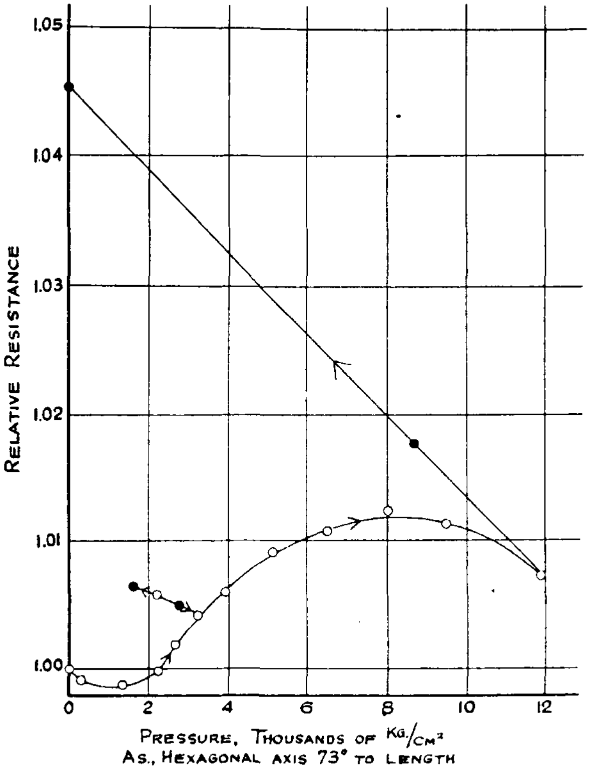

Figure 5. The relative resistance at 30° C of single crystal arsenic, hexagonal axis 73° to length, on the initial application of pressure.

three steel separators to make up the length required by the piezometer. The ends were not perfectly flat, and the irregular effects may have been geometrical, due to rocking of one piece on another. The linear results are reproduced below, but the average compressibility, 0 to 12000, is much more secure than the results for the separate ranges. The 40° orientation gave results like those for any normal material; there was no evidence of separate episodes, the hysteresis, although perceptible, was slight, and there were no important seasoning effects on the first application of pressure. The average deviation of single points at both temperatures from smooth curves through the centers of the hysteresis loops was 0.4% of the maximum effect; the deviation would be much less if the two branches of the hysteresis loops were treated separately.

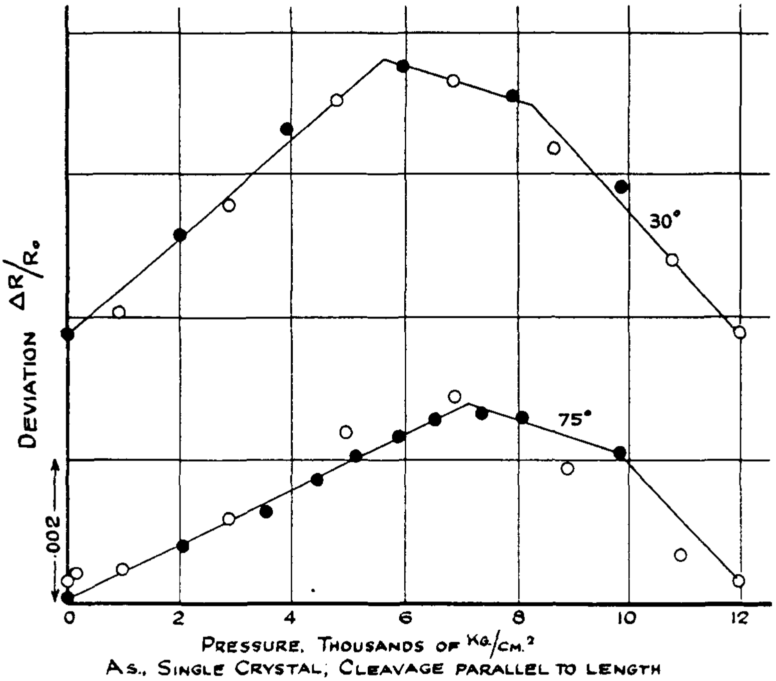

Figure 6. Deviation from linearity of the change of relative resistance of single crystal arsenic, hexagonal axis perpendicular to the length, at 30° and 75°. The open circles are for increasing pressure and the solid circles for decreasing pressure.

The resistance samples covered a smaller range of orientations: there were two of 90°, one of 83°, and one of 73°. The results of the resistance measurements agreed in that the whole pressure range breaks up into sub ranges, but did not agree with regard to the first seasoning effects of pressure. One 90° and the 83° orientation showed no seasoning effects at all, but the other two specimens showed very large seasoning effects of the same general character. The effect of the first application of pressure on the resistance of the 73° specimen is reproduced in Figure 5. It is the same sort of thing as was previously obtained with polygrained material, and accounts for the apparently positive coefficient at low pressures of the virgin specimen, followed

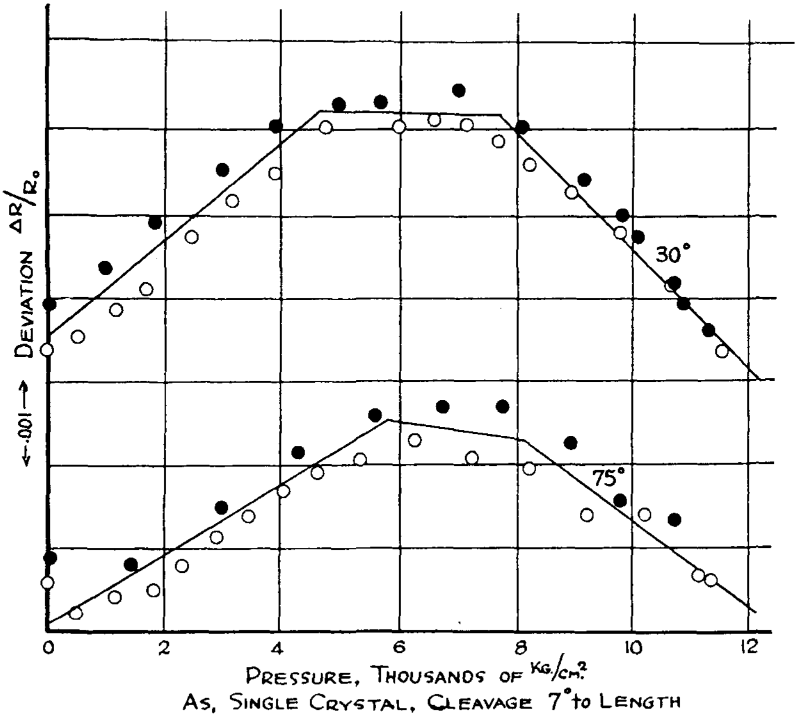

Figure 7. Deviation from linearity of the change of relative resistance of single crystal arsenic, hexagonal axis 83° to the length, at 30° and 75°. The open circles are for increasing pressure and the solid circles for decreasing pressure.

by a permanent negative coefficient. After the first application of pressure, there was very little hysteresis between resistance and pressure, but the behavior became steady almost at once. Unfortunately, the first 90° specimen, which showed no initial seasoning effects, did later develop irregularities, and on removing from the pressure cylinder fell apart into several pieces, so that the results had to be discarded. Within the somewhat wider limits of error, the results on it agreed with those of the other 90° orientation obtained after seasoning was complete. In figures 6 and 7 the deviations from linearity of resistance as a function of pressure for the second 90° orientation and the 83° orientation are given. It is not difficult, knowing already from the compressibility measurements that there are linear ranges, to persuade oneself that there are linear ranges here, which are indicated in the figures. It must be remembered that in general resistance has proved much less sensitive to this sort of irregularity than compressibility.

The numerical results now follow.

Compressibility.

First 90° orientation.

30°

0– 4050, $-\Delta l/l_0 = -1.46 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$
4050– 6300, $-\Delta l/l_0 = 1.76 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$
6300–12000, $-\Delta l/l_0 = 1.88 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$
Average, 0–12000, 1.850×10^{-7}

75°

0– 3400, $-\Delta l/l_0 = 1.41 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$
3400– 7600, $-\Delta l/l_0 = 1.69 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$
7600–12000, $-\Delta l/l_0 = 1.94 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$
Average, 0–12000, 1.680×10^{-7}

Second 90° orientation.

30°

0– 1850, $-\Delta l/l_0 = 0.43 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$
1850– 5200, $-\Delta l/l_0 = 2.30 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$
5200–12000, $-\Delta l/l_0 = 1.88 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$
Average, 0–12000, 1.786×10^{-7}

75°

0– 4300, $-\Delta l/l_0 = 1.70 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$
4300– 5300, $-\Delta l/l_0 = 1.12 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$
5300–12000, $-\Delta l/l_0 = 1.90 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$
Average, 0–12000, 1.771×10^{-7}

79° orientation.

30°

$-\Delta l/l_0 = 1.826 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$

75°

$-\Delta l/l_0 = 1.867 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$

56°.5 orientation.

30°

0– 7100, $-\Delta l/l_0 = 5.87 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$
7100–12000, $-\Delta l/l_0 = 3.19 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$
Assuming no breaks:
0–12000, $-\Delta l/l_0 = 4.77 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$

75°

0– 4800, $-\Delta l/l_0 = 7.61 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$
4800–12000, $-\Delta l/l_0 = 3.35 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$
Assuming no breaks:
0–12000, $-\Delta l/l_0 = 5.04 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$

40° orientation.

At 30°, $-\Delta l/l_0 = 16.75 \times 10^{-7}p - 36.1 \times 10^{-12}p^2$
At 75°, $-\Delta l/l_0 = 17.38 \times 10^{-7}p - 37.4 \times 10^{-12}p^2$

The figures just given for the 40° orientation are only approximate, for the relation differs markedly from the second degree. The second degree formulas are so constructed as to give the correct changes of length at 0, 6000 and 12000. At intermediate points the actual changes of length can be obtained by combining with the second degree formulas a deviation curve. These deviation curves can be drawn with sufficient accuracy from the following specifications. At 30° the change of length is greater numerically at 3000 by 0.000177 than given by the second degree formula and at 9000 0.000164 less.

At 75° the actual change of length is greater numerically at 3000 by 0.000239 than given by the formula, and at 9000 0.000164 less. The maximum deviation from linearity at each temperature comes at 4600 instead of 6000 as it would if the relation were second degree.

Resistance:

90° orientation.

Specific resistance at 30°, 27.5×10^{-6}

Average temperature coefficient, 0–100°, + 0.00479.

30°

$$0\text{–}5500,\ \Delta R/R_0 = -1.74 \times 10^{-6}p$$
$$5500\text{–}8200,\ \Delta R/R_0 = -2.68 \times 10^{-6}p$$
$$8200\text{–}12000,\ \Delta R/R_0 = -3.30 \times 10^{-6}p$$

75°

$$0\text{–}7100,\ \Delta R/R_0 = -1.56 \times 10^{-6}p$$
$$7100\text{–}9800,\ \Delta R/R_0 = -2.08 \times 10^{-6}p$$
$$9800\text{–}12000,\ \Delta R/R_0 = -2.58 \times 10^{-6}p$$

83° orientation.

Specific resistance at 30°, 27.6×10^{-6}

Average temperature coefficient, 0–100°, .00483

30°

$$0\text{–}4800,\ \Delta R/R_0 = -1.33 \times 10^{-6}p$$
$$4800\text{–}7400,\ \Delta R/R_0 = -1.81 \times 10^{-6}p$$
$$7400\text{–}12000,\ \Delta R/R_0 = -2.34 \times 10^{-6}p$$

75°

$$0\text{–}6100,\ \Delta R/R_0 = -1.08 \times 10^{-6}p$$
$$6100\text{–}8200,\ \Delta R/R_0 = -1.43 \times 10^{-6}p$$
$$8200\text{–}12000,\ \Delta R/R_0 = -1.96 \times 10^{-6}p$$

73° orientation.

Specific resistance at 30°, 30.4×10^{-6}

Average temperature coefficient, 0–100°, .00472

30°

$$\begin{aligned} 0\text{–}\ 2550,\ \Delta R/R_0 &= -\ 2.77 \times 10^{-6}p \\ 2550\text{–}\ 8200,\ \Delta R/R_0 &= -\ 2.59 \times 10^{-6}p \\ 8200\text{–}12000,\ \Delta R/R_0 &= -\ 2.88 \times 10^{-6}p \end{aligned}$$

75°

$$\begin{aligned} 0\text{–}\ 3000,\ \Delta R/R_0 &= -\ 2.80 \times 10^{-6}p \\ 3000\text{–}12000,\ \Delta R/R_0 &= -\ 2.30 \times 10^{-6}p \end{aligned}$$

The thermal expansions were also determined as usual, and were as follows:

Orientation	Mean linear expansion, 30°–75°	
	Extreme values	Average
90°, 1st	3.2 to 3.6 × 10^{-6}	3.4 × 10^{-6}
90°, 2nd	3.4 to 6.8	5.1
79°	8.0 to 8.8	8.4
56°.5	13.6 to 14.4	14.0
40°	27.5 to 29.3	28.4

Disregarding for the present the various abnormalities, certain broad features stand out. There is great difference of properties in different directions. In the cleavage plane the compressibility is less than that of iron, whereas across the plane the compressibility is more than ten times as high. This estimate is based on a rough linear extrapolation according to the $\cos^2$ law of the compressibilities of the orientations listed above, which gives for the average compressibility up to 12000 kg. the figure 19×10^{-7} for the 0° orientation. The previous conclusion about large differences of compressibility in different directions is therefore verified, but the difference turns out to be more extreme than was surmised. In the same way the linear expansion for the 0° orientation may be estimated to be about 43×10^{-6} against 5 or less for the 90° orientation, again showing an enormously greater deformability across the cleavage planes.

The *initial* linear compressibility for the 0° orientation may be estimated by a similar linear extrapolation to be about 27×10^{-7}. This would make the initial volume compressibility 31.6×10^{-7}. The only previous measurement of this is by Richards,[8] on polycrystalline material, who found 44×10^{-7}, so much higher than 31.6 as to be beyond the possibility of experimental error. The difference is probably to be explained as an effect of internal strains arising from the very great differences of compressibility in different directions, and again emphasizes the conclusion that bulk measurements on haphazard aggregates of strongly non-isotropic crystals are of doubtful significance.

The average volume expansion, calculated from the linear expansions in the 90° and 0° orientations is 53×10^{-6}, or 17.7×10^{-6} for the average linear expansion of the haphazard aggregate. I. C. T. gives for this 4.7×10^{-6} from Fizeau, measured on sublimed mixed crystals. It would appear that Fizeau's orientations must have been almost exclusively the 90° orientation.

Now consider the abnormalities. These are of two sorts: seasoning effects, and breaks in the direction of the curves after complete accomodation. The seasoning effects are not universally present; one 90° and one 79° compressibility sample, and one 90° and one 83° resistance sample showed no such effects. The seasoning effects are so striking in some of the other samples, however, that it is difficult to think that the effect is entirely spurious. It may possibly be connected with the rate of cooling, and may mean that there is some other modification at high temperatures that gets carried down to room temperature into its region of instability by rapid cooling, and that the reaction then runs when the atoms are joggled, as it were, by an application of pressure. The matter requires further examination.

The other abnormality, ranges in which the behavior is linear with sharp changes in direction but no discontinuities at the points separating the ranges, is doubtless a real phenomenon, for it is shown by all the high orientations, both in compressibility and resistance. The precise details are not consistent, however, which means that this is a sensitive phenomenon. In general there seem to be three linear ranges at the 90° orientation; as the orientation changes toward zero, these phenomena rapidly disappear, and the crystal becomes normal.

Beryllium. This material I owe to the courtesy of Dr. R. Mehl, at the time Director of the Department of Metallurgy of the Naval Research Laboratory at Anacostia, D. C. The material was in the form of a lump approximately 2.5 cm. on a side; the original source was presumably the now defunct Beryllium Corporation of America. The method of formation of this lump is not known to me, but the result was a piece containing one large crystal grain, as could be established by the cleavage, which was quite perfect. This specimen was also distinguished from specimens made earlier in the art by its apparent complete homogeneity, no blow holes, slag inclusions, or other imperfections being evident. Two specimens were worked out of the lump by grinding, one parallel and one perpendicular to the axis, each about 7.5 mm. long. Beryllium crystallizes in the hexagonal system, and has therefore a single axis of six-fold symmetry.

The compressibility results were unusual for a single crystal in that there was relatively large hysteresis between the readings obtained with increasing and decreasing pressure. The hysteresis was about the same for both specimens at both temperatures. The width of the hysteresis loop at the maximum amounted to about 2% of the total deformation. The individual points on the ascending and descending branches for the parallel specimen were regular within at most one tenth of the width of the loop; the irregularity of the other orientation was three or four times greater. The sign of the hysteresis in the changes of length was the same in both directions, so that there is also a volume hysteresis of approximately the same amount. The direction of the volume hysteresis is normal, that is, a lag, as is demanded by thermodynamics.

The numerical results follow:

Hexagonal axis parallel to the length,

At 30°, $-\Delta l/l_0 = 2.20 \times 10^{-7}p - .70 \times 10^{-12}p^2$,
At 75°, $-\Delta l/l_0 = 2.30 \times 10^{-7}p - .70 \times 10^{-12}p^2$.

Hexagonal axis perpendicular to the length,

At 30°, $-\Delta l/l_0 = 2.82 \times 10^{-7}p - 1.67 \times 10^{-12}p^2$,
At 75°, $-\Delta l/l_0 = 2.82 \times 10^{-7}p - 1.70 \times 10^{-12}p^2$.

From these results the volume compressibility may be calculated to be:

At 30°, $-\Delta V/V_0 = 7.84 \times 10^{-7}p - 4.24 \times 10^{-12}p^2$,
At 75°, $-\Delta V/V_0 = 7.94 \times 10^{-7}p - 4.31 \times 10^{-12}p^2$.

I have previously found[9] for the volume compressibility of a polycrystalline rod $8.55 \times 10^{-7}p - 3.88 \times 10^{-12}p^2$ at both temperatures. The results are not far different from those above, but indicate that the orientation of the former sample was not completely random.

The mean linear thermal expansion between 30° and 75° was found to be:

Hex. axis parallel to length, 12.3 to 14.5, average 13.4×10^{-6},
Hex. axis perpendicular, 15.2 to 17.0, average 16.1×10^{-6}.

This gives for the volume expansion 45.6×10^{-6}. The thermal expansion of Be is not listed in I. C. T.

The axial ratio of beryllium is known by X-ray analysis to be 1.585. The ratio for *spheres* in hexagonal close packing is 1.63. It would be expected therefore that the compressibility of beryllium in the two directions would not differ greatly, as turns out to be the case.

Compounds.

Gulonic Lactose. This organic substance crystallizes in the orthorhombic system, so that measurements of the linear compressibility in three different directions are necessary to exhaustively characterize the behavior under pressure. I am indebted to Mr. Phelps of the Bureau of Standards for several large clear crystals which had been grown by slow evaporation from aqueous solution. From these crystals rods varying in length from 6 to 9 mm. were cut in the three independent crystal directions "a," "b," and "c," in the nomenclature of Dana, and the compressibility measured with the lever piezometer in the usual way. Pressure was transmitted directly to the specimen with kerosene or petroleum ether, previous trials having shown no perceptible amount of solubility. Regular readings were made without incident at 30° and 75°. The average arithmetical departure from second degree curves of single readings at 30° and 75° were respectively: "a" direction, 0.22% (14 readings), and 0.29% (13 readings); "b" direction, 0.16% (13 readings), and 0.16% (15 readings); "c" direction, 0.05% (13 readings), and 0.07% (13 readings) The results are as follows:

30° 75°

"a" direction, $-\Delta l/l_0 = 9.69 \times 10^{-7}p - 10.3 \times 10^{-12}p^2$; 10.66, − 11.9

"b" direction, $-\Delta l/l_0 = 33.95 \times 10^{-7}p - 70.2 \times 10^{-12}p^2$; 36.67, − 81.5

"c" direction, $-\Delta l/l_0 = 17.02 \times 10^{-7}p - 39.9 \times 10^{-12}p^2$; 17.36, − 40.1

and by calculation, for the volume,

$$-\Delta V/V_0 = 60.66 \times 10^{-7}p - 132.0 \times 10^{-12}p^2;\ 64.69,\ -145.7$$

Rhamnose. I owe this material like the gulonic lactose, to Mr. Phelps of the Bureau of Standards, who placed at my disposal several clear crystals grown from aqueous solution. This is monoclinic and strictly specimens in four directions would be required to completely characterize the volume distortion under pressure. Unfortunately there was not enough material available to permit getting the four directions, but only the crystallographic "a," "b," and "c" directions could be obtained. However, judging by analogy with sucrose, to be described next, and which is very similar, the missing Y direction is so much like the "a" direction that little is lost by not making a measurement on it. The specimen in the "a" direction was 4 mm. long, in the "b" direction 1.0 cm. long, and in the "c" direction 1.1 cm. The average arithmetical deviations from second degree curves of single readings at 30° and 75° were respectively: "a" direction 0.38% (16 readings), and 0.23% (15 readings); "b" direction, 0.25% (16 readings), and 0.13% (14 readings); and "c" direction, 0.15% (14 readings), and 0.16% (14 readings). The results follow:

30° 75°

"a" direction, $-\Delta l/l_0 = 35.42 \times 10^{-7}p - 61.0 \times 10^{-12}p^2; 37.63, -71.0$

"b" direction, $-\Delta l/l_0 = 21.98 \times 10^{-7}p - 38.4 \times 10^{-12}p^2; 24.82, -53.4$

"c" direction, $-\Delta l/l_0 = 13.78 \times 10^{-7}p - 33.0 \times 10^{-12}p^2; 14.66, -35.3$

and by calculation, for the approximate change of volume:

$$-\Delta V/V_0 = 71.18 \times 10^{-7}p - 148.1 \times 10^{-12}p^2; 77.11, -178.1$$

Sucrose (*Cane Sugar*). The source of this material was the ordinary "rock candy" of commerce. In a pound a large number of clear single crystals were found of sufficient size. This crystallizes in the monoclinic system, like rhamnose, so that four orientations are necessary. The four orientations used were the three crystallographic axes "a," "b," and "c," and the direction perpendicular to b and c, the "Y" axis of Voigt. Since the a axis is inclined at 103° to the c axis, there is a difference of only 13° between it and the Y axis, so that not much difference in compressibility in these directions is to be anticipated.

The runs on the four orientations were made without incident of any kind. The "a" sample was 5.0 mm. long, the "b" sample 9.4 mm., the "c" 7.3 mm., and the "Y" 5.6 mm. In spite of the small lengths, the compressibility is so great that large deflections were obtained, and the results are among the most regular. The average deviations from smooth curves at 30° and 75° respectively were as follows: "a" direction, 0.40 and 0.21%; "b" direction 0.43 and 0.29%; "c" direction 0.37 and 0.32%, and "Y" direction 0.47 and 0.33%. In the following the results are reproduced by second degree formulas, the constants being so chosen as to pass through the end points, and to locate at 6000 kg/cm^2 the maximum deviation from linearity. But this does not always represent the results within experimental error, and in a number of cases small corrections must be applied; in general these corrections have the effect of somewhat increasing the compressibility at low pressures and decreasing it at high pressures as compared with the compressibilities of the second degree formulas. The corrections are given below in the form of deviations from the second degree formulas.

"a" direction.

30°. The maximum deviation from linearity is at 5300 instead of at 6000. At 3000, $\Delta l/l_0$ is greater numerically by 0.00005 than is given by the formula, at 6000 less by 0.00007, and at 9000 less by 0.00022.

75°. As at 30°, except that at 9000 the compressibility is less by 0.00030 instead of 0.00022.

"b" direction.

30°. The second degree formula is sufficiently good.

75°. The maximum deviation from linearity is at 5200 instead of at 6000. At 3000, $\Delta l/l_0$ is 0.00011 greater numerically than given by the formula, and at 9000 less by 0.00017.

"c" direction.

The second degree formula is sufficiently good at both temperatures.

"Y" direction.

30°. The second degree formula is sufficiently good.

75°. The maximum deviation is at 5500 instead of 6000. $\Delta l/l_0$ at 3000 is greater numerically by 0.00014 than given by the formula, and at 75° less by 0.00026.

The second degree results are:

"a" direction.

At 30°, $-\Delta l/l_0 = 32.74 \times 10^{-7}p - 69.9 \times 10^{-12}p^2$.

At 75°, $-\Delta l/l_0 = 33.80 \times 10^{-7}p - 76.3 \times 10^{-12}p^2$.

"b" direction.

At 30°, $-\Delta l/l_0 = 14.49 \times 10^{-7}p - 24.0 \times 10^{-12}p^2$.

At 75,° $-\Delta l/l_0 = 16.98 \times 10^{-7}p - 34.6 \times 10^{-12}p^2$.

"c" direction.

At 30°, $-\Delta l/l_0 = 22.81 \times 10^{-7}p - 44.4 \times 10^{-12}p^2$.

At 75°, $-\Delta l/l_0 = 24.74 \times 10^{-7}p - 52.9 \times 10^{-12}p^2$.

"Y" direction.

At 30°, $-\Delta l/l_0 = 32.00 \times 10^{-7}p - 77.5 \times 10^{-12}p^2$.

At 75°, $-\Delta l/l_0 = 33.14 \times 10^{-7}p - 78.6 \times 10^{-12}p^2$.

For the volume compressibility, from "b," "c," and "Y":

At 30°, $-\Delta V/V_0 = 69.30 \times 10^{-7}p - 161.1 \times 10^{-12}p^2$.

At 75°, $-\Delta V/V_0 = 74.86 \times 10^{-7}p - 184.2 \times 10^{-12}p^2$.

The results are seen to be numerically very close to those for rhamnose, except for a curious reversal of the b and c directions.

The thermal expansions were found to be:

Direction	Mean Linear expansion, 30°–75°.	
	Extreme values	Average
a	22.5 to 23.3 $\times 10^{-6}$	22.9 $\times 10^{-6}$
b	59.2 to 60.4	59.8
c	55.5 to 58.5	57.0
Y	21.3 to 25.3	23.3

The mean volume expansion is thus 140.1 $\times 10^{-6}$.

$MnCl_2$. This material crystallizes in the hexagonal system, but due to a misapprehension of some sort I originally supposed it to be cubic and it was measured with this idea. I probably would not have measured it if I had realized that it was hexagonal, for the values are only a sort of average, of somewhat indefinite significance. However, the measurements having been made, it seemed worth while to at least record them here.

The material in powdered condition was obtained from Eimer and Amend. It was compressed into a slug in the split mold at room temperature. (See a preceding paper for a description of the technique of forming the compressed slugs.) The material is very hygroscopic, and although precautions were taken to prevent the absorption of moisture during preparation of the specimen, treatment in the dessicator and weighing before and after heating indicated a water content of the compressibility sample of about 1%. Runs were made at 30°, 75°, and 30° again. There was a slight plastic deformation during the runs. This may have been due to two causes: in the first place there may have been some recrystallization, due to the fact that the crystal system is not cubic. In the second place, the water content may have allowed some plastic flow under the compressive force exerted by the lever of the piezometer, an effect shown very much more prominently by the copper halides, in connection with which it has been discussed in detail in a preceding paper. This plastic flow was greatest during the first run at 30°, and during the first two low pressure readings at 75°, and amounted to about 4% of the maximum effect. It had entirely disappeared from the second run at 30°, which was the only run at 30° used in the computations. The two first points at 75° were also discarded in the computations. Because of the distortions in the specimen it did not seem worth while to try for the thermal expansion, although this had been measured for most of the other compressed slugs. The average arithmetical departure from a smooth curve of a single one of the 14 readings of compressibility at 30° was 0.14% of the maximum effect, and the corresponding figure for 13 readings at 75° was 0.18%. The results for compressibility are:

$$\text{At } 30°,\ -\Delta V/V_0 = 54.09 \times 10^{-7}p - 96.3 \times 10^{-12}p^2,$$
$$\text{At } 75°,\ -\Delta V/V_0 = 55.62 \times 10^{-7}p - 106.5 \times 10^{-12}p^2.$$

These volume compressibilities were calculated on the assumption of equal linear compressibility in all directions.

$ZnCl_2$. This crystallizes with the same hexagonal structure as $MnCl_2$; I also thought at first that this was cubic, and measured the linear compressibility in that misapprehension. The results are recorded here in the same spirit as those for $MnCl_2$. $ZnCl_2$ absorbs normally very large amounts of water. The water was driven off by heating as far as possible, until decomposition appeared to be just beginning, and after that it was kept in a dessicator and worked under Nujol. But according to Mellor's Chemical Handbook, it is impossible to remove all the water by heating, and the measurements bore this out.

The material was formed into a compressed slug by the technique of most of the materials of the preceding paper. Measurements were attempted only at 30°. There were enormous permanent changes of zero after each application of pressure, of which three were made altogether. Whether the permanent changes are a regrowth phenomenon connected with the water still remaining and the compressive force exerted by the lever of the piezometer, or whether they are a consequence of the non-cubic nature of the crystal, I did not attempt to decide. It seemed significant, however, that the three sets of readings with increasing pressure agreed approximately, the permanent change taking place only on decreasing pressure. The readings on the second application of increasing pressure were made with the usual care, and showed an unusually high degree of regularity, the average arithmetical deviation from a second degree curve of a single one of the 8 readings being only 0.04% of the maximum effect. The volume change was calculated from the change of length by assuming equal compressibility in all directions, and is given for what it is worth:

$$\text{At } 30°, -\Delta V/V_0 = 41.40 \times 10^{-7}p - 107.7 \times 10^{-12}p^2.$$

Al_2O_3, Synthetic Sapphire. The material was obtained from G. Everett March, a dealer in synthetic jewels of Chicago, who specially selected it from his stock. It was furnished in the form of a disc about 1.3 cm. in diameter and 3 mm. thick, the crystal axis being stated to be parallel to the plane of the disc. In appearance it was perfectly transparent and flawless. The statement about the orientation of the axis was most kindly checked for me by Professor E. S. Larsen of the Department of Mineralogy of Harvard University by optical examination, and the precise position of the axis determined. The disc was then cut without incident with a saw charged with diamond powder into rods for the compressibility measurements, a single rod 1.26 cm. long for the specimen parallel to the direction of the axis, and two pieces each 0.42 cm. long for the perpendicular orientation. These two pieces were placed end to end for the measurements.

The compressibility of the parallel sample gave an irregular curve, which could be reproduced within the error of measurement by a broken straight line, rather than the usual smooth curve. The experimental points at 30° and 75° are reproduced in Figure 8. At 30°, two complete runs were made, the first to 12000 and back with 12 readings, followed by the second with 30 readings. There was also an initial seasoning application of 12000, after which the change of zero was slightly less than after the third application of 12000, and within the limit of error of the other readings. It appears, therefore, that the effects are reproducible, approximately single valued and without hysteresis, and are not accomodation effects, but are truly characteristic of the material. At 75°, only a single application of pressure was made, but with the readings much more closely spaced than usual, 44 in all instead of the usual 14. There now appears to be a slight hysteresis between increasing and decreasing pressure, but it probably is not significant, and is of different sign at the higher and lower ends of the pressure range.

In order to reproduce the results within experimental error, it is sufficient to give the pressures at which the breaks in the curves of Figure 8 occur, and the changes of length at these pressures. Between the successive points of break the relation between pressure and change of length is linear.

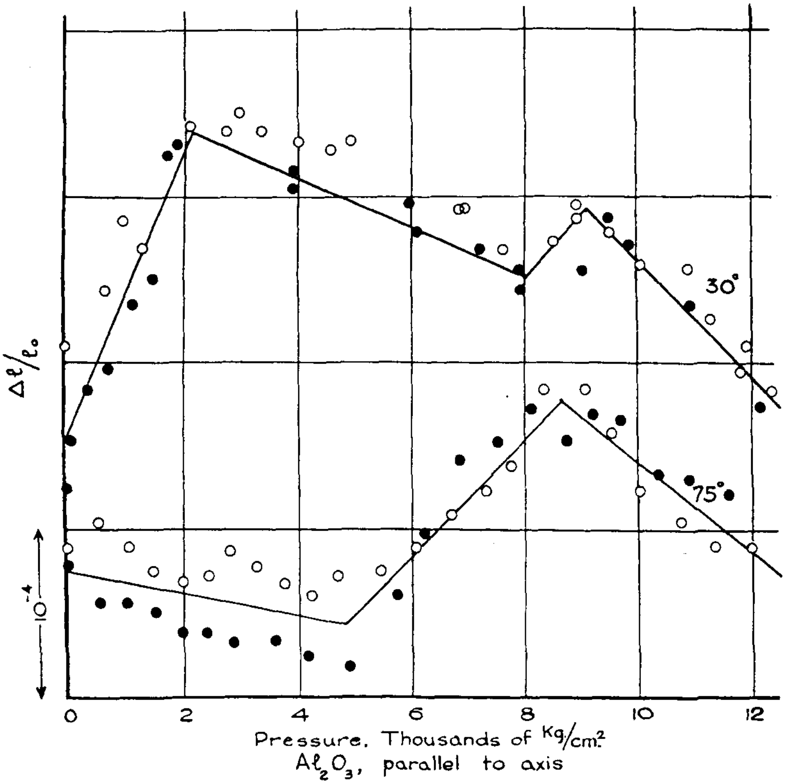

FIGURE 8. Deviations from linearity of the linear compression of Al_2O_3, crystal axis parallel to the length. The open circles are for increasing pressure and the solid circles for decreasing pressure.

At 30° the initial linear compressibility in the parallel direction is 3.9×10^{-8}, about one fifth that of iron, and at 75° it is 13.1×10^{-8}, an enormous temperature effect.

The linear thermal expansion at atmospheric pressure was determined incidentally from the zeroes of the 30° and 75° runs, as explained

TABLE II.

Al_2O_3, Parallel Orientation.

Pressure kg/cm²	$-\Delta l/l_0$
30°	
2100	0.000082
8000	.000862
9100	.000998
12000	.001362
75°	
4900	.000652
8600	.000965
12000	.001448

in the preceding paper. The mean linear thermal expansion in the parallel direction between 30° and 75° was thus found to be 9.5×10^{-6}. This is the mean of two figures, one with increasing temperature and the other with decreasing temperature, 9.7 and 9.3 respectively.

The perpendicular orientation gave nothing so striking as the parallel orientation. The experimental points are reproduced in Figure 9. There seems to be a slight amount of hysteresis at both temperatures, more, perhaps, at 75° than at 30°, as was also the case for the parallel orientation. This small amount of hysteresis was ignored in making the computations. At 30° the results lie within experimental error on a single straight line, but at 75° there would seem to be no question but that two lines, as shown, are required, with a break at 6400 kg. The results may be reproduced as follows:

Al_2O_3, perpendicular orientation.

30°

$$-\Delta l/l_0 = 1.11 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$$

75°

Below 6400, $-\Delta l/l_0 = 1.39 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$

Above 6400, $-\Delta l/l_0 = .000899 + 1.01 \times 10^{-7}(p - 6400) - 0.70 \times 10^{-12}(p - 6400)^2$.

The mean linear thermal expansion at atmospheric pressure between 30° and 75° for the perpendicular orientation was 11.6×10^{-6}, being the mean of the two values 11.7 and 11.5 with increasing and decreasing temperature. I. C. T. gives for the linear expansion at 40° 6.2 and 5.4×10^{-6} parallel and perpendicular to the axis respectively, both about half the values found for this specimen.

The initial volume compressibility at 30°, obtained by combining the results for the two orientations, is 3.17×10^{-7}. Madelung and Fuchs[10] found at 0° for synthetic sapphire 3.8. It is evident that the value 3.17 would be considerably altered if a linear extrapolation were made for the temperature effect, but in view of the many irregularities such an extrapolation is hazardous; it is probable, however, that this specimen is distinctly less compressible than that of Madelung and Fuchs.

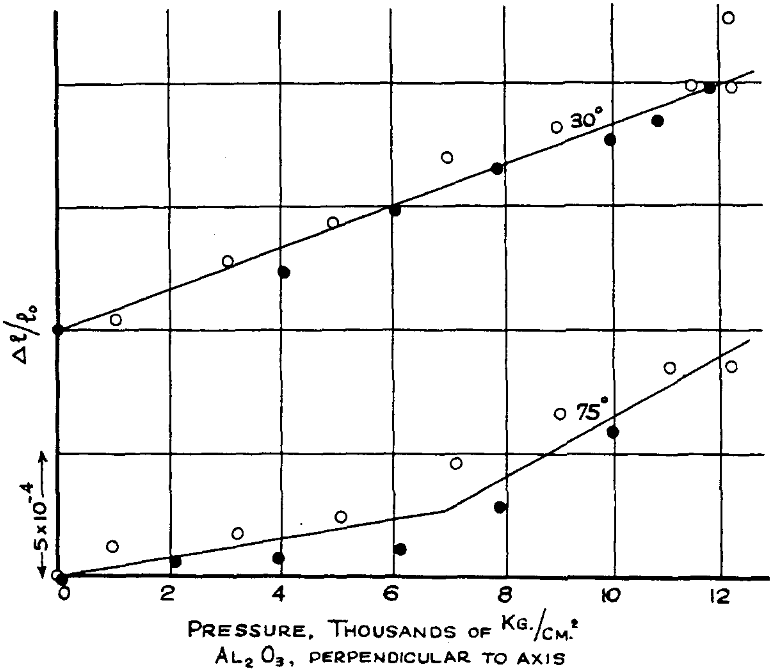

FIGURE 9. Deviations from linearity of the linear compression of Al_2O_3, crystal axis perpendicular to the length. The open circles are for increasing pressure and the solid circles for decreasing pressure.

It was felt to be highly desirable to check these results on a specimen from another source, and to see in particular whether the breaks in the line repeat, but I was not successful in getting other suitable specimens. In this I had the assistance of Mr. B. W. St. Clair of the General Electric Co. at Lynn, who has been engaged in research on sapphires for use in bearings of scientific instruments, and of Mr. R. H. Bird of Waltham, who specializes in the manufacture of sapphire bearings, and who has on hand a considerable stock of artificial sapphires. It appeared in the first place that artificial Al_2O_3 as it is usually supplied by the manufacturer has already experienced a spontaneous fracture due to some sort of internal strain. As it comes from the furnace, the Al_2O_3 is usually in the shape of a much elongated pear shaped drop, up to perhaps 5 cm. long and 2 cm. in diameter. This usually fractures spontaneously along some longitudinal plane. No process of annealing has yet been successful in removing the internal strains or in avoiding the fracture. The crystal axis is not simply related to the plane of fracture, but is usually inclined to it and to the length of the drop by something of the order of 45°. Mr. St. Clair was kind enough to select from his large stock two drops smaller than usual which had not experienced the usual spontaneous fracture. That each of the drops was a single crystal was beautifully evident by the appearance of the reflection pattern from the minute natural pits of the surface, a method of examination which I have often applied to metal crystals. The orientation of the crystal axes was again kindly determined by Professor Larsen, who found it to be within the expected range. He also found evidences of high internal strain, the refraction patterns being much confused, making the location of the axis difficult. These specimens were now properly mounted to give rods of the proper orientations, and Mr. Bird undertook to cut them on his diamond wheel, but both drops broke spontaneously into many pieces after the cut had penetrated only a small fraction of the total diameter. No further attempt to get other artifical specimens was made. It would appear that the specimen measured above was quite exceptional, both with regard to the location of the crystal axis with respect to the length, and with regard to freedom from internal strain, a fact which was established by Professor Larsen by his microscopic examination, and is further vouched for by the absence of any seasoning effects on the first application of pressure. Repetition of the experiment had best be made on a natural crystal, which presumably is not subject to the internal strains of the artifical crystal, but I was not able to obtain such a crystal.

Cu_5Cd_8. This is the first intermetallic compound whose compressibility I have yet measured, and it is but the beginning of an extensive projected program. This program is likely to prove arduous if many of the other compounds have as complicated behavior as this.

The compound was formed from electrolytic copper from the Bureau of Standards and Kahlbaum's cadmium. They were reduced to fine shavings in a milling machine using a new cutter; weighed quantities were then mixed together and compressed into a slug like the compressibility samples, which was then sealed into an evacuated quartz tube. The slug was melted at 650° C and then annealed by holding at 490° for 7 days. The outside surface did not fill the mold perfectly, but it is highly probable that there could have been no internal

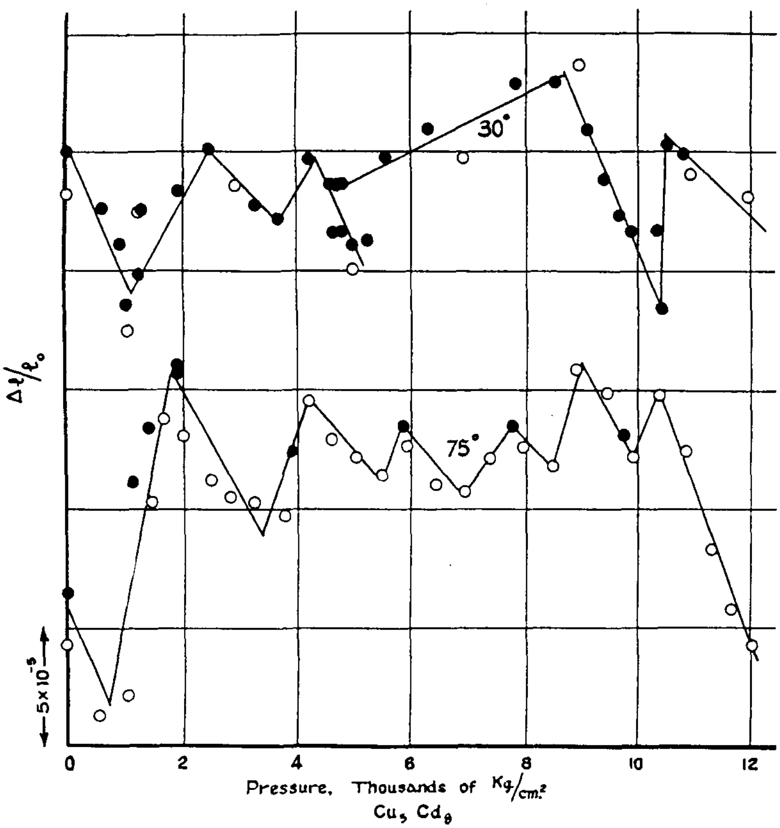

FIGURE 10. Deviations from linearity of the linear compression of Cu_5Cd_8. The open circles are for increasing pressure and the solid circles for decreasing pressure. The straight lines are to be taken merely as suggestive, and in fact a literal interpretation of the lines at 30° is not possible.

cavities, for there were no appreciable seasoning effects on the first application of 12000. The annealing produced some grain growth, but the largest grains, as far as could be told from the reflection pattern of the outside surface, were not more than 2 or 3 mm. on a side. Since Cu_5Cd_8 is cubic, grain size of this magnitude should not be a factor in the final result. The specimen was formed to shape for the compressibility measurements by turning the outside surface with a diamond tool, a steel tool not being hard enough, and grinding the ends flat in the usual way.

Runs were made at 30° and 75°, and both of the runs showed an astonishing amount of fine structure. The experimental points are reproduced in Figure 10, in which the $\Delta l/l_0$ differential with respect to iron is plotted against pressure. The lines drawn in the diagram are to be taken only as roughly suggestive. It is difficult to know how much of this structure is real, and certainly difficult to know how much is significant. Detailed examination of the distribution of the points will show however that in most cases any possible hysteresis between points with increasing and decreasing pressure is much less than the scale of the structure suggested, and that there must be some sort of structure. This is by far the most complicated case of structure yet encountered. If the structure is entirely ignored, then the following average results are found for the compressibility:

$$\text{At } 30^\circ, \ -\Delta V/V_0 = 11.94 \times 10^{-7}p - 25.8 \times 10^{-12}p^2,$$
$$\text{At } 75^\circ, \ -\Delta V/V_0 = 12.29 \times 10^{-7}p - 26.1 \times 10^{-12}p^2.$$

The average linear thermal expansion at atmospheric pressure between 30° and 75° is 24.0×10^{-6}; the two runs with increasing and decreasing temperature gave 24.4 and 23.6 respectively.

The mean compressibility of the compound Cu_5Cd_8, calculated by the rule of mixtures from the compressibilities of its components, is 18.6×10^{-7} against 11.9 actual. The difference is in the direction to be expected in a compound, the forces of chemical affinity effectively increasing the internal pressure and thus decreasing the compressibility.

Alloys. A systematic attack on the problem of the compressibility of alloys would be a most elaborate undertaking, and I have not yet any such program in contemplation. However, the labor of making the compressibility measurements is comparatively so small that I have been glad to make them when the specimens of the alloys have already been prepared, usually by someone else for another purpose; the measurements here recorded were made on such specimens.

There are measurements of the Ag-Au series, and on a number of the binary alloys of Fe with W and of W with Co, selected from a rather complicated set of the ternary alloys of Fe-W-Co. The measurements brought to light a complicated situation not at all to be suggested by the known phase diagrams, so that it appears from these results that a study of compressibility may become a useful tool in the hands of the metallurgist by means of which he may discover internal changes in the composition of alloys not accessible by other methods of measurement.

Ag-Au Alloys. Three alloys of this series were measured, containing 25, 50, and 75 atomic percent of Ag. I am much indebted for them to Dr. R. Mehl, who at the time was Superintendent of the Department of Metallurgy at the Naval Research Laboratory, Anacostia, D. C., and who had them prepared under his direction. They were made up from weighed amounts, melted in vacuum, and annealed at about 920° for 40 hours. They were then forged into bars about 2.5 cm. long and 3 mm. in diameter and annealed again. The composition was not checked after preparation, so that there may have been some departure from the nominal composition, but Dr. Mehl states that any such departure could have been only slight. The bars as supplied were not quite the most convenient length for the compressibility measurements. They were accordingly lengthened by forging an additional 15%, annealed at 900° for 30 minutes, and then turned in a jeweller's lathe to true cylinders for the measurements. After the compressibility measurements had been completed, they were drawn out through steel dies, with frequent annealings in an open gas flame, to a final diameter of 0.048 cm., for the measurements of resistance. After the final size was reached they were subject to a final annealing to 900° in an electric furnace, protected by quartz tubing.

Alloy No. 1, Ag 25 A%, Au 75 A%. Measurements were made of the compressibility in the regular way at 30° and 75°. The observed displacements of the slider of the potentiometer, multiplied by a suitable constant to give approximately the differential change of length as compared with Fe, are given in Figure 11. It would seem that there can be no question that there is a break in the curve for 75° at approximately 8000 kg/cm^2. If it were not for the data at 75° it is doubtful whether the single markedly divergent point of the 30° series at approximately 7000 would have been interpreted as a genuine break, but in view of the 75° curve such an interpretation seems fairly secure. Table III gives the changes of volume at various

TABLE III.

Relative Changes of Volume of Alloy Ag 25A%, Au 75A%.

Pressure kg/cm²	$\Delta V/V_0$ 30°	$\Delta V/V_0$ 75°
2000	.001281	.001254
4000	2532	2502
6000	3738	3726
8000	4959	5010
10000	6138	6207
12000	7293	7389

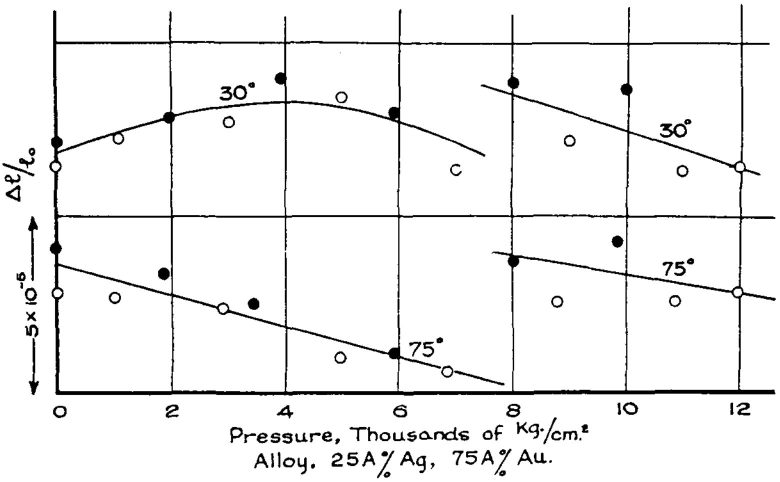

Figure 11. The differential linear compression of the alloy 25% Ag, 75% Au. The open circles are for increasing pressure and the solid circles for decreasing pressure.

pressures, obtained by combining the data of Figure 11 with the known compressibility of iron. At each temperature there is a break just below 8000, indicated by a horizontal line in the table; at 30° the break in $\Delta V/V_0$ is approximately 0.000039, and at 75° 0.000069. The break is in the direction corresponding to a decrease of volume with increasing pressure. This of course is necessary thermodynamically; if the sign were the opposite of that shown it would mean either that the dimensions of the rod were not changing equally in all directions, which would be curious in a fine grained material belonging to the cubic system like this alloy, or else that the apparent discontinuity is really not abrupt, but is spread over a sufficient pressure range so that the net compressibility in the range (correcting the data of the figure for the compressibility of iron) is still negative.

The mean linear thermal expansion at atmospheric pressure between 30° and 75° was 15.3×10^{-6}; the two values obtained with increasing and decreasing temperature were 15.4 and 15.2. The agreement between these results is good, and indicates, among other things, that any permanent effects after a pressure run were very small.

The electrical resistance of wire drawn from the compressibility sample, as already explained, was measured in the regular way at 30° and 75°. Special search was made at 75° for a break in the resistance corresponding to the break in the compressibility curve. No break could be established, but the points over the entire pressure range at 75° were unusually irregular, which may have been evidence of an internal instability of some sort. The average departure from a second degree curve at 75° of a single one of the 36 readings was 0.83% of the maximum pressure effect, whereas the corresponding figure at 30° for 15 readings was 0.16%. One might at first be inclined to explain ths greater irregularity at 75° by the inadequacy of the temperature control, which is always more difficult at the higher temperature. The irregularity in the resistance just recorded for 75° would demand an *average* temperature fluctuation in the bath of 0.14°. The temperature of the bath was read with a Beckmann thermometer at the time of making each pressure reading; the *extreme* fluctuation from the mean was only .04°, and the average fluctuation somewhat less than 0.01°.

An irregularity of the change of resistance of 0.83% corresponds to 1/7000 of the total resistance. Compared with this, the break in the change of length at 75° found during the compressibility measurements was only 2 parts in 100000 on the total length. It is evident, therefore, that the compressibility measurements are very much more sensitive to small changes than are the resistance measurements. If the irregularities observed in the resistance at 75° are due to some internal change, this change must have a much smaller effect on the dimensions than on the resistance. This sort of thing is not impossible, as shown, for example, by a comparison of the numerical magnitudes of the ordinary pressure coefficient of resistance with the compressibility.

The changes of resistance given by the best smooth curves are:

$$\text{At } 30°, \; -\Delta R/R\,(0, 30°) = 1.275 \times 10^{-6}p - 0.55 \times 10^{-11}p^2,$$

$$\text{At } 75°, \; -\Delta R/R\,(0, 75°) = 1.380 \times 10^{-6}p - 0.72 \times 10^{-11}p^2.$$

The average temperature coefficient of resistance at atmospheric pressure between 0° and 100°, given by linear extrapolation of the readings at 30° and 75°, was 0.001048. The specific resistance at 30° was 8.65×10^{-6}.

Beckman[11] has measured the pressure coefficient of the Ag-Au series over a pressure range of 3000 kg., and gives curves which allow comparison with the results above by interpolation. Beckman's initial pressure coefficient at room temperature is 14.2×10^{-7} against 12.7 above. His temperature coefficient is 0.0010, and specific resistance at 30° 8.5×10^{-6}.

Alloy No. 2, Ag 50 A%, Au 50 A%. The initial application of a seasoning pressure of 12000 produced a permanent differential change of length almost 50% greater than the subsequent elastic change under 12000. After this single application of seasoning pressure, the subsequent permanent changes of dimensions were not large, as shown by Figure 12. The differences between the displacements of the slider of the potentiometer and those demanded by a linear relation with pressure, multiplied by such a constant as to give approximately the differential $\Delta l/l_0$ as compared with iron, are plotted in Figure 12. At 30° there is evidently some sort of break between 6000 and 8000. At pressures higher than this there is no perceptible hysteresis, while below it there is distinct hysteresis. At 75° the irregularities are so great that it is not quite evident what is happening; with increasing pressure there seem to be two or three distinct episodes, which are smoothed out with decreasing pressure.

The results in the following Table IV were obtained by assuming at 30° a break between 6000 and 8000 kg. of 0.000045 on $\Delta V/V_0$, and for the other pressures values calculated from the curve of Figure 12.

TABLE IV.

PROPORTIONAL CHANGES OF VOLUME OF ALLOY AG 50A%, AU 50A%.

30°

Pressure kg/cm²	$-\Delta V/V_0$
2000	.001437
4000	2868
6000	4284
8000	5694
10000	6996
12000	8286

75°

$$-\Delta V/V_0 = 7.398 \times 10^{-7}p - 2.28 \times 10^{-12}p^2.$$

At 75° the irregularities were ignored, and the differential $\Delta l/l_0$ was taken to be linear in pressure.

The mean linear thermal expansion at atmospheric pressure between 30° and 75° was 16.6×10^{-6}. The two figures obtained with increasing and decreasing temperature were 16.7 and 16.5 respectively, close agreement.

The resistance measurements gave smooth results, with no trace of the breaks of the compressibility measurements, except possibly for the lowest two points at 75°, which fell off a smooth curve by about 1% of the total pressure effect, and were ignored in the calculations. At 30° the average arithmetical departure from a smooth curve of a single one of the 14 readings was 0.12% of the maximum effect, and at 75° the corresponding figure for 12 readings (2 discards) was 0.10%. The results were:

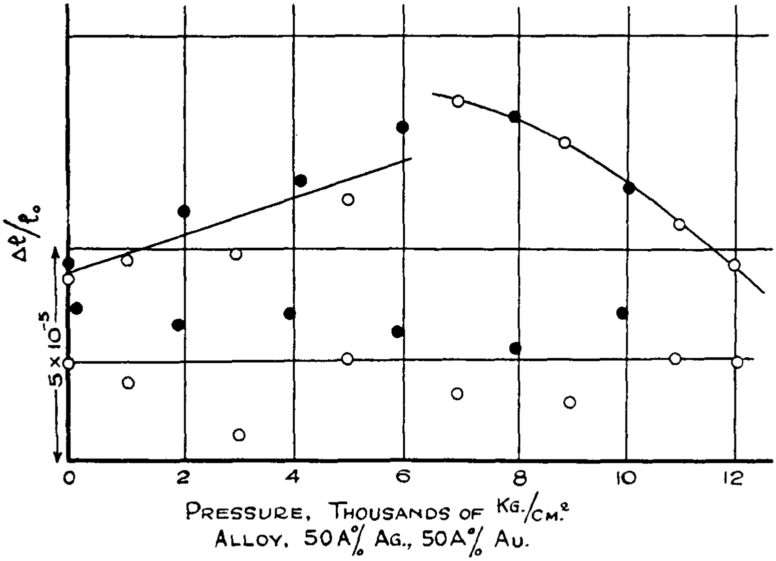

Figure 12. The differential linear compression of the alloy 50% Ag, 50% Au. The open circles are for increasing pressure and the solid circles for decreasing pressure. The upper points are for 30°, the lower for 75°.

$$\text{At } 30°, \; -\Delta R/R\ (0, 30°) = 1.213 \times 10^{-6}p - 4.3 \times 10^{-12}p^2,$$
$$\text{At } 75°, \; -\Delta R/R\ (0, 75°) = 1.277 \times 10^{-6}p - 5.0 \times 10^{-12}p^2.$$

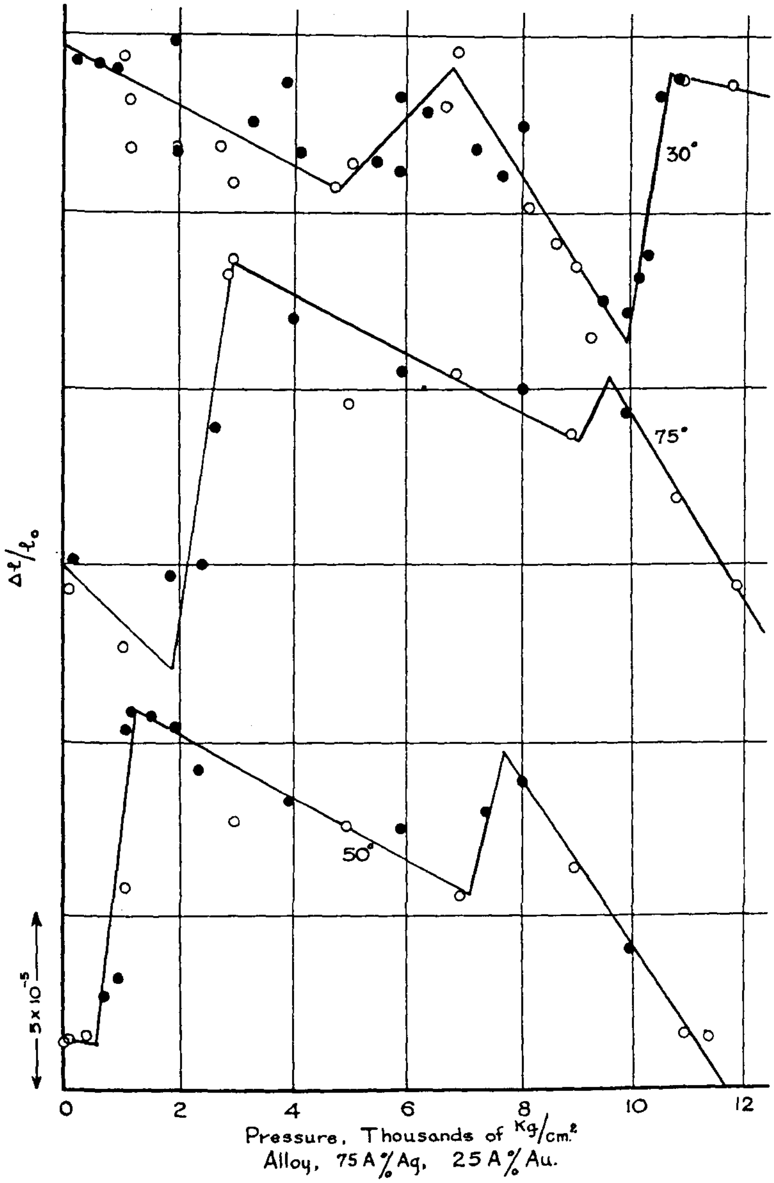

FIGURE 13. Deviations from linearity of the linear compression of the alloy 75% Ag, 25% Au. The open circles are for increasing pressure and the solid circles for decreasing pressure.

The specific resistance at 30° was 10.9×10^{-6}, and the mean temperature coefficient 0°–100°, extrapolated linearly from 30° and 75°, was 0.000774. Beckman's[11] values are: for the initial pressure coefficient 1.4×10^{-6}, for the specific resistance 10.7×10^{-6}, and for the mean temperature coefficient 0.0007.

Alloy No. 3, Ag 75 A%, Au 25 A%. The initial seasoning application of 12000 to the compressibility sample produced a permanent change of l/l_0 of 10^{-5}, which is less than 10% of the subsequent elastic differential effect. This is somewhat surprising in view of the complicated state of affairs revealed by the later measurements. Readings were first made in the regular way at 30° and 75°, and then, because of the complications, a run was inserted at 50°. The difference between the actual displacement of the slider of the potentiometer and the displacements that would have been found if the relation with pressure had been linear, multiplied by such a constant as to give approximately the differential change of length with reference to iron, are given in Figure 13. The results can be reproduced nearly within the limits of error by the broken straight line drawn. In Table V the volumes are given at the various break points of these lines; between the break points the volumes are to be found by linear interpolation.

TABLE V.

PROPORTIONAL CHANGES OF VOLUME OF ALLOY AG 75A%, AU 25A%.

Pressure kg/cm²	$-\Delta V/V_0$
30°	
4900	00388
6900	558
10000	788
10700	860
12000	958
50°	
560	00047
1200	00127
7100	596
7700	657
12000	970
75°	
1900	00152
3000	248
9100	771
9600	813
12000	989

The mean linear thermal expansion at atmospheric pressure between 30° and 75° was 20.2×10^{-6}; the two values obtained with increasing and decreasing temperature were 20.3 and 20.1.

The resistance measurements showed no trace of the anomalies of the compressibility measurements. At 30° the average arithmetical deviation from a second degree curve of a single one of the 16 readings was 0.15% of the maximum effect, and at 75° the corresponding figure for 14 readings was 0.20%. The results are:

$$\text{At } 30°, \ -\Delta R/R\ (0, 30°) = 1.301 \times 10^{-6}p - 5.1 \times 10^{-12}p^2,$$
$$\text{At } 75°, \ -\Delta R/R\ (0, 75°) = 1.410 \times 10^{-6}p - 6.9 \times 10^{-12}p^2.$$

The mean temperature coefficient of resistance at atmospheric pressure between 0° and 100°, by linear extrapolation of readings at 30° and 75°, was 0.000888, and the specific resistance at 30°, 8.64×10^{-6}.

Beckman[3] found for the *initial* pressure coefficient of resistance 1.5×10^{-6}, for the temperature coefficient 0.0008, and for the specific resistance at 30° 8.7×10^{-6}.

Co-Fe-W alloys. Mr. Bruce A. Rogers had made up some 30 of the ternary alloys with the special object of studying the magnetic behavior. The alloys were melted in the laboratories of the Western Electric Co., in Chicago, in a vacuum furnace. The magnetic measurements were made in the Cruft Laboratory of Harvard University and will be reported elsewhere. Mr. Rogers was kind enough to place this material at my disposal, and from it I selected three binary alloys of Fe and W and three binary alloys of W and Co as most interesting. It seemed to me not worth the labor to measure all the ternary alloys, in view of the fact that a study of the compressibility of even binary alloys has hardly yet begun. I did, however, measure that one of the ternary alloys which the magnetic work of Mr. Rogers had suggested to be the most likely to show interesting effects. It did not disclose any peculiarities in the compressibility corresponding to the peculiarities in the magnetic behavior, and the subject was not pushed further.

The percentage compositions given in the following are nominal and approximate weight percentages; they were determined from the amounts of the constituents before alloying. The actual compositions of the final alloys were determined by special analysis made by Mr. Walter M. Saunders of pieces milled from the forgings and were as follows:

Nominal Composition weight per cent.			Actual Composition weight per cent.		
W	Co	Fe	W	Co	Fe
10	0	90	10.05	—	—
20	0	80	20.03	—	—
30	0	70	27.93	—	—
10	90	0	9.83	87.38	2.30
20	80	0	15.15	82.82	2.30
30	70	0	30.01	67.40	2.46
30	10	60	30.16	10.75	—

In addition the common impurities were determined in an alloy selected at random, which should be roughly typical of the lot, and were found to be: Carbon, 0.02%, Manganese 0.007, Phosphorus 0.004, Sulphur 0.009, Silicon 0.03, Nickel trace, Copper trace.

After melting, which was done in a Northrup induction furnace, the contents of the crucible were forged to flat discs about 1 cm. thick and 10 cm. in diameter. In the following, "unannealed" refers to the condition as left by forging. The annealing was done by Mr. Rogers by heating either in a vacuum or in an atmosphere of hydrogen at approximately atmospheric pressure to 1050° for 5 hours and slowly cooling in the furnace.

Fe 90%, W 10%. This alloy was measured both before and after annealing. Before annealing the Rockwell hardness numbers, B scale, measured at four places varied from 57.5 to 65.8. After annealing they were from 49.6 to 56.1.

Unannealed specimen. This was subjected to a preliminary seasoning application of 11500, and a second one of 9000. The total permanent change of zero after the second application was about twice the total elastic effect at 30° shown in the figure. This seasoning appeared to be effective and there were no further permanent changes. The potentiometer displacements, multiplied by the proper constant to give roughly the differential $\Delta l/l_0$ with respect to Fe, are shown in Figure 14. It is to be noticed in this and all similar diagrams of this paper that the exact calculations cannot be made from the diagrams alone, because certain important corrections are not indicated, which often are of the same order of magnitude as the displacements of the figures. At 30° the points lie smoothly and the differential compression with respect to Fe is linear in the pressure. At 75°, on the other hand, there are two distinct episodes in the behavior. Above 8000 the relation between pressure and differential compression is linear, without hysteresis, and with the same slope as at 30°, meaning a smaller compressibility than that of iron. Below 8000, on the other hand, there is hysteresis, and the average compressibility is greater than that of iron. It is especially to be noticed that this hysteresis in the length is in the abnormal direction, following, instead of lagging behind the pressure. It is thermodynamically impossible that the volume change should lag in this way, which means that the changes of length giving rise to this hysteresis cannot be uniform in all directions. The maximum width of the hysteresis loop corresponds to a $\Delta l/l_0$ of 0.000110.

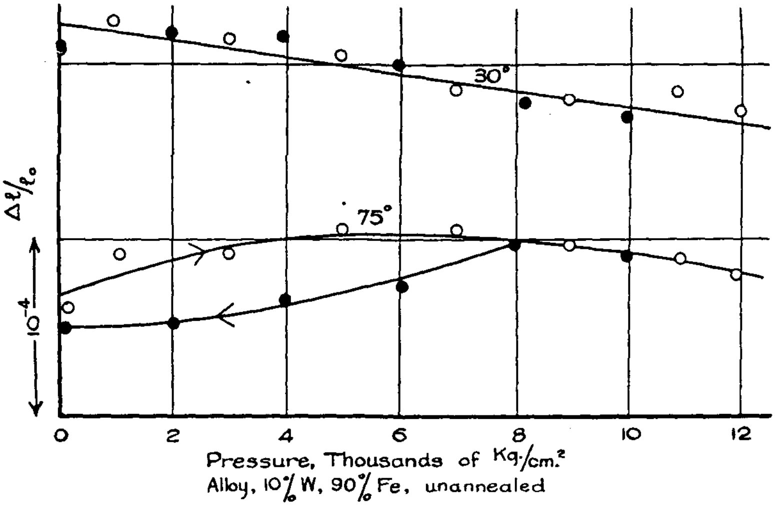

FIGURE 14. The differential linear compression of the alloy, 10% W, 90% Fe, unannealed. The open circles are for increasing pressure and the solid circles for decreasing pressure.

In the following the formula ignores the hysteresis, assuming the linear relation shown by the straight line drawn through the center of the loop. The results follow:

At 30°, $-\Delta V/V_0 = 5.712 \times 10^{-7}p - 2.22 \times 10^{-12}p^2$.

At 75°. Between 0 and 8000 kg/cm², ignoring hysteresis,

$$-\Delta V/V_0 = 5.617 \times 10^{-7}p - 2.22 \times 10^{-12}p^2.$$

Between 8000 and 12000,

$$-\Delta V/V_0 = .004353 + 5.462 \times 10^{-7}(p - 8000) - 2.22 \times 10^{-12}(p - 8000)^2.$$

Annealed Specimen. After the measurements recorded above the specimen was annealed at 1050° in vacuum for 5 hours.

The curves of approximate differential compressibility corresponding to Figure 14 are shown in Figure 15. Notice that the scale is different. The general character is not much altered. A slight amount of hysteresis appears at 30°. This agrees with the general experience of Mr. Rogers, who finds that the results of annealing are often anomalous, the material after annealing being apparently further from a state of internal equilibrium than before. This probably indicates the action of some very slow internal change at the annealing temperature, such that a very much longer annealing than that actually given would have been necessary to attain complete equilibrium, and even then there is every reason to think that there are further changes at lower temperatures, so that behavior at room temperature need not be entirely free from hysteresis, even after infinitely long seasoning at a single temperature. The 75° curve of Figure 15 is generally similar to that of Figure 14. There is hysteresis at the lower pressures; the pressure at which the hysteresis loop

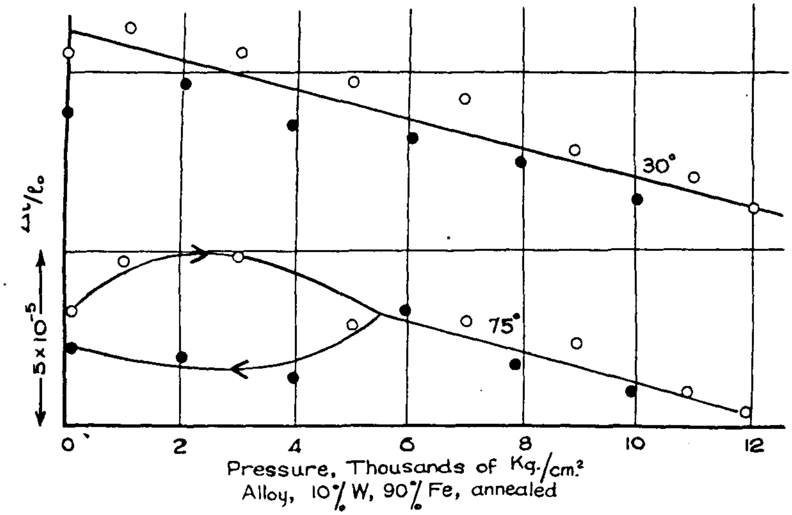

FIGURE 15. The differential linear compression of the alloy 10% W, 90% Fe, annealed. The open circles are for increasing pressure and the solid circles for decreasing pressure.

closes has been displaced from 8000 to about 6000. The maximum width of the hysteresis loop corresponds to a $\Delta l/l_0$ of 0.000049. Again the hysteresis is in the abnormal direction. The results follow:

At 30°, $-\Delta V/V_0 = 5.433 \times 10^{-7}p - 2.19 \times 10^{-12}p^2$.

At 75°, entirely ignoring the hysteresis loop and using the slope given by the straight line in Figure 15,

$$-\Delta V/V_0 = 5.535 \times 10^{-7}p - 2.19 \times 10^{-12}p^2.$$

In addition to the effects already discussed it is to be noticed that annealing has decreased the compressibility somewhat.

Fe 80%, W 20%, unannealed. The Rockwell hardness, C scale, varied from 3.8 to 8.2. This was seasoned by a preliminary application of 12000, but the accompanying change of dimensions was not recorded. The displacements of the potentiometer slider of the regular runs, multiplied by such a constant as to give roughly the differential $\Delta l/l_0$ with respect to iron, are plotted in Figure 16. There is practically no hysteresis, but a very distinct pattern and departure from linearity. The experiment was now tried of shortening the specimen and repeating the measurements at 30°. It was shortened by 0.002 inches, which is 13 times the differential change of length under 12000.

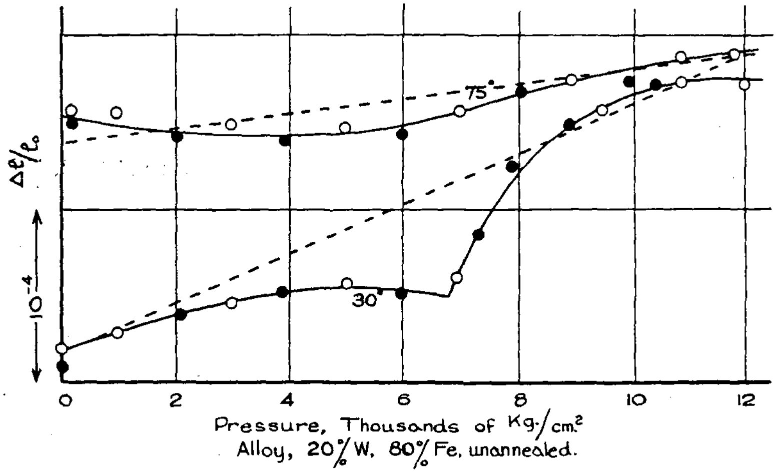

FIGURE 16. The differential linear compression of the alloy 20% W, 80% Fe, unannealed, first run. The open circles are for increasing pressure and the solid circles for decreasing pressure.

This changed the position of the lever of the piezometer so that an entirely different part of the slide wire was engaged, and provided a test as to whether the pattern was due to irregularities in the slide wire or to irregularities in the pivot of the lever at that particular part of its motion. The first run with the shortened specimen gave irregular results with a break at approximately the same pressure as the second. The second run gave the curve of Figure 17, which has essentially the same character as the 30° of Figure 16 although the pressure of the break is somewhat displaced.

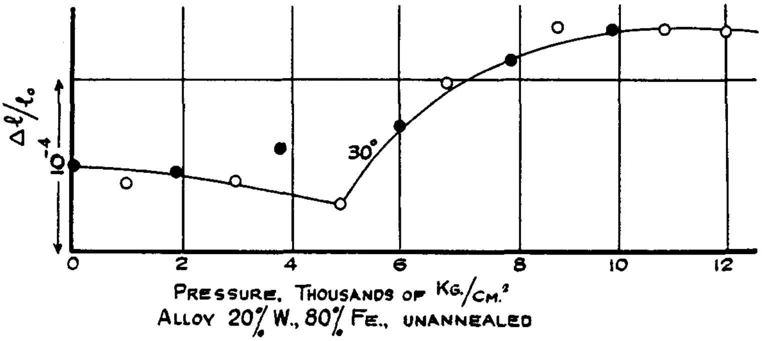

FIGURE 17. The differential compression of the alloy 20% W, 80% Fe, unannealed, shortened, second run. The open circles are for increasing pressure and the solid circles for decreasing pressure.

Smoothed results were computed by entirely ignoring the breaks and using for 30° and 75° the two straight lines indicated in Figure 12. These give:

$$\text{At } 30^\circ, -\Delta V/V_0 = 5.139 \times 10^{-7}p - 2.19 \times 10^{-12}p^2,$$
$$\text{At } 75^\circ, -\Delta V/V_0 = 5.559 \times 10^{-7}p - 2.19 \times 10^{-12}p^2.$$

Superposed on the ΔV given by the formulas are small scale effects determined by the curves. At 30° the maximum departure is at 7000 and is 0.000780 for $\Delta V/V_0$, the actual ΔV at 7000 being less arithmetically than given by the formula. At 75° the maximum deviation is at about 6000, the $\Delta V/V_0$ here being less arithmetically by 0.00039 than given by the formula.

Annealed specimen. The specimen was now annealed for five hours in vacuum at 1050°, and the behavior was very materially changed. Its Rockwell hardness, C scale, varied from 2.0 to 8.5. The displacements, multiplied by the proper constant to give approximately

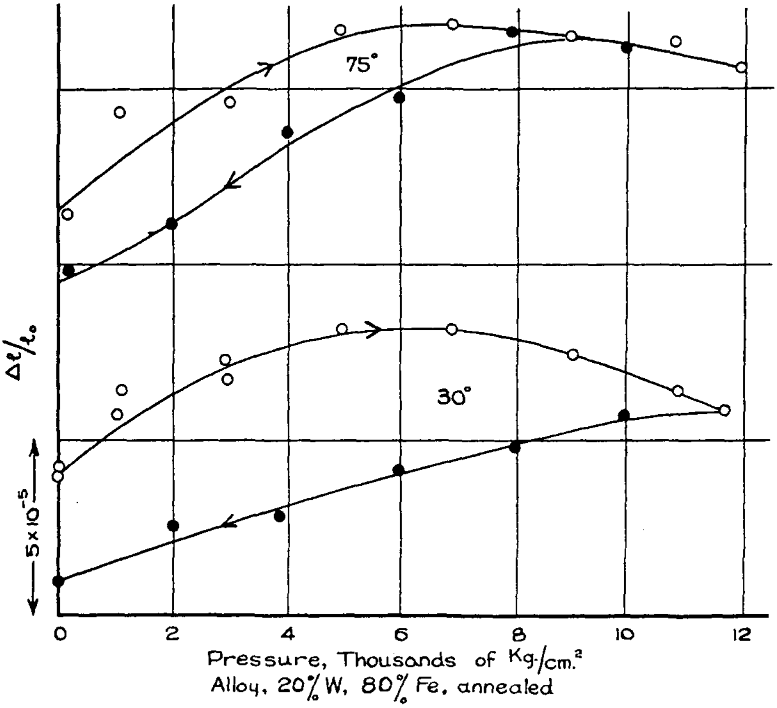

FIGURE 18. The differential linear compression of the alloy 20% W, 80% Fe, annealed. The open circles are for increasing pressure and the solid circles for decreasing pressure.

TABLE VI.

PROPORTIONAL CHANGES OF LENGTH OF ALLOY FE 80%, W 20%, ANNEALED.

Pressure kg/cm²	$-\Delta l/l_0$	
	30°	75°
2000	.000348	.000354
4000	701	708
6000	.001057	.001066
8000	1414	1431
10000	1772	1790
12000	2126	2146
10000	1784	1790
8000	1442	1434
6000	1097	1085
4000	747	732
2000	390	381
0	−29	−17

$\Delta l/l_0$ are reproduced in Figure 18. There is now large hysteresis. Further, this hysteresis is in the abnormal direction, indicating that this is not a true bulk effect, but there must be preferential directions. The only satisfactory way to reproduce these results is by a table, giving the changes of length with increasing and decreasing pressure.

The average compressibility has been considerably increased by the annealing; the total change of length at 30° for 12000 kg. before annealing was 0.00195 against 0.00213 after annealing. It is not surprising that a change of internal structure great enough to change the total compressibility by this amount should wipe out the pattern shown in Figure 16, and replace it by a different pattern as in Figure 18.

Fe 70%, W 30%. Two batches of alloy of this approximate composition were made; in one the W content was less than 30% by an inconsiderable amount, and in the other it was slightly greater than 30%. These will be referred to as the "W low" and the "W high" specimens respectively.

Unannealed Specimens. The Rockwell hardness, C scale, varied from 7.2 to 11.2, and from 21.5 to 24.7 for the "low" and the "high" specimens, respectively. They were both measured first in the unannealed condition, and the results are shown in Figure 19, in which the approximate differential $\Delta l/l_0$ is given against pressure, as usual. It will be seen that both specimens give essentially the same results; a curve smooth within limits of error and with only small hysteresis at 30°, but a 75° curves with a distinct break in the neighborhood of 2000 kg. It will be noticed that the "high" specimen shows evidence of a displacement of the break to very low pressures at 30°. It may well be that as a consequence of slight difference of chemical composition the break is displaced a little further with the "low" specimen, below atmospheric pressure, and so does not show on the diagram. The essential agreement of the results with these two specimens affords further welcome proof of the reality of these small effects. If one disregards to a first approximation the discontinuity at 75°, computing from the smooth curve, and taking the average for the two specimens, the following will be found:

$$\text{At } 30°,\ -\Delta V/V_0 = 4.752 \times 10^{-7}p + 1.53 \times 10^{-12}p^2,$$
$$\text{At } 75°,\ -\Delta V/V_0 = 4.401 \times 10^{-7}p + 3.66 \times 10^{-12}p^2.$$

Notice the abnormal sign of the second degree term, the significance being a compressibility increasing with increasing pressure. At 75° the results given by the formula should be corrected by the small amounts suggested by the curve of Figure 19 to get ΔV to a

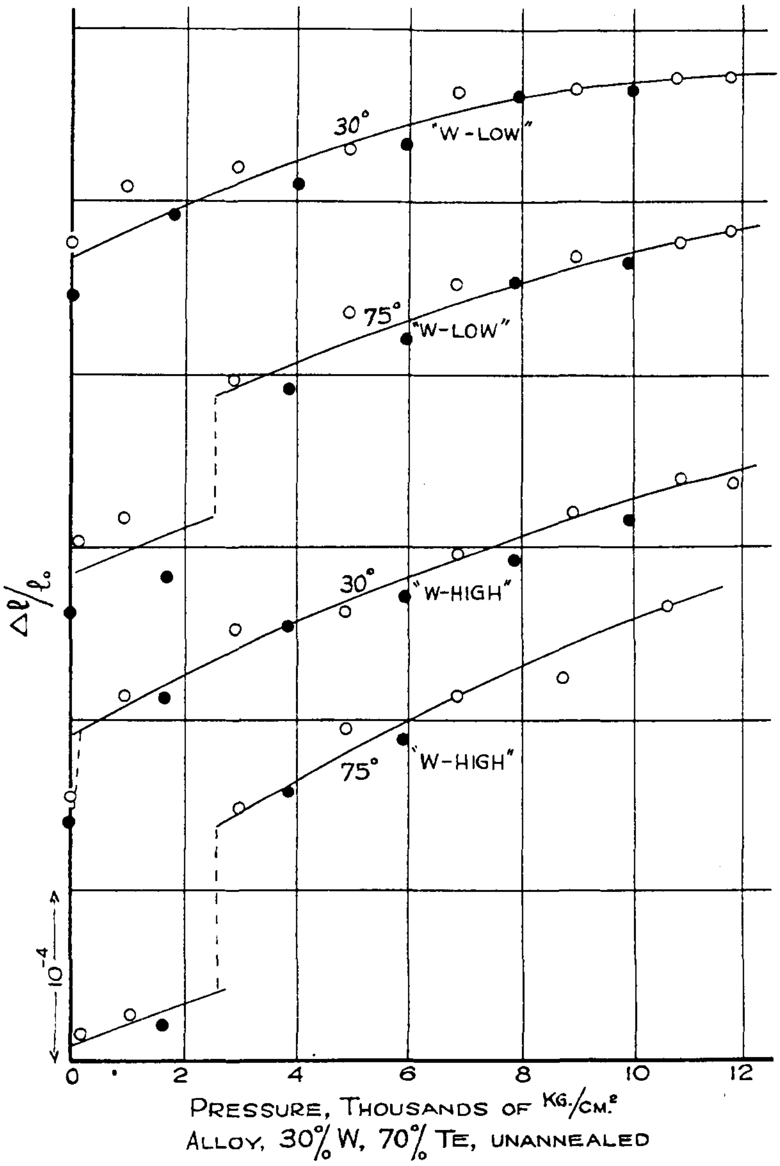

FIGURE 19. The differential linear compression of the alloy 30% W, 70% Fe, unannealed. The open circles are for increasing pressure and the solid circles for decreasing pressure.

precision high enough to show the small scale effects. But the accurate numerical values of the small scale effect can not be taken from the figure, as already explained, because the figure neglects certain important corrections. The accurate value of the discontinuity in $\Delta l/l_0$ at 75° is 0.000155 in the direction of a decrease of length, which involves a decrease of volume as demanded by thermodynamics. If account is taken of the discontinuity, the initial compressibility at 75° becomes 4.47 instead of 4.40×10^{-7}.

Annealed specimens. The specimens were now annealed at 1050° in vacuum for five hours. The Rockwell hardness, C scale, was now 7.5 to 9.1 and 26.8 to 27.0, respectively. The behavior is entirely altered, as shown in Figure 20, and although the behavior of the two specimens still show resemblances, they are more different than they were before annealing. This suggests again that the annealing time is not long enough, and that important internal changes take place during the cooling, which was not slow enough. The best way to reproduce the results is by a table, giving the two specimens separately.

TABLE VII.

RELATIVE CHANGES OF LENGTH OF TWO SPECIMENS OF ALLOY FE 70%, W 30%, AFTER ANNEALING.

Pressure kg/cm^2	"W low" specimen $-\Delta l/l_0$		"W high" specimen $-\Delta l/l_0$	
	30°	75°	30°	75°
2000	.000344	.000349	.000300	.000328
4000	674	695	619	657
6000	.001018	.001036	956	976
8000	1354	1369	.001289	.001316
10000	1696	1698	1616	1650
12000	2030	2048	1936	1975
10000	1710	1726	1636	1656
8000	1379	1396	1313	1328
6000	1045	1062	984	996
4000	713	725	640	688
2000	383	373	323	350
0	−42	−14	−43	−10

The effect of annealing this alloy has been to decrease somewhat the mean compressibility up to 12000.

W, Co Alloys. These alloys were all measured only in the annealed condition. The three different alloys were annealed in hydrogen at 985° for five hours and cooled slowly in the furnace.

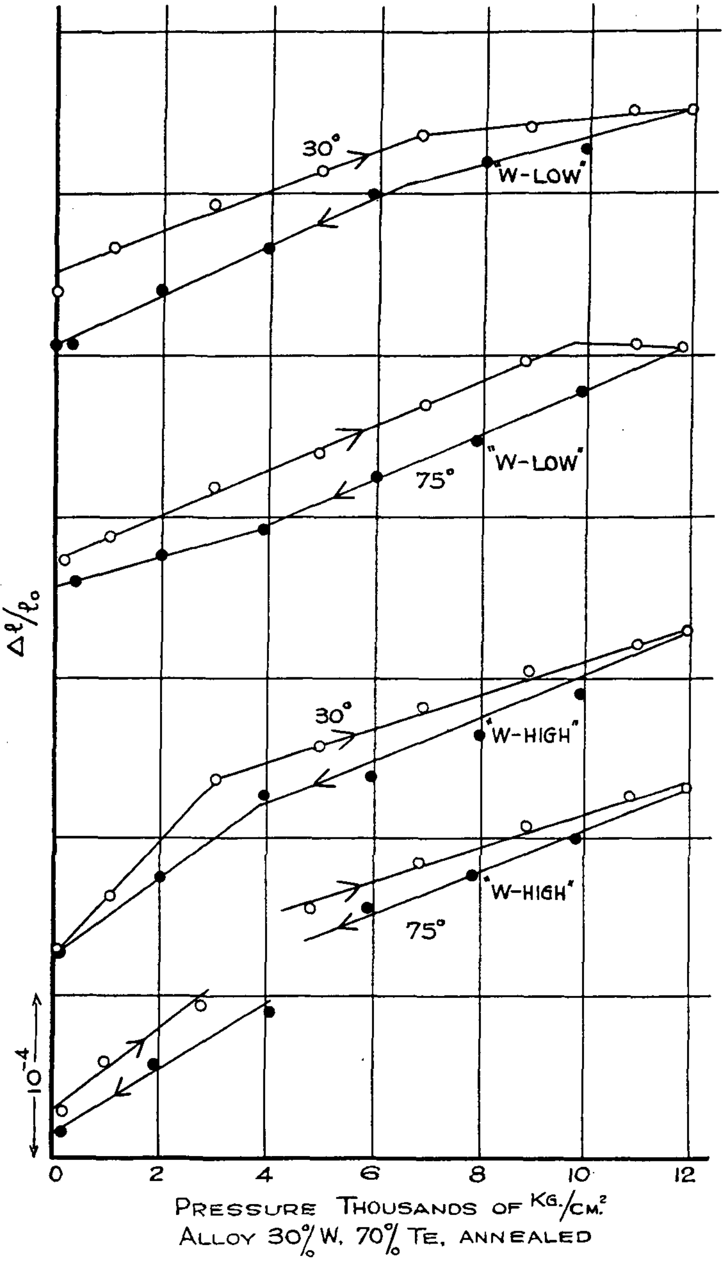

FIGURE 20. The differential linear compression of the alloy 30% W, 70% Fe, annealed. The open circles are for increasing pressure and the solid circles for decreasing pressure.

W 10%, Co 90%. The Rockwell hardness, C scale, was between 6.8 and 11.0. The first seasoning application of 12000 to this changed the zero by an amount just detectible on the slide wire and less than the normal irregularity of a single reading. Runs were made at 30°

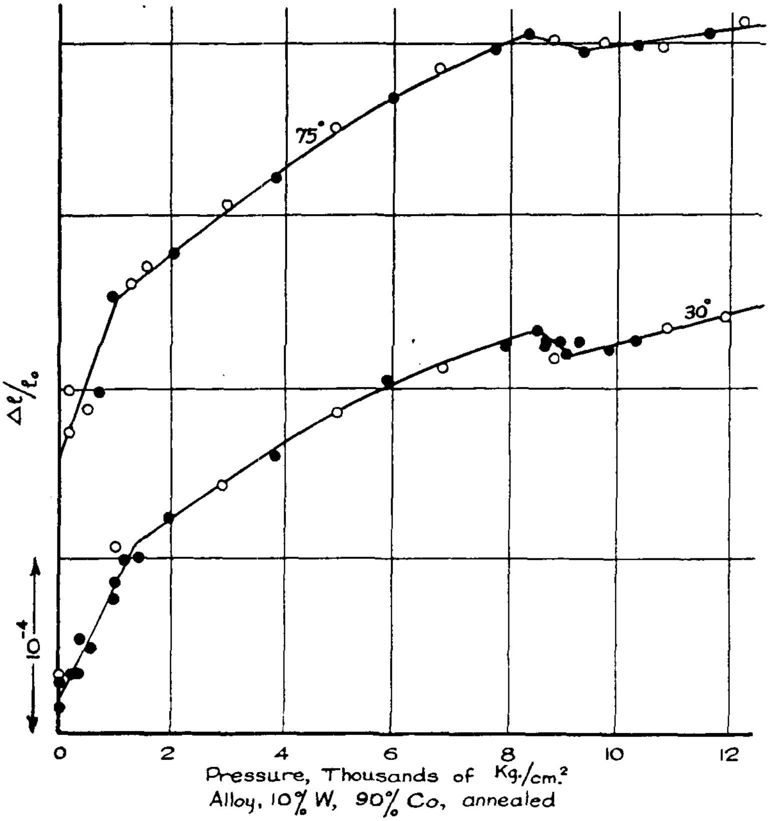

FIGURE 21. The differential linear compression of the alloy 10% W, 90% Co, annealed. The open circles are for increasing pressure and the solid circles for decreasing pressure.

and 75°. The approximate differential changes of length with respect to iron are shown in Figure 21. There is practically no hysteresis, in marked contrast to the Fe-W series. Obviously the results cannot be reproduced by a simple formula, and the changes of volume are therefore given as a function of pressure in Table VIII.

TABLE VIII.

PROPORTIONAL CHANGES OF VOLUME OF ALLOY W 10%, Co 90%.

Pressure	$-\Delta V/V_0$	
kg/cm²	30°	75°
1000	.000351	.000300
2000	792	786
3000	.001269	.001263
4000	1761	1755
5000	2235	2232
6000	2724	2721
7000	3210	3207
8000	3702	3702
9000	4236	4230
10000	4740	4764
11000	5214	5268
12000	5694	5763

W 20%, Co 80%. The Rockwell hardness, C scale, was between 22.8 and 28.5. On the first application of 12000 seasoning pressure there was a permanent change of l/l_0 of 0.000017. The readings of the regular runs at 30° and 75°, multiplied by such a constant as to give approximately the differential $\Delta l/l_0$ with respect to iron, are shown in Figure 22. There is no hysteresis within the limits of error. There is some sort of a break at a mean pressure of 5100 at 30° and 5600 at 75°. The magnitude of the break means a $\Delta l/l_0$ of 0.000021, and the direction corresponds to an *increase* of length. If this increase of length were equal in all directions, and were truly abrupt, it would mean an increase of volume, which is impossible thermo-

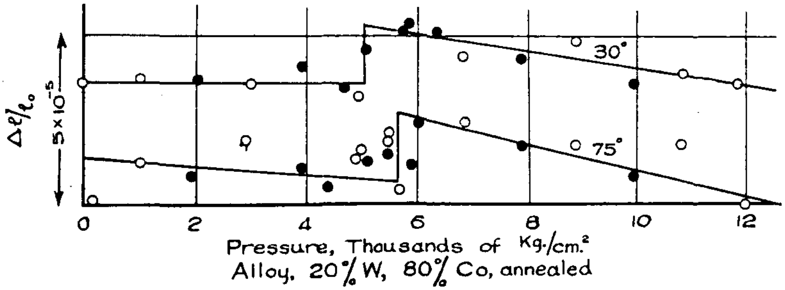

FIGURE 22. The differential linear compression of the alloy 20% W, 80% Fe, annealed. The open circles are for increasing pressure and the solid circles for decreasing pressure.

dynamically. There are two possible solutions; either the increase of length does not take place uniformly in all directions, there being compensating decreases in other directions, or else the change is not truly abrupt, but is spread over a range of pressure wide enough to give a total decrease of volume on combining the compressibility of Fe with the differential compressibility of the alloy. The numerical values demand that the transition region be at least 100 kg. wide, and this does not seem to be at all impossible.

Above and below the region of the break the relation between pressure and differential $\Delta l/l_0$ is approximately linear, although at the lower pressures at 75° there may be some departures. Assuming the linear relation, the following are the numerical results of the measurements:

At 30°, Below 5100, $-\Delta l/l_0 = 1.691 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$.

At 5100, or in a range centering on 5100, there is an increase of l/l_0 of 0.000017.

Above 5100, $-\Delta l/l_0 = .000030 + 1.717 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$.

At 75°, Below 5600, $-\Delta l/l_0 = 1.719 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$.

At 5600, or in a range centering on 5600, there is an increase of l/l_0 of 0.000021.

Above 5600, $-\Delta l/l_0 = .000033 + 1.744 \times 10^{-7}p - 0.70 \times 10^{-12}p^2$.

W 30%, Co 70%. The Rockwell hardness, C scale, was between 44.5 and 46.0. The permanent change of relative length on the first seasoning application of 12000 was − 0.000027. Runs were made at 30°, 75°, and at 30° again up to 5000, repeating the first part of the first run, which was essentially checked. The slider displacements at 30° and 75°, multiplied by such a constant as to give approximately the differential $\Delta l/l_0$ are shown in Figure 23. The pattern is more complicated than for the two previous alloys, but again there is no hysteresis within limits of error. The results are reproduced in Table IX. In this table are given the pressures at which occur the various cusps of the broken lines of Figure 23. Between the pressures of the cusps the relation between $\Delta l/l_0$ and pressure can be assumed to be linear with sufficient accuracy, although a very slight amount of curvature is indicated in the figure in the two segments at the highest pressures at 75°.

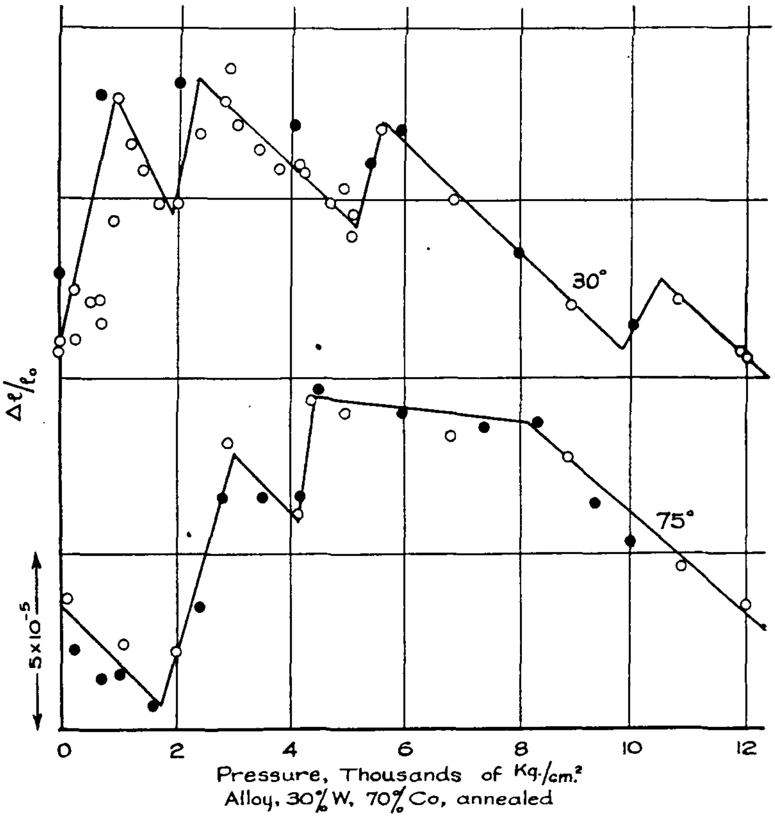

FIGURE 23. The differential linear compression of the alloy 30% W, 70% Co, annealed. The open circles are for increasing pressure and the solid circles for decreasing pressure.

TABLE IX.

RELATIVE CHANGES OF LENGTH OF ALLOY W 30%, Co 70%.

Pressure kg/cm²	$-\Delta l/l_0$	Pressure kg/cm²	$-\Delta l/l_0$
30°		75°	
880	.000065	1660	.000325
1900	.000235	2990	427
2350	278	4110	613
5180	735	4420	630
5630	771	8400	.001228
9880	.001438	12000	1797
10520	1504		
12000	1723		

Fe 60%, W 30%, Co 10%. As already stated, this is that one of the ternary alloys which Mr. Rogers had found to show the most complicated magnetic behavior. Three sets of measurements were made on this alloy: in the unannealed condition after forging, after annealing at 670° C for 5¼ hours and cooling in the furnace, and after annealing for 5 hours in vacuum at 1050° and cooling in the furnace. The first measurements on the unannealed specimen showed nothing of interest, the relation between pressure and potentiometer setting being linear within experimental error. At 30° the average arithmetical deviation from a straight line of a single one of the 13 readings is 0.53% of the maximum effect, and at 75° the corresponding figure is 0.31% for 14 readings. The only exception is the single reading at 30° and 12000 kg/cm², which departs from a line by six times the average deviation of the other points, and may indicate a real effect. Neglecting this point, the results are:

Before annealing.

$$\text{At } 30°, \; -\Delta V/V_0 = 5.31 \times 10^{-7}p - 2.2 \times 10^{-12}p^2,$$
$$\text{At } 75°, \; -\Delta V/V_0 = 5.62 \times 10^{-7}p - 2.2 \times 10^{-12}p^2.$$

After the first annealing at 670° the results were little changed. Most of the readings again lay on a straight line; at 30° the average arithmetical deviation from a straight line of a single one of the 12 readings was 0.33% of the total effect, and at 75° the corresponding figure for 15 readings was 0.36%. Again the only exception was at 30°, where now the three highest points, at 10000, 11000, and 12000 lay above the curve by 10 times the average deviation of the other points, corresponding to a discontinuity in $\Delta l/l_0$ of 0.000075. The effect is doubtless real, and means some sort of internal change at the high pressure end of the 30° isotherm, the effect of annealing at 670° being to depress somewhat the pressure at which this change occurs. Neglecting this transition, and assuming a linear relation, the results are:

After annealing at 670°,

$$\text{At } 30°, \; -\Delta V/V_0 = 5.24 \times 10^{-7}p - 2.2 \times 10^{-12}p^2,$$
$$\text{At } 75°, \; -\Delta V/V_0 = 5.28 \times 10^{-7}p - 2.2 \times 10^{-12}p^2.$$

The effect of annealing at 670° has been to decrease the mean compressibility slightly.

The second annealing at 1050° materially altered the behavior, and introduced hysteresis effects and permanent changes, just as the annealing of the Fe-W alloys did. In all, five applications of pressure were made at 30° and 75°, the results of which are shown in Figure 24. There is no point in trying to reproduce the details of the hysteresis. If the mean curves indicated by the dotted lines are used in the computation, the results given in Table X will be found.

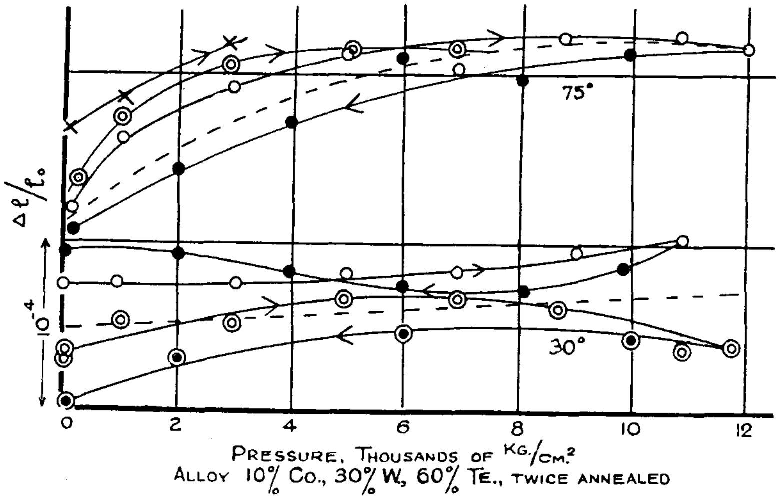

Figure 24. The differential linear compression of the alloy 10% Co, 30% W, 60% Fe, twice annealed. The significance of the points is as follows. At 30°: open circle, first increase; solid circle, first decrease; double open circle, second increase; double circle, solid core, second decrease. At 75°: Cross, first increase; double circle, second increase; single open circle, third increase; single solid circle, third decrease.

TABLE X.

Relative Changes of Length of Alloy Fe 60%, W 30%, Co 10% after Annealing at 1050°.

Pressure	$-\Delta l/l_0$	
kg/cm²	30°	75°
2000	.000365	.000338
4000	714	681
6000	.001077	.001027
8000	1425	1375
10000	1767	1726
12000	2104	2075

At 30° the results can also be indicated by the formula:

$$-\Delta V/V_0 = 5.51 \times 10^{-7}p - 2.3 \times 10^{-12}p^2.$$

The annealing has now increased the compressibility by a slight amount.

Discussion.

This discussion will be almost entirely concerned with the anomalous behavior of the various substances; the normal substances at the beginning of the paper have already been sufficiently commented on in the detailed exposition of the data. All that can be attempted with regard to the discontinuities is a discussion of the thermodynamics involved and a few very general remarks.

The basic thermodynamics has already been discussed in connection with the anomalies of NH_4Cl.[12] It was there shown that if there is a break in direction (discontinuity in the first derivative) of the curves of volume against pressure at constant temperature, then there is a relation between the increments of pressure and temperature at which the break occurs which is the exact analogue of Clapeyron's equation, and is $d\tau/dp = \tau dv/dQ$. dv/dQ is the ratio of volume change to heat absorbed by whatever the process is which is responsible for the break. There is now one important remark to be made about the thermodynamics of the situation which was not made in the previous analysis. It is well known that the discontinuities in the volume itself, such as occur when a system changes from one phase to another, are subject to the thermodynamic restriction that the phase stable at the higher pressure must have the smaller volume, or in other words, any discontinuity in volume brought about by increasing pressure must be a volume decrease. It might at first impulse be thought that there should be a similar restriction on any discontinuities in the direction of the isotherm (that is, in the derivative of the volume with respect to pressure at constant temperature), so that only discontinuities of the kind indicated in Figure 25 should be possible, that is, only discontinuities in which the internal change brought about by increasing pressure runs with decreasing volume. This however, is not the case. The relation above is a geometrical one, the only intervention of thermodynamics in the derivation of the formula occurring when $\left(\frac{\partial v}{\partial \tau}\right)_p$ is replaced by $-\frac{1}{\tau}\left(\frac{\partial Q}{\partial p}\right)_\tau$. This thermodynamic relation contains no restriction as to sign, and the generalized Clapeyron equation may be regarded as a geometrical one determining dQ in terms of any possible values of $d\tau$, dp, and dv. A detailed examination will show that the only restriction as to sign which thermodynamics imposes is merely in demanding that the slope of the isotherms be negative, that is, that the compressibility be negative. In Figure 26 are shown the various geometrical possibilities with regard to the relative location of the various isotherms; to each of these geometrical possibilities corresponds a case with $d\tau$ either positive or negative, so that physically there are 16 qualitatively different cases.

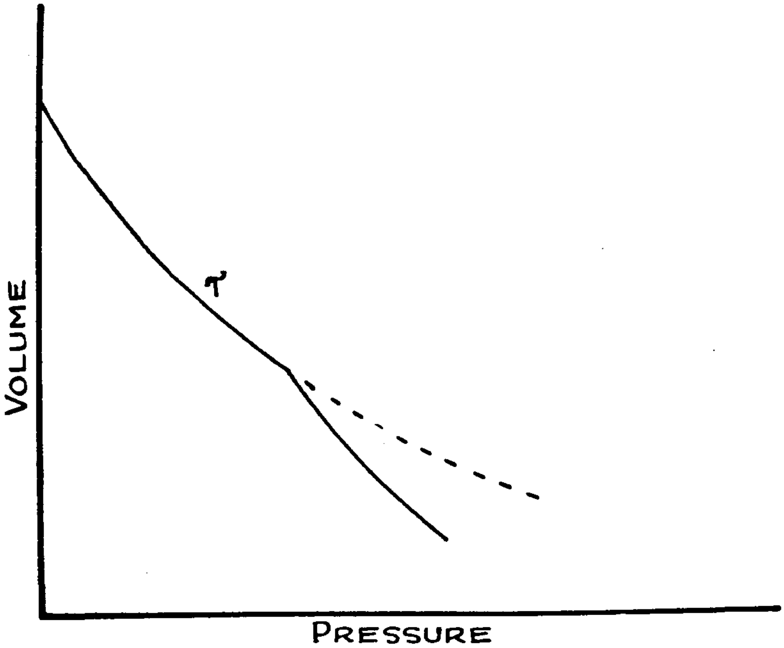

FIGURE 25. One possible kind of discontinuity in the volume-pressure isotherm.

From the point of view of a single isotherm, all this merely means that the compressibility may either increase or decrease on passing with increasing pressure through a point of discontinuity in the derivative. This is only natural, because of course it is well known that there is no thermodynamic reason why the compressibility of a substance

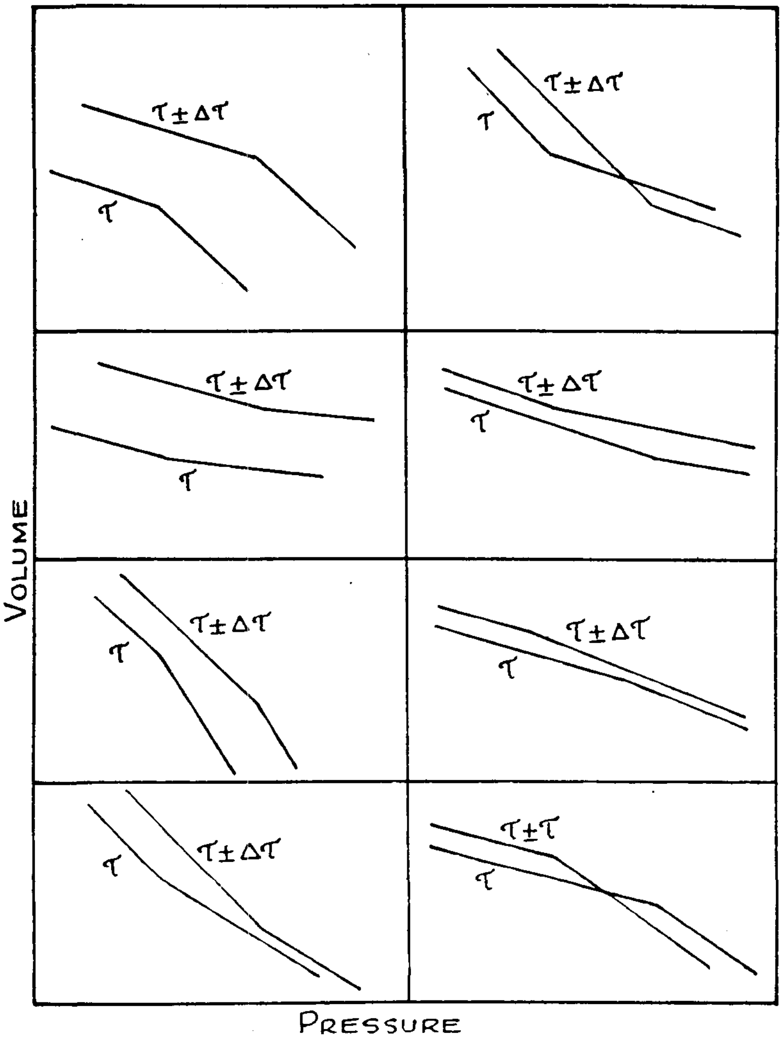
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FIGURE 26. Shows the various possibilities with regard to the change with temperature of the discontinuity in the slope of an isotherm.

should not increase with increasing pressure, and such cases actually exist, although they are not very common.

The substances of this paper afford many examples of discontinuities in which the system on the high pressure side of the discontinuity has a greater compressibility than on the low pressure side. To take a single example, every other one of the breaks of the W 30%, Co 70% alloy must obviously be of this kind.

A complete thermodynamic description of the discontinuities measured in this paper would demand that the isotherms be spaced close enough to permit an evaluation of $d\tau/dp$. This evidently is not possible in most of the cases. Thus in Figure 23 for the W 30% Co 70% alloy, it is not obvious how the breaks of the 30° curve are to be made to correspond to the breaks of the 75° curve, and failing such a correspondence it is not possible to calculate $d\tau/dp$ at any of the breaks. In some cases, however, the calculation is possible. Thus the two breaks at low and high pressures respectively of the W 10% Co 90% alloy obviously correspond, so that $d\tau/dp$ can be found for both these breaks. The Ag 75% Au 25% alloy shows how the breaks may be correlated when the isotherms are closely enough spaced, although this would have been impossible otherwise. If only the 30° and the 75° isotherms were known for this alloy, the proper correlation would have been exceedingly questionable, but with the aid of the 50° isotherm there cannot be much doubt. Thus, the break at 1900 kg. at 75° has been displaced to 560 at 50°, and at 30° had dropped out to negative pressures. Similarly the 3000 point at 75° is displaced to 1200 at 50°, and is also out of the picture at 30°. The 9100 break at 75° moves to 7100 at 50° and to 4900 at 30°. The 9600 break at 75° goes to 7700 at 50° and 6900 at 30°. The two breaks at the highest pressures at 30°, 10000 and 10700, are presumably displaced to pressures higher than 12000 at 50° and 75° and so could not be detected. These two examples show the possibility of either sign for $d\tau/dp$; it is negative for the two breaks of the W 10% Co 90% alloy, and positive for all the breaks of the Ag-Au alloy.

With regard to the numerical values of $d\tau/dp$ there seems no reason to anticipate that any sort of restriction will hold. It is known that there is no restriction on the $d\tau/dp$ of possible transitions from one solid phase to another. The mechanism of the discontinuity in the derivative is doubtless more complicated in many cases than the mechanism of a phase change, and it would be most surprising if as wide a range of variation of relations between the various parameters were not possible.

The most interesting question to be answered in connection with these discontinuities is what the mechanism may be which is responsible. The most important clue is doubtless the fundamental fact that we are not concerned here with discontinuities in the volume, but with discontinuities in the derivatives. This means that whatever internal change is responsible is spread over a range of pressure and temperature, instead of being abrupt, as in an ordinary polymorphic transition or phase change of a pure substance. This means that changes in the type of lattice structure may be ruled out as the explanation of these discontinuities. Direct experimental proof has been given by X-ray analysis that no change of lattice structure occurs during the anomalous volume changes of NH_4Cl.

Phase changes in impure substances, when the impurity is not equally soluble in the two phases, are spread over a range instead of being abrupt. However, phase changes in the presence of impurity are accompanied by hysteresis effects, due to diffusion of the impurity from one phase to the other, and such hysteresis effects are absent from many of the discontinuities examined above. In fact, it is one of the most striking characteristics of most of these phenomena that they are free from hysteresis. The only exception is the Fe-W alloys, which all show hysteresis.

The existence of phenomena of hysteresis are not difficult to understand in alloys where the phenomena of solid solubility are met, for we have the possibility of the limits of solubility being altered by pressure, and so internal changes being initiated by changes of pressure. It is doubtful, however, whether anything as simple as this can explain the pressure effects, even in the Fe-W alloys. The phase diagram for Fe-W as given by Sykes[13] indicates that the region between 10 and 30% W is a two phase region, the alloy breaking up into a mixture of WFe_2 and a solid solution of about 8% W in αFe. The only possibility here would seem to be a change produced by pressure in the solubility limits of the solid solution of W in αFe, but in this case alloys of different compositions should show the same sort of anomaly, the effect of changing the gross composition being merely to change the numerical magnitude of the effect because the fractional part played by the solid solution in the whole phenomenon varies as its relative amount with regard to the WFe_2 varies. As a matter of fact, the pressure effects on the 10%, 20%, and 30% alloys are quite different in character. This suggests that the correct phase diagram of W-Fe is much more complicated than at present known, and that there must be many small scale effects not yet found in the region now supposed to be occupied only by mechanical mixtures. This also suggests the possibility of a pressure exploration as a tool in detecting in alloys small scale structure not at present detectible by other methods.

For those substances which do not show hysteresis the conclusion would seem to be justified that the mechanism responsible for the discontinuity must be some intimate affair of the structure elements, and cannot be anything like those changes of lattice type that involve tearing down and building up again the lattice structure with comparatively large motions of some of the elements of the lattice or like anything else that involves motion of the elements of the structure through large distances. There are, of course, types of lattice rearrangement in which only a very slight change of position of some of the elements is demanded, as for example, in the inversion of quartz, which is accomplished by a slight angular displacement of each of the oxygens. But even such a change of lattice type would not satisfy our requirements, because it is accompanied by a discontinuity in the volume instead of in the derivative. Doubtless the explanation of Pauling[14] for the anomalies of NH_4Cl exemplifies the sort of thing involved. It is supposed that at a certain temperature the NH_4 radicals begin to acquire rotational motion, instead of the oscillational motion with which they are content at low temperatures, and that above a certain temperature practically all have acquired this rotation. This sort of thing cannot be general, however, but must be regarded only as suggesting one type of thing that might happen. For example, RbCl, described in another paper, has a break which cannot well be due to this sort of thing, because of the spherical symmetry of the ions. It would seem that the anomaly of RbCl must demand some internal change in the ions themselves, such perhaps as a change in the size of one of the electron shells. When one considers that the accurate solution of the wave mechanics problem demands that the wave equation be set up for the whole complicated system with every particle reacting on every other, and that the system cannot be regarded as built up of rigid building blocks, one sees enormous possibilities for small scale complications.

In conclusion, I must acknowledge my indebtedness to my assistant Mr. L. H. Abbot for making most of the readings of this paper, and to the Rumford Fund of the American Academy of Arts and Sciences and to the Milton Fund of Harvard University for financial assistance.

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REFERENCES.

[1] P. W. Bridgman, Proc. Amer. Acad. 58, 151, 166, 1923; 59, 109, 1923; 60, 385, 1925; 62, 207, 347, 1928; 64, 50, 1929; 66, 255, 1931; Amer. Jour. Sci. 10, 483, 1925; 15, 287, 1928.

[2] P. W. Bridgman, Rec. Trav. Chim. Pays-Bas, 51, 627, 1932.

[3] F. Simon, Ann. d. Phys. 68, 241, 1922.

[4] P. W. Bridgman, Proc. Amer. Acad. 59, 109, 1923.

[5] P. W. Bridgman, Proc. Amer. Acad. 60, 315, 1925.

[6] Fifth reference under 1.

[7] P. W. Bridgman, Proc. Amer. Acad. 56, 113, 1920; 64, 57, 1929.

[8] T. W. Richards, Jour. Amer. Chem. Soc. 37, 1646, 1915.

[9] P. W. Bridgman, Proc. Amer. Acad. 62, 224, 1927.

[10] E. Madelung und R. Fuchs, Ann. d. Phys. 65, 289, 1921.

[11] Bengt Beckman, Inaugural Dissertation, 1911, Uppsala, Akademiska Bokhandeln.

[12] P. W. Bridgman, Phys. Rev. 38, 182, 1931.

[13] W. P. Sykes., Mining and Metallurgy, Feb., 1926.

[14] L. Pauling, Phys. Rev. 36, 430, 1930.

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THE EFFECT OF PRESSURE ON THE ELECTRICAL RESISTANCE OF SINGLE METAL CRYSTALS AT LOW TEMPERATURE.

BY P. W. BRIDGMAN.

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Introduction.

In this paper measurements are presented of the effect of hydrostatic pressure up to 7000 kg/cm^2 at $-78°$ and $-183°$ (solid CO_2 and liquid O_2 temperatures) on the electrical resistance of the non-cubic metals Zn, Cd, Sn, Bi, Sb, As, and Te in the single crystal form. Except for As, two orientations were examined for each metal, so that the effect of pressure on resistance for all directions of current flow in the crystal is thereby determined. These measurements constitute an extension of measurements previously made in the same range of pressure and temperature for a number of metals crystallizing in the cubic system.[1] The apparatus and general method were the same as formerly and therefore need not be described again.

The preparation of the single crystals requires a short discussion since the small size demanded a modification of the method formerly used. The length must not be greater than 2.5 cm., and the diameter must be proportionally small to give a resistance high enough for the requisite accuracy. The diameter of most of the crystal rods was of the order of 1 mm. They were cast in pyrex tubing of this diameter by the same method as that by which my other crystals have been made,[2] namely by slow lowering out of a furnace. The pyrex tubing was bent at a number of places so as to form a broken line, each segment being about 2.5 cm. long, with the orientations of the segments spaced more or less uniformly over the entire possible range. In this way a single casting gives crystal rods of many orientations. The formation of single crystal castings of this small diameter is much easier than the formation of larger diameters, since the chance of occurrence of new nuclei is much less with the small diameter, so that the probability of a switch from one orientation to another is much decreased. In fact, all the castings attempted were single grains without any switches. There are, however, compensating difficulties in the use of such small crystals. If the metal has a well developed cleavage, as do Zn, Bi, Sb, As, and Te, the orientation of the rod may be determined at once by locating the cleavage plane. But Cd and Sn have no cleavage. The former method was to determine the reflection pattern of each rod, but this method is hardly practical now because of the difficulty of locating on such small rods enough regular reflections to permit a certain identification of the axis. A simple modification of the reflection method was found feasible. The reflection pattern of the entire casting, as it comes from the furnace, was determined by the method previously applied to single rods. This was always easy, because out of the dozen or more directions there were sure to be some favorably situated with regard to any particular reflection direction, so that the entire reflection pattern could be located, and the axis certainly determined. This axis was then marked, and the casting broken into the separate straight rods, noting the orientation of each rod as it was broken off. This method does not permit as clean a location of each separate direction in the reflection pattern as was formerly possible with the individual bare rods, because of the confusion incident to reflection through the glass tubing, but this is compensated by the possibility of locating all the faces, so that the finally determined orientation should be as accurate as before, within a degree or less.

Another detail of technique that had to be modified because of the small size of the crystal is the method of removing from the glass mold. The glass cannot be cracked away, as was formerly done, without danger of producing a serious amount of bending or other distortion of the rod. The glass was therefore ground off; this was conveniently done by rubbing the glass tube with the crystal in it on a plate glass with carborundum powder until the metal was just reached, and then rubbing through again from the opposite side, when the glass tube drops off. The naked crystal rod, which is exceedingly deformable, was then slipt at once into a protecting trough, made by grinding off one side of a glass tube of internal diameter slightly greater than the diameter of the crystal. The current leads were then attached by soldering at the ends, and the potential leads by soldering at the sides through the open side of the trough.

Because of the low resistance of the specimens, which is much enhanced at the low temperatures, much larger measuring currents were necessary than formerly; these reached at the maximum about 0.4 amp. This large current did not, however, introduce undesirable parasitics, and the accuracy was in fact in many cases better than before.

The methods of computation, by which the effect of pressure on the measured resistance in two different arbitrary directions is to be reduced to the effect of pressure on the specific resistance parallel and perpendicular to the crystal axis, are somewhat complicated, and involve several steps. The effect of pressure on the *measured* resistance (that is, the resistance of a rod with terminals attached to the rod) of a crystal rod of given orientation must first be converted into the effect of pressure on the specific resistance in the same direction. This involves two corrections; one for the effect of pressure on the linear dimensions of the rod, and the other for the effect of pressure on the orientation of the rod, for obviously the angle between the length of the rod and the crystal axis changes with pressure. The method of computing both these corrections has been previously given.[3] The fractional increase of resistance under unit increment of pressure arising from change of dimensions is:

$$2\,(S_{11} + S_{12}) \cos^2\theta + 2S_{13} \sin^2\theta + S_{33}\,(1 - 2\cos^2\theta),$$

and the corresponding fractional increase arising from the change of angle produced by pressure is:

$$2\,\frac{\rho_{90^\circ} - \rho_{0^\circ}}{\rho} \sin^2\theta \cos^2\theta\,[S_{33} + S_{13} - S_{11} - S_{12}].$$

Here the S's are the elastic constants in Voigt's notation, θ is the angle between the crystal axis and the length of the rod, and ρ is specific resistance. The elastic constants have been previously determined at room temperature and for low stresses, and are tabulated in the paper just referred to. The correction for the change of angle is usually much smaller than that for the change of dimensions, and is in many cases negligible. The maximum correction is for Cd, where both corrections together amount to 37%. Strictly, the elastic constants are functions of pressure and temperature, but the variation is probably not large, and since it has not been determined experimentally, the best that could be done was to assume that the correction is linear with pressure and the same at all temperatures.

The effect of pressure on *specific resistance* having thus been determined for the two given orientations, it remains to find the effect of pressure on the specific resistance in the parallel and perpendicular directions, these being the two independent parameters of greatest simplicity. This may be done by means of the general relations connecting resistance and orientation

$$\rho_{\theta 1} = \rho_{0^\circ} \cos^2 \theta_1 + \rho_{90^\circ} \sin^2 \theta_1$$

$$\rho_{\theta 2} = \rho_{0^\circ} \cos^2 \theta_2 + \rho_{90^\circ} \sin^2 \theta_2,$$

which give

$$\rho_{0^\circ} = \frac{\rho_{\theta 1} \sin^2 \theta_2 - \rho_{\theta 2} \sin^2 \theta_1}{\cos^2 \theta_1 \sin^2 \theta_2 - \cos^2 \theta_2 \sin^2 \theta_1}$$

$$\rho_{90^\circ} = \frac{\rho_{\theta 1} \cos^2 \theta_2 - \rho_{\theta 2} \cos^2 \theta_1}{\cos^2 \theta_2 \sin^2 \theta_1 - \cos^2 \theta_1 \sin^2 \theta_2}.$$

These equations connecting ρ_{0° and ρ_{90° with $\rho_{\theta 1}$ and $\rho_{\theta 2}$ hold at every pressure. But $\rho_{\theta 1}$ and $\rho_{\theta 2}$ are now known as functions of pressure because of the corrections for distortion which have just been discussed, so that ρ_{0° and ρ_{90° may at once be found also as functions of pressure. In particular, if $\rho_{\theta 1}$ and $\rho_{\theta 2}$ are linear or second degree functions of the pressure then ρ_{0° and ρ_{90° are also.

It is to be noticed that the calculation demands the absolute values of $\rho_{\theta 1}$ and $\rho_{\theta 2}$ as well as the pressure coefficients. The relative resistance of the different orientations at atmospheric pressure was measured as a function of temperature, as well as the pressure effect, and this, combined with the previously measured resistance[4] at atmospheric pressure at temperatures above 0° C gives the required absolute values at lower temperatures.

Because of the necessary approximations in reducing from measured to specific resistance parallel and perpendicular, I have thought it best to record in the following both the actual results for the effect of pressure on the measured resistance as well as the values computed therefrom for specific resistance parallel and perpendicular. The measured resistances stand; if in the future the dependence of elastic constants on temperature and pressure should be determined, the specific resistances as given in the following may be recomputed and improved.

The routine of the pressure measurements was the same as in the preceding paper. One general difference between the previous measurements with cubic metals, which were usually in the form of drawn wires, and the present measurements needs special comment, namely the fact that seasoning effects on the very first application of pressure were much less for the crystals than for the wires. This is in accord with general experience that a single crystal, if it has been properly handled, is in a state of complete ease, and needs no seasoning. The tendency for the zero point to lie off the curve drawn through the pressure points was also much less marked for the single crystals than for the wires, and there was no consistency in the direction of deviation.

Detailed Data.

Zinc. The single crystals were made from spectroscopically pure Zn obtained from the New Jersey Zinc Co. This was obtained through the courtesy of Mr. Cyr, and I take this opportunity of expressing my indebtedness to him. The purity is doubtless materially higher than that of the Zn used in the previous work, although there was no reason to suspect an amount of impurity in the "Kahlbaum" zinc previously used sufficient to produce appreciable error. Before casting, the metal was filtered in the molten condition, and thoroughly outgassed in a diffusion pump vacuum. This preliminary outgassing plays an important part in permitting the formation of geometrically perfect crystals without blow holes. The same procedure was followed with all the other metals.

Runs were made on two samples, the axis being perpendicular and at 18° to the length. Various accidents, which culminated in the rupture of the connecting pipe, made readings on the perpendicular sample, particularly at CO_2 temperature, less complete than usual. However the entire pressure range was covered and it did not appear that the uncertainty in the results was great enough to demand another set-up. The run at CO_2 temperature on the perpendicular sample was not good enough to justify more than drawing the best straight line through the experimental points, but the other three runs, one on the perpendicular and two on the 18° orientation, gave curves distinctly not linear, but of the second degree in the pressure. The average deviation of the points of these three runs from second degree curves was a small fraction of one per cent.

The numerical results are now given in Table I.

TABLE I.

ZINC.

Angle between axis and length	Temp. °C	Relative Effect of Pressure on Measured Resistance	Average Coefficient of Measured Resistance 0–7000 kg/cm²	Relative Effect of Pressure on Specific Resistance	Specific Resistance at 0 Pressure	Relative Specific Resistance at 0 Pressure
90°	0°				5.386×10^{-6}	1.0000
	− 78°.44	$-5.38 \times 10^{-6} p$	-5.38×10^{-6}	$-6.67 \times 10^{-6} p$	3.657	.6788
	−182°.98	$-7.53 \times 10^{-6} p + 6.9 \times 10^{-11} p^2$	-7.04	$-8.82 \times 10^{-6} p + 6.9 \times 10^{-11} p^2$	1.322	.2455
18°	0°				5.568	1.0000
	− 78°.54	$-11.06 \times 10^{-6} p + 13.1 \times 10^{-11} p^2$	-10.14	$-10.34 \times 10^{-6} p + 13.1 \times 10^{-11} p^2$	3.784	.6797
	−182°.95	$-12.76 \times 10^{-6} p + 17.5 \times 10^{-11} p^2$	-11.53	$-12.04 \times 10^{-6} p + 17.5 \times 10^{-11} p^2$	1.665	.2922
90°	0°				5.386	1.0000
	− 78°.30			As above	3.660	.6794
	−182°.80				1.325	.2459
0°	0°				5.589	1.0000
	− 78°.30			$-10.72 \times 10^{-6} p + 14.4 \times 10^{-11} p^2$	3.804	.6807
	−182°.80			$-12.31 \times 10^{-6} p + 18.4 \times 10^{-11} p^2$	1.703	.3047

Cadmium. The material was from Kahlbaum, grade "Kahlbaum." The procedure in casting the single crystals was canonical in every way, except that the furnace was filled with CO_2 during the melting and filling of the mold in order to avoid oxidation.

Previous measurements on cadmium at temperatures above 0° C had disclosed an unusual situation in that there are two polymorphic transitions,[5] the lower one taking place at about 3000 kg/cm^2. It is a peculiarity of these transitions that when the original modification is restored by a release of pressure the original crystal orientation is completely lost, the originally single crystal becoming a haphazard aggregate. It follows that no pressure measurements of significance can be made on the normal low pressure modification unless the pressure is continually kept below 3000 kg/cm^2, a consideration that was not properly appreciated soon enough in making the former measurements.

On the first applications of pressure at CO_2 temperature unusual care was therefore taken. The pressure steps were made unusually short and 12 readings were made in the first 3000 kg. instead of the usual 4. No discontinuities were found, however, and beyond 3000 the pressure steps were made longer, still with no trace of discontinuity. The same was found with both crystals at both temperatures: no evidence whatever of a transition was found. The obvious explanation is that the transition is suppressed at low temperatures by internal viscosity.

Runs on both orientations at both temperatures were accomplished without accident of any kind. The average deviation from smooth curves of all 71 points of the 4 runs, including the zero point deviations which were by far the largest, was 0.3% of the maximum pressure effect. Within experimental error the measurements can be reproduced by second degree curves, except for the 25° orientation at CO_2 temperature. Here the curvature at the low pressure end of the range is somewhat less and at the high pressure end somewhat greater than indicated by the second degree formula, but the deviation is so small that no attempt was made to reproduce it by the introduction of higher terms. The second degree formula given in the table is that corresponding to a curve drawn through the two end and the mid points.

The numerical results are collected in Table II.

Tin. The material was "Kahlbaum" stock, treated and crystallized in the usual way. The axis of the first specimen was inclined at 82° to the length. There were zero point irregularities of the order of 5% of the maximum effect. The average numerical deviation of

TABLE II.

CADMIUM.

Angle between axis and length	Temp. ° C	Relative Effect of Pressure on Measured Resistance	Average Coefficient of Measured Resistance 0–7000 kg/cm²	Relative Effect of Pressure on Specific Resistance	Specific Resistance at 0 Pressure	Relative Specific Resistance at 0 Pressure
89°	0°				6.289×10^{-6}	1.0000
	− 78.35	$-7.67\times10^{-6}p+6.1\times10^{-11}p^2$	-7.24×10^{-6}	$-10.38\times10^{-6}p+6.1\times10^{-11}p^2$	4.339	.6900
	−182°.68	$-9.01\times10^{-6}p+11.8\times10^{-11}p^2$	− 8.19	$-11.72\times10^{-6}p+11.8\times10^{-11}p^2$	1.793	.2850
25°	0°				7.359	1.0000
	− 78°.28	$-12.90\times10^{-6}p+20.8\times10^{-11}p^2$	−11.44	$-13.21\times10^{-6}p+20.8\times10^{-11}p^2$	5.078	.6902
	−182°.83	$-13.13\times10^{-6}p+15.9\times10^{-11}p^2$	−12.01	$-13.44\times10^{-6}p+15.9\times10^{-11}p^2$	2.123	.2886
90°	0°				6.289	1.0000
	− 78°.30			$-10.37\times10^{-6}p+6.1\times10^{-11}p^2$	4.340	.6899
	−182°.80			$-11.71\times10^{-6}p+11.8\times10^{-11}p^2$	1.791	.2850
0°	0°				7.593	1.0000
	− 78°.30			$-13.72\times10^{-6}p+23.4\times10^{-11}p^2$	5.240	.6903
	−182°.80			$-13.75\times10^{-6}p+16.6\times10^{-11}p^2$	2.195	.2891

the other points from smooth curves was 0.3% of the maximum effect. At liquid O_2 temperature the relation between pressure and resistance was of the second degree within the limits of error. At CO_2 temperature there was perceptibly too much curvature at the higher pressures, as had also been found with cadmium, but the effect was so small that no attempt was made to get it into a formula. The other sample, with the axis at 17° to the length, showed on the first application of pressure at CO_2 temperature consistent seasoning effects much larger than normal, the initial zero lying below the final zero by about 10% of the maximum effect. The points with decreasing pressure were alone retained; these lay on a straight line with an average deviation of 0.7% of the maximum effect. At liquid O_2 temperature the seasoning effects entirely disappeared, and there was no zero disturbance, the average deviation of all points from a second degree curve being 0.3% of the maximum effect.

The numerical results are given in Table III.

Bismuth. This material had been prepared by a double electrolytic deposition; the original source of the material was a mixture of bismuth "Kahlbaum" and electrolytic bismuth from the U. S. Metals Refining Co. The regular casting procedure was followed. Runs were made on three samples. Three runs, at liquid O_2, CO_2, and liquid O_2 temperatures again, were made on the first sample, the axis of which was at 65° to the length. Both runs at liquid O_2 temperature showed considerable difference between the points obtained with increasing and decreasing pressure, the points of either set separately lying on a smooth curve. The maximum difference between ascending and descending branches was about 8% of the maximum effect. At CO_2 temperature the same phenomenon was exhibited, but in much less degree. This is unusual, for the irregularity is usually greatest at the higher temperature. It is also most unusual for the irregularity not to disappear on the second application of pressure at liquid O_2 temperature. No explanation was found of this anomalous behavior, so that it seemed best to discard the results for this crystal altogether. The results, however, were not inconsistent within their own irregularity with the results for the other two samples. The other two orientations were at 78° and 14.5° to the axis. The readings with these were very much more regular than for the first sample. There was, however, still more irregularity of the zero than with any of the other metals of this series. The maximum zero irregularity was shown by the 14.5° orientation at liquid O_2 temperature, where the maximum discrepancy of six

TABLE III.

TIN.

Angle between axis and length	Temp. ° C	Relative Effect of Pressure on Measured Resistance	Average Coefficient of Measured Resistance 0–7000 kg/cm²	Relative Effect of Pressure on Specific Resistance	Specific Resistance at 0 Pressure	Relative Specific Resistance at 0 Pressure
82°	0°				13.01×10^{-6}	1.0000
	− 78°.30	$-9.36\times10^{-6}p+10.2\times10^{-11}p^2$	-8.64×10^{-6}	$-10.02\times10^{-6}p+10.2\times10^{-11}p^2$	8.684	.6674
	−182°.90	$-10.19\times10^{-6}p+16.7\times10^{-11}p^2$	− 9.01	$-10.85\times10^{-6}p+16.7\times10^{-11}p^2$	3.239	.2490
17°	0°				9.430	1.0000
	− 78°.38	$-9.79\times10^{-6}p$	− 9.79	$-10.33\times10^{-6}p$	6.367	.6752
	−182°.73	$-11.06\times10^{-6}p+12.2\times10^{-11}p^2$	−10.20	$-11.60\times10^{-6}p+12.2\times10^{-11}p^2$	2.478	.2628
90°	0°				13.08	1.0000
	− 78°.30			$-10.16\times10^{-6}p+10.4\times10^{-11}p^2$	8.736	.6678
	−182°.80			$-10.84\times10^{-6}p+16.8\times10^{-11}p^2$	3.255	.2489
0°	0°				9.088	1.0000
	− 78°.30			$-10.38\times10^{-6}p+13.8\times10^{-11}p^2$	6.147	.6764
	−182°.80			$-11.66\times10^{-6}p+11.9\times10^{-11}p^2$	2.404	.2646

readings at 100 kg. or less was 4% of the total resistance. At higher pressures the regularity was much better. The 78° sample showed in general much the same sort of behavior as the discarded 65° sample, there being a difference between the ascending and descending points, both of which by themselves lay on smooth curves. The maximum difference between ascending and descending points at CO_2 temperature was 2.7% of the maximum effect, and at liquid O_2 temperature 4.2%. Since it has been my general experience that the descending points are much more likely to be correct, the ascending points were discarded in the final computation. The 14.5° orientation had no such consistent difference between ascending and descending points, the irregularities being distributed more at random between the two sets. The average numerical departure from a smooth curve was 0.7% at both temperatures. The corresponding departure of the points that were retained of the 78° orientation was so small as to be inappreciable on the scale of graphical construction employed. At liquid O_2 temperature the points of the 78° specimen lay sensibly on a second degree curve, and similarly the points of the 14.5° specimen at CO_2 temperature. The other two runs deviated very noticeably from the second degree, and in each case the deviation was in the same direction, in that the curvature is greater at the lower pressure and less at the higher pressure than would be demanded by a second degree relation. No attempt was made to find a formula for the non-second degree results, but in the following they are tabulated at 1000 kg. intervals.

The numerical results are given in Table IV.

Antimony. The material was the "Kahlbaum" grade. Except for the fact that the casting was made in quartz instead of pyrex, the procedure was the same as with the other metals. The extreme ease of cleavage makes the orientations with cleavage across the rod very difficult to handle, particularly in these small diameters, and the highest orientation that was measured was 41° (49° between cleavage plane and length). The other orientation was 87°. All four runs departed very markedly from the second degree relation and always in the same direction, the curvature being greater at the low pressure end of the curves; this is in the same direction as the departures shown by bismuth. The same irregularity with increasing pressure exhibited by bismuth was shown by this also, although not to so marked a degree. There was also some scattering of the zero, although again not nearly as much as by bismuth. The maximum deviation between ascending and descending points of the 41° speci-

TABLE IV.

BISMUTH.

Angle between axis and length	Temp. °C	Relative Effect of Pressure on Measured Resistance		Average Coefficient of Measured Resistance 0–7000 kg/cm²	Relative Effect of Pressure on Specific Resistance		Specific Resistance at 0 Pressure	Relative Specific Resistance at 0 Pressure
		p	*Resis*		*p*	*Resis*		
78°	0°						101.3×10^{-6}	1.0000
	$-78^\circ.30$	0	1.0000	$+18.11\times10^{-6}$	0	1.0000	70.87	.6998
		1000	1.0115		1000	1.0100		
		2000	1.0257		2000	1.0228		
		3000	1.0426		3000	1.0382		
		4000	1.0619		4000	1.0561		
		5000	1.0825		5000	1.0752		
		6000	1.1040		6000	1.0953		
		7000	1.1268		7000	1.1166		
	$-182^\circ.75$	$+15.31\times10^{-6}p+239\times10^{-11}p^2$		$+32.06$	$+13.86\times10^{-6}p+239\times10^{-11}p^2$		36.28	.3583
14°.5	0°						125.1	1.0000
	$-78^\circ.30$	$+17.14\times10^{-6}p+167\times10^{-11}p^2$		$+28.86$	$+17.32\times10^{-6}p+167\times10^{-11}p^2$		86.06	.6877
		p	*Resis*		*p*	*Resis*		
	$-182^\circ.85$	0	1.0000	$+39.03$	0	1.0000	41.44	.3312
		1000	1.0235		1000	1.0237		
		2000	1.0528		2000	1.0532		
		3000	1.0874		3000	1.0879		
		4000	1.1269		4000	1.1276		
		5000	1.1712		5000	1.1721		
		6000	1.2200		6000	1.2211		
		7000	1.2732		7000	1.2745		

TABLE IV.—Bismuth.—*Continued.*

Angle between axis and length	Temp. ° C	Relative Effect of Pressure on Measured Resistance	Average Coefficient of Measured Resistance 0–7000 kg/cm²	Relative Effect of Pressure on Specific Resistance		Specific Resistance at 0 Pressure	Relative Specific Resist-ance at 0 Pressure
	0°			*p*	*Resis*	100.1	1.0000
90°	− 78°.30			0	1.0000	70.08	.7001
				1000	1.0095		
				2000	1.0216		
				3000	1.0365		
				4000	1.0537		
				5000	1.0721		
				6000	1.0916		
				7000	1.1114		
	−182°.80			0	1.0000	36.02	.3598
				1000	1.0159		
				2000	1.0365		
				3000	1.0618		
				4000	1.0820		
				5000	1.1268		
				6000	1.1664		
				7000	1.2111		

TABLE IV.—Bismuth.—*Continued.*

Angle between axis and length	Temp. °C	Relative Effect of Pressure on Measured Resistance	Average Coefficient of Measured Resistance 0–7000 kg/cm²	Relative Effect of Pressure on Specific Resistance		Specific Resistance at 0 Pressure	Relative Specific Resistance at 0 Pressure
				p	*Resis*		
0°	0°					126.8	1.0000
	−78°.30			0	1.0000	87.14	.6873
				1000	1.0195		
				2000	1.0424		
				3000	1.0687		
				4000	1.0982		
				5000	1.1316		
				6000	1.1586		
				7000	1.2084		
	−182°.80			0	1.0000	41.78	.3295
				1000	1.0241		
				2000	1.0542		
				3000	1.0895		
				4000	1.1297		
				5000	1.1750		
				6000	1.2244		
				7000	1.2783		

men was 1.2% of the maximum at CO_2 temperature, and the maximum deviation by the 87° specimen 7% at liquid O_2 temperature. The descending points only were used, as in the case of bismuth, and again the average deviation from a smooth curve was inappreciable on the graphical scale employed, and was not more than a fraction of a tenth of a per cent.

The results are given in Table V.

Arsenic. It has not yet been found possible to prepare single crystals of dimensions suitable for the low temperature apparatus in which the cleavage plane runs across the rod at any considerable angle. These measurements refer to only a single orientation with the cleavage plane parallel to the length. In a previous paper[6] measurements have been given at 0° and 95° C for the resistance of single crystal arsenic up to 12000 kg/cm^2. The specimen used here was the same as one of the pieces previously measured. Arsenic is highly anomalous; both compressibility and electrical resistance in the direction of the cleavage plane (that is, the direction measured here) showing breaks, the interval from 0 to 12000 kg/cm^2 consisting of ranges in each of which the dimensions or resistance vary linearly with pressure, with different slopes in the different ranges. These anomalous effects rapidly disappear as the orientation swings toward the perpendicular.

Exactly the same phenomenon was found at CO_2 and liquid O_2 temperatures. One break in the slope of the curve giving resistance as a function of pressure occurs at each temperature in the pressure range up to 7000. Previously two breaks had been found up to 12000, which means three linear ranges. The middle range was shorter and somewhat more in doubt as compared with the two extreme ranges. It is not possible to say whether the two ranges found now at the lower temperatures correspond to the two low pressure ranges of the higher temperatures or whether the former middle range has been so displaced by temperature as to be absorbed by the low and the high pressure ranges. The latter seems to me somewhat the more probable.

At CO_2 temperature there were considerable seasoning effects at the low pressures, as is usual, so that the points obtained with increasing pressure did not show a very distinct break. The points with decreasing pressure, however, lay without question on two rather well marked different lines. At liquid O_2 temperature the seasoning effects were less, and the location of the break between the two separate ranges could be established with more certainty.

TABLE V.

ANTIMONY.

Angle between axis and length	Temp. ° C	Relative Effect of Pressure on Measured Resistance		Average Coefficient of Measured Resistance 0–7000 kg/cm²	Relative Effect of Pressure on Specific Resistance		Specific Resistance at 0 Pressure	Relative Specific Resistance at 0 Pressure
		p	*Resis*		*p*	*Resis*		
	0°						38.63×10⁻⁶	1.0000
87°	− 78°.26	0	1.0000	+15.10×10⁻⁶	0	1.0000	24.67	.6390
		1000	1.0055		1000	1.0039		
		2000	1.0154		2000	1.0122		
		3000	1.0290		3000	1.0240		
		4000	1.0457		4000	1.0391		
		5000	1.0651		5000	1.0569		
		6000	1.0852		6000	1.0753		
		7000	1.1057		7000	1.0942		
	−182°.76	0	1.0000	+33.49	0	1.0000	8.70	.2251
		1000	1.0032		1000	1.0016		
		2000	1.0148		2000	1.0115		
		3000	1.0393		3000	1.0344		
		4000	1.0770		4000	1.0704		
		5000	1.1240		5000	1.1158		
		6000	1.1763		6000	1.1664		
		7000	1.2344		7000	1.2229		

TABLE V.—Antimony.—*Continued.*

Angle between axis and length	Temp. ° C	Relative Effect of Pressure on Measured Resistance		Average Coefficient of Measured Resistance 0–7000 kg/cm²	Relative Effect of Pressure on Specific Resistance		Specific Resistance at 0 Pressure	Relative Specific Resistance at 0 Pressure
		p	*Resis*		*p*	*Resis*		
41°	0°						33.28	1.0000
	−78°.32	0	1.0000	+22.61	0	1.0000	21.20	.6374
		1000	1.0115		1000	1.0133		
		2000	1.0272		2000	1.0309		
		3000	1.0475		3000	1.0530		
		4000	1.0716		4000	1.0790		
		5000	1.0985		5000	1.1077		
		6000	1.1277		6000	1.1387		
		7000	1.1593		7000	1.1722		
	−182°.70	0	1.0000	+37.66	0	1.0000	7.525	.2262
		1000	1.0057		1000	1.0075		
		2000	1.0210		2000	1.0247		
		3000	1.0490		3000	1.0544		
		4000	1.0917		4000	1.0991		
		5000	1.1430		5000	1.1522		
		6000	1.2010		6000	1.2120		
		7000	1.2636		7000	1.2765		
90°	0°						38.66	1.0000
	−78°.30				0	1.0000	24.69	.6390
					1000	1.0039		
					2000	1.0121		
					3000	1.0239		
					4000	1.0390		
					5000	1.0567		
					6000	1.0750		
					7000	1.0940		

TABLE V.—ANTIMONY.—*Continued.*

Angle between axis and length	Temp. ° C	Relative Effect of Pressure on Measured Resistance	Average Coefficient of Measured Resistance 0–7000 kg/cm²	Relative Effect of Pressure on Specific Resistance		Specific Resistance at 0 Pressure	Relative Specific Resistance at 0 Pressure
				p	*Resis*		
90°	−182°.80			0	1.0000	8.706	.2251
				1000	1.0016		
				2000	1.0114		
				3000	1.0343		
				4000	1.0703		
				5000	1.1157		
				6000	1.1662		
				7000	1.2227		
0°	0°					29.23	1.0000
	− 78°.30			0	1.0000	18.55	.6362
				1000	1.0228		
				2000	1.0498		
				3000	1.0823		
				4000	1.1193		
				5000	1.1590		
				6000	1.2027		
				7000	1.2510		
	−182°.80			0	1.0000	6.631	.2269
				1000	1.0134		
				2000	1.0378		
				3000	1.0743		
				4000	1.1275		
				5000	1.1884		
				6000	1.2573		
				7000	1.3298		

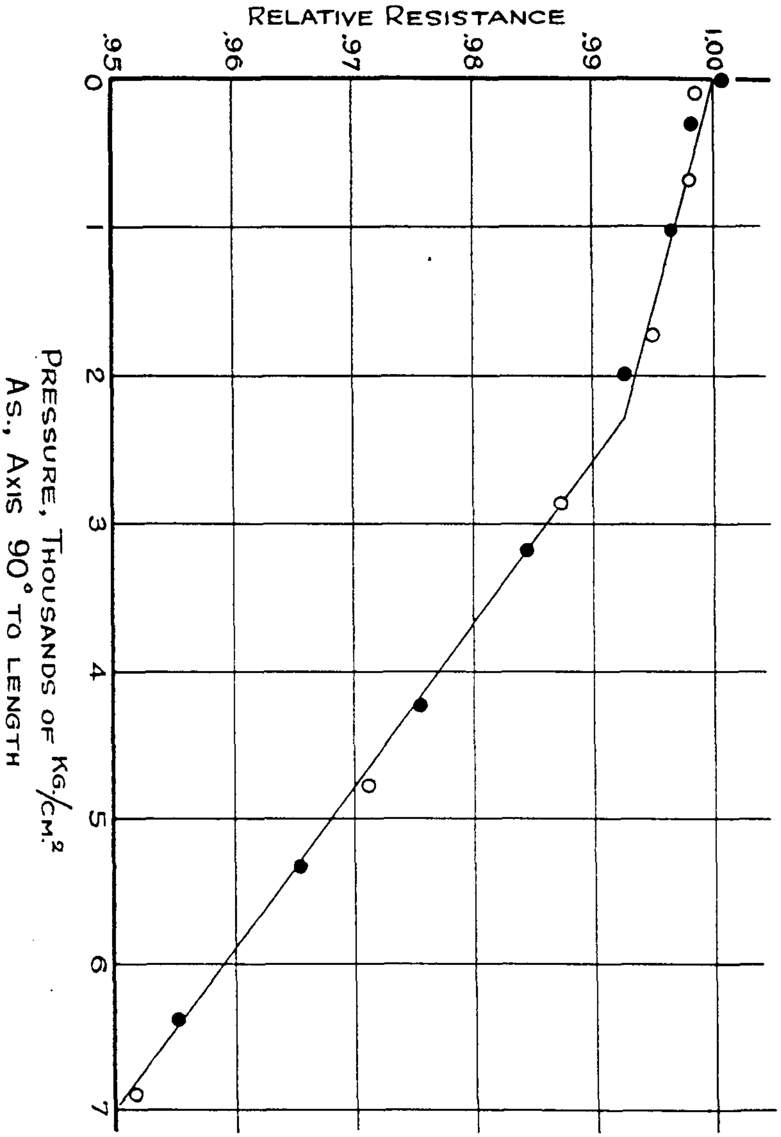

FIGURE 1. Relative resistance at − 82.8° of single crystal arsenic, hexagonal axis perpendicular to the length.

The experimental points at liquid O_2 temperature are shown in Figure 1. The ranges and the coefficients are as follows:

As, axis perpendicular to length.

$$-78.3°, 0-3540, \Delta R/R\ (0, -78°.3) = -1.92 \times 10^{-6}p.$$

$$3540-7000, \Delta R/R\ (0, -78°.3) = -3.24 \times 10^{-6}p.$$

$$-182.8°\ 0-2360, \Delta R/R\ (0, -182°.8) = -3.34 \times 10^{-6}p.$$

$$2360-7000, \Delta R/R\ (0, -182°.8) = -8.86 \times 10^{-6}p.$$

The relative resistances at atmospheric pressure at these temperatures were respectively .6501 and .2114 in terms of the resistance at 0° C at atmospheric pressure as unity.

The pressure at which the first break occurs is a more or less well defined function of pressure. The numerical values are as follows:

Pressure	Temperature
7100 kg/cm^2	75° C
5500	0
3540	− 78.3
2360	− 182.8

If these points are plotted, a fair straight line will be obtained passing through 0° Abs.

Tellurium. The crystals were made from "Kahlbaum" material. They were cast in pyrex tubes of about 1.2 mm. diameter like the other metals. One casting proved sufficient, so that the two pieces measured below are different orientations from the identical crystal. The furnace was filled with a protecting atmosphere of CO_2 during the filling of the tube, and the rate of lowering was about 10 cm. per hour. The cleavage system of tellurium differs from that of the other metals of this paper in that there are three planes of easy cleavage, instead of one, each of the three planes being parallel to the trigonal axis, so that the intersections of the three planes form triangular prisms with edges parallel to the axis. Unless the axis is very nearly parallel to the length, one of the three cleavage planes will cut across the rod at a fairly high angle, which means, in rods of this small diameter, very easy fracture. The usual method of grinding away the glass tubes proved to be too severe, resulting in frequent breaking. The method finally adopted was to dissolve away the glass in hydrofluoric acid, the tellurium fortunately proving not to be attacked. The crystals obtained in this way were highly satisfactory in appearance, with clean mirror-like cleavage planes, and were much better than I have ever obtained before. The improvement is probably to be ascribed partly to higher purity in the material, and partly to the smaller diameter.

Some difficulty was encountered in attaching the leads. Tellurium cannot be soldered, but it is known to stick directly to copper or platinum, when these are heated hot enough to melt their way into the tellurium. Platinum leads were attached by fusing in this way, but although they proved satisfactory on the initial application of pressure, high resistance developed on subsequent applications, doubtless an effect of cracking due to differential compressibility between tellurium and platinum. Experiment showed that silver very readily alloys with tellurium, in fact, the alloying action is so rapid that fine silver wires cannot be fused into the tellurium, but melt back and drop off. Finally fine platinum wire (0.005 cm. in diameter) was first coated with silver by fusing the silver, and then melted into the tellurium with a diminutive loop of nichrome heated to incandescence. This was entirely satisfactory; apparently the thin layer of alloy on the surface of the platinum is so tenacious as not to be entirely cracked by the differential compressibility.

The specific resistance of tellurium is so high that a modification of the regular potentiometer method of measurement was necessary. The simplest modification imaginable proved satisfactory. The potential drop across the tellurium under a known current was equalized to the potential drop produced by the same current across a variable resistance. An easy modification of the regular potentiometer connections proved feasible, which it is not necessary to describe in further detail.

The effect of pressure on the resistance of single crystal tellurium has not been measured previously because suitable specimens had not been available. The usual measurements at − 78.3° and −182.8° were therefore supplemented by a run at 0° C in the low pressure apparatus, pressure transmitted with nitrogen, to a maximum of 7000 kg/cm^2. The results obtained were so interesting, showing very large changes of resistance with both temperature and pressure, that it appeared desirable to extend the pressure and temperature range. The specimens were therefore set up in the regular high pressure apparatus, and the run at 0° repeated to 12000, petroleum ether transmitting the pressure, and a similar run was made at 95°.

The effects are so large that the logarithm of the resistance may be conveniently shown as a function of pressure rather than the re-

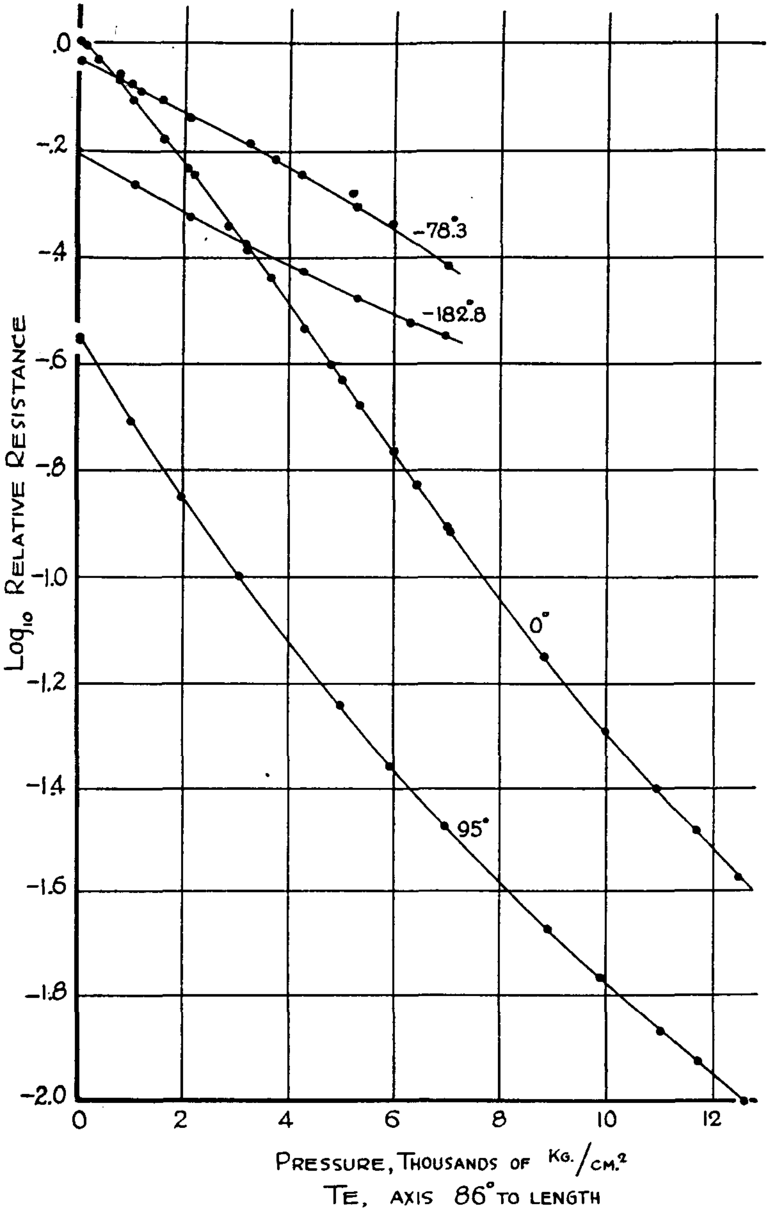

FIGURE 2. Logarith to the base 10 of the relative resistance of single crystal tellurium, axis 86° to the length, as a function of pressure at several temperatures.

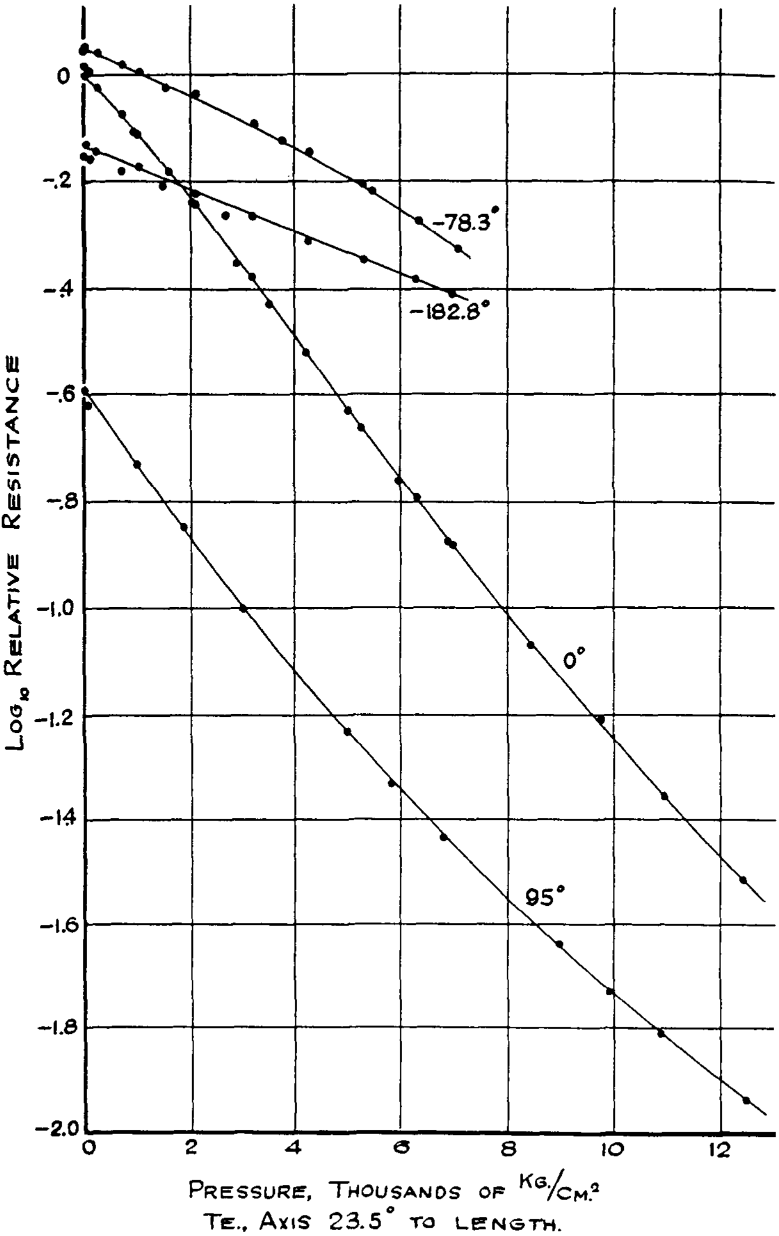

Figure 3. Logarithm to the base 10 of the relative resistance of single crystal tellurium, axis 23.5° to the length, as a function of pressure at several temperatures.

sistance itself. These are shown for the two orientations, axes at 86° and 23.5° to the length, in Figures 2 and 3. The ascending and descending branches lay so closely together in general that no attempt was made to separate them in the figures. The low temperature run of the 23.5° orientation does, however, show some divergence; the points with decreasing pressure are the high lying points. It is to be noticed that the two independent runs at 0° lie on the same curve

TABLE VI.

TELLURIUM.

Pressure kg/cm²	$\log_{10}(R/R_0)$ (measured)				$\log_{10}(\rho/\rho_0)$ (specific)			
	−182°.8	−78°.3	0°	95°	−182°.8	−78°.3	0°	95°
86° between axis and length								
0	−.213	−.032	.000	−.541	−.213	−.032	.000	−.541
1000	−.262	−.079	−.107	−.702	−.262	−.079	−.107	−.702
2000	−.310	−.127	−.229	−.853	−.310	−.127	−.229	−.853
3000	−.357	−.177	−.356	−.990	−.357	−.177	−.356	−.990
4000	−.406	−.230	−.490	−1.122	−.405	−.229	−.489	−1.121
5000	−.454	−.287	−.632	−1.245	−.453	−.286	−.631	−1.244
6000	−.503	−.351	−.772	−1.363	−.502	−.350	−.771	−1.362
7000	−.553	−.419	−.908	−1.478	−.552	−.418	−.907	−1.477
8000			−1.038	−1.582			−1.037	−1.581
9000			−1.164	−1.681			−1.163	−1.680
10000			−1.290	−1.776			−1.289	−1.774
11000			−1.408	−1.865			−1.406	−1.863
12000			−1.520	−1.952			−1.518	−1.950
23°.5 between axis and length								
0	−.135	+.047	.000	−.582	−.135	+.047	.000	−.582
1000	−.175	+.005	−.116	−.727	−.170	+.010	−.111	−.722
2000	−.215	−.038	−.239	−.863	−.205	−.028	−.229	−.853
3000	−.255	−.086	−.365	−.994	−.240	−.071	−.350	−.979
4000	−.295	−.137	−.498	−1.119	−.275	−.117	−.478	−1.099
5000	−.336	−.193	−.635	−1.237	−.311	−.168	−.610	−1.212
6000	−.376	−.257	−.768	−1.347	−.346	−.227	−.738	−1.317
7000	−.416	−.330	−.898	−1.450	−.382	−.296	−.864	−1.416
8000			−1.022	−1.548			−.983	−1.509
9000			−1.134	−1.645			−1.090	−1.601
10000			−1.246	−1.735			−1.198	−1.687
11000			−1.357	−1.818			−1.304	−1.765
12000			−1.465	−1.896			−1.408	−1.839

indistinguishably. The numerical values of the logarithm of the relative resistance are given in Table VI in terms of the resistance at atmospheric pressure at 0° C of the two orientations. The table also contains the logarithms of the relative specific resistances, that is, the measured resistance corrected for the effect of pressure on dimensions. The relative specific resistances at atmospheric pressure have not, however, been corrected for the effect of thermal contraction. It will be noticed that the correction for distortion is almost negligible for the 86° orientation, but quite perceptible for the 23.5° orientation.

TABLE VII.

Tellurium.

Pressure kg/cm^2	$\text{Log}_{10}\,\rho$ 0° between axis and length				$\text{Log}_{10}\,\rho$ 90° between axis and length			
	−182°.8	−78°.3	0°	95°	−182°.8	−78°.3	0°	95°
0	−.564	−.389	−.450	−1.046	−.417	−.236	−.204	−.745
1000	−.592	−.413	−.561	−1.178	−.466	−.283	−.311	−.907
2000	−.625	−.449	−.678	−1.303	−.515	−.331	−.433	−1.057
3000	−.658	−.489	−.799	−1.425	−.562	−.381	−.560	−1.194
4000	−.690	−.533	−.923	−1.541	−.610	−.434	−.693	−1.325
5000	−.721	−.585	−1.053	−1.652	−.658	−.491	−.835	−1.448
6000	−.752	−.642	−1.177	−1.752	−.707	−.555	−.975	−1.566
7000	−.790	−.712	−1.300	−1.848	−.757	−.623	−1.111	−1.681
8000			−1.416	−1.935			−1.241	−1.785
9000			−1.519	−2.027			−1.367	−1.884
10000			−1.622	−2.111			−1.493	−1.978
11000			−1.723	−2.186			−1.610	−2.067
12000			−1.827	−2.257			−1.722	−2.154

Table VII contains the logarithms of the actual specific resistances for the 0° and the 90° orientations, calculated by the same methods as for the other metals.

Discussion.

This discussion will at first not be concerned with tellurium, which is only partially metallic in character, reserving this for later. Considering now only the true metals, the general effect of lowering temperature is the same as that found previously for the cubic metals.[1] In all cases, that is, for every orientation of every metal, the pressure coefficient at liquid oxygen temperature is greater *numerically* than at CO_2 temperature; this holds both for the more normal metals and for arsenic, the resistance of which decreases with pressure, and also for bismuth and antimony with positive pressure coefficients. Excepting arsenic, the largest temperature effect is shown by the 90° orientation of antimony, the pressure coefficient of which at liquid O_2 temperature is more than double that at CO_2 temperature. These remarks apply to the linear term in the power series development in those cases in which such a development is possible. The second degree terms do not show any such regularity, but may either increase or decrease with decreasing temperature. Arsenic is anomalous because of the breaks in its pressure coefficient. Its coefficient in the upper pressure range at liquid O_2 temperature is of the order of three times greater than in any other range.

It is significant that in those cases in which departure from the second degree relation can be established the curvature becomes greater at increased pressure, provided the resistance decreases. On the other hand, bismuth and antimony, the resistance of which increases with pressure, have a decreasing curvature with increasing pressure. This result for the normal metals would suggest that the suspected minimum of resistance is nearer than would be indicated by an extrapolation of the second degree curves. The accuracy of the previous measurements for the cubic metals was not sufficient to indicate any such effect for them, and in fact most of the former results were linear.

A question which could not be put with regard to the cubic metals now presents itself, the question namely as to the relative effect of pressure on resistance in different directions. Is the effect of pressure to make the resistance in different directions more nearly alike, or does it accentuate the difference? The question can be answered at once by forming the derivative:

$$\frac{1}{F}\frac{dF}{dp} \equiv \frac{1}{\frac{\rho_{0^\circ}}{\rho_{90^\circ}}}\frac{d}{dp}\left(\frac{\rho_{0^\circ}}{\rho_{90^\circ}}\right) = \frac{1}{\rho_{0^\circ}}\frac{d\rho_{0^\circ}}{dp} - \frac{1}{\rho_{90^\circ}}\frac{d\rho_{90^\circ}}{dp}$$

We therefore have merely to take the difference of pressure coefficients in the two directions. When the ratio $\rho_{0^\circ}/\rho_{90^\circ}$ $(\equiv F)$ is greater than unity, a negative value for the right hand side of the equation means that the resistance becomes more nearly equal under pressure. The results are given in Table VIII.

TABLE VIII.

Effect of Pressure on Ratio of Resistance in Different Directions in the Crystal.

Substance	Temp.	$\frac{1}{F}\frac{dF}{dp}$	$\frac{\rho_{0^\circ}}{\rho_{90^\circ}}$ at 0° C
Zn	− 78°.3 −182°.8	$-4.05\times10^{-6}p+14.4\times10^{-11}p^2$ $-3.49\times10^{-6}p+11.5\times10^{-11}p^2$	1.038
Cd	− 78°.3 −182°.8	$-3.35\times10^{-6}p+17.3\times10^{-11}p^2$ $-2.04\times10^{-6}p+4.8\times10^{-11}p^2$	1.207
Sn	− 78°.3 −182°.8	$-0.22\times10^{-6}p+3.4\times10^{-11}p^2$ $-0.83\times10^{-6}p-4.9\times10^{-11}p^2$	.695
		$\frac{\Delta F}{F}$	
Bi	− 78°.3 −182°.8	3000, +.0322; 7000, +.0970 +.0530 +.0672	1.267
Sb	− 78°.3 −182°.8	3000, +.0584; 7000, +.1570 +.0400 +.1071	.756

It is seen that pressure tends to equalize the resistance in different directions for Zn, Cd, and Sb, but for Bi and Sn it accentuates the differences. The effect of pressure in equalizing resistance becomes less marked at low temperatures for Zn, Cd, and Sb, whereas for Bi and Sn the effect of pressure in unequalizing resistance becomes greater at lower temperatures.

The behavior of antimony demands special consideration. The resistance of the 90° orientation at − 182.8° apparently passes through a very flat minimum at approximately 400 kg/cm^2, as may be found from a rough graphical construction of the data in the table. The initial pressure coefficient of resistance is therefore negative, and doubtless would become increasingly so at lower temperatures. This is to be contrasted with the maximum resistance at 50° previously found for the same orientation. It is obvious that if the high temperature curves previously found are to fit together with the low temperature curves now found there must be a reversal of curvature of the low temperature curves at sufficiently high pressure. It is this which is evidently forshadowed by the decreasing curvature with increasing pressure already commented on.

Finally, with regard to tellurium, the most striking feature is the magnitude of the effect, the resistance at 95° at 12000 kg/cm^2 being only about one per cent of the resistance at 0° at atmospheric pressure. This is two or three times greater than the change which I found in black phosphorus, the substance with the largest pressure effect which I have yet measured. To a rough first approximation the logarithm of resistance is linear in the pressure, that is, the rate of proportional change with pressure is roughly the same at all pressures. Furthermore, the rate of change is approximately the same at 0° and 95°, and also the same at − 78.3° and − 182.8°, the rate at the two latter temperatures being very much less than at the two higher temperatures. This difference of rate at the two lower temperatures demands a crossing of the curves and a consequent reversal of sign of the temperature coefficient as pressure rises. One might be inclined to suspect a polymorphic transition between 0° and − 78.3, but there is no evidence for it, as shown by a special run in which resistance was measured as a function of temperature at atmospheric pressure, and also by the fact that there were no irreversible changes on running the temperature back and forth. The special temperature run showed for the 23°.5 orientation a smooth maximum of resistance at about − 45° C.

To a second approximation, the logarithm of the resistance is not linear in pressure. At 0° there is reversal of curvature; at low pressures the curve is concave downward, but at high pressures it is concave upward, while the 95° curve is concave upward at all pressures. Curvature in this direction is necessary if pressure is to pass through a minimum with rising pressure, but any such minimum must be very remote as far as any present indications are concerned, the immediate effect of pressure at these temperatures being a continuous decrease of resistance towards that characteristic of true metals. The curve at − 78.3° is in some respects not unlike that at 0°; it starts out concave downward, and apparently will cross the − 182.8° curve at pressures not very far beyond those reached, with consequent additional complications in the behavior of the temperature coefficient.

The effect of pressure on the resistance of tellurium is not greatly different in the different directions, as may be seen by comparison of figures 2 and 3. There is some difference in the order of the curves for different temperatures at low pressures, indicating that the reversal of sign of the temperature coefficient of resistance occurs at different temperatures in different directions, and there are also differences in the numerical values, but on the whole the resemblances in the different directions are more striking than are the differences. This leads to the conclusion that the very large effect of pressure on the resistance is not intimately connected with the crystal structure.

Examination of Table VII shows that the difference between different directions in the crystal becomes less marked as pressure increases at all temperatures.

I am much indebted to Mr. Charles Chase for setting up the apparatus and regulating the temperature controls for the low temperature measurements, and to Mr. L. H. Abbot for making the measurements of the resistance of tellurium at the higher pressures. I am also indebted to the Milton Fund of Harvard University for financial assistance.

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REFERENCES.

[1] P. W. Bridgman, Proc. Amer. Acad. 67, No. 8, 1932.
[2] P. W. Bridgman, Proc. Amer. Acad. 60, 305, 1925.
[3] Reference 2, page 320.
[4] Reference 2, and also: P. W. Bridgman, Proc. Amer. Acad. 63, 351, 1928.
[5] Reference 2, page 346.
[6] P. W. Bridgman, Proc. Amer. Acad. (immediately preceding paper).

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RADIOACTIVITY MEASUREMENTS.

I. The Radium Content of the Keweenawan Basalts and Some Accessory Minerals.

By Wm. D. Urry.

RADIOACTIVITY MEASUREMENTS.

I. THE RADIUM CONTENT OF THE KEWEENAWAN BASALTS AND SOME ACCESSORY MINERALS.

BY WM. D. URRY.[1]

In determining the age of the Keweenawan Basalts in the copper mines of the Upper Peninsula, Keweenaw Point, Michigan, a series of accurate measurements on the radium content was carried out. In so far as the data required to calculate the age, namely, the helium radium and thorium content of these rocks, is not yet completed, the thorium remaining to be determined, it is not proposed to discuss the question of the geological age here but rather to give the results of the analyses for radium since these values also have an important bearing in other researches.

The field work was carried out by the author in July, 1931, thus ensuring an exact knowledge of the position and relation of the samples, most of which were obtained underground at that time.[2] The determination of the radium content was made in co-operation with Professor Dr. F. Paneth in his private laboratory in Koenigsberg. i. Pr. using the same emanation method as described by Paneth and Koeck,[3] and employed by them to determine the very small quantities of radium in the iron meteorites.

In Keweenawan times many hundreds of basaltic flows were laid down interspersed with strata of felsitic conglomerate and sandstone. These flows are of the "Plateau" type. The accompanying diagrams are included to facilitate the comprehension of the following data. Figure 1 shows very roughly the stratigraphical arrangement taking a cross section of the Keweenaw Point on a line about North-West and South-East. The oldest flows are those nearest the Keweenaw Fault and the youngest those nearest the Freda sandstone. The thicker lines represent very roughly the relative positions of the mines

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[1] Contribution from the Johns Hopkins University, Baltimore, Md.

[2] The author is greatly indebted to the officials of the various mines and the staff of the Michigan College of Mining and Technology for their aid in collecting the specimens and instruction in the detailed geological aspect of the district.

[3] F. Paneth and W. Koeck, Zeit. Physik. Chemie. Bodenstein-Festband. 145 (1931). The standard solution was obtained from Prof. W. Bothe, of the Physikalisch-Technischen Reichsanstalt, Charlottenburg.

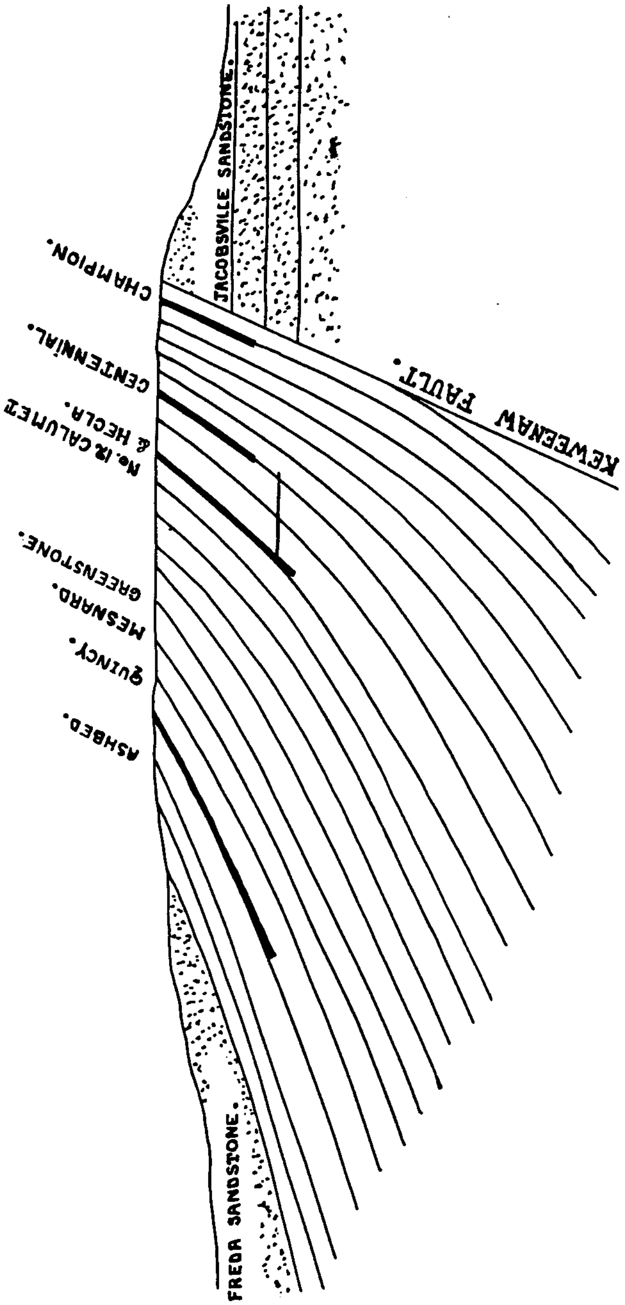
JACOBSVILLE SANDSTONE.
CHAMPION.
CENTENNIAL.
No. 12 CALUMET & HECLA.
GREENSTONE.
MESNARD.
QUINCY.
ASHBED.
FREDA SANDSTONE.
KEWEENAW FAULT.

FIG. 1.—Cross-section sketch of the Keweenaw Point, Michigan.

although they naturally do not all fall on such a cross section but are spaced over many miles along the "Strike."[4] The beds on Keweenaw Point now dip with varying angles to the horizon ranging from 80° in the lower beds to 10° at some points in the later flows. Except where vertical, the mine shafts follow the angle of dip. This is explained by reference to Figure 2, an enlarged cross section of the flows. As a lava is laid down, gas bubbles rise to the top and some are entrapped in the solidifying lava, forming a permeable top to the flow. It is here that the mineralizing solutions have deposited along with several other types of minerals, the native copper. Hence the mine shafts follow the so-called "Amygdaloid lode" downwards and the levels or drifts follow the amygdaloid outwards in the horizontal direction. If the mine shaft is sunk in the amygdaloid in flow A, then flow B is termed the hanging trap to this lode and the portion of flow A underlying its amygdaloid, the foot trap. It will at once be seen that there is no geological stratigraphical difference between the lode and the foot trap, both being the same flow, although the term foot trap continues to be used in the district. The hanging trap, however, constitutes the flow directly above that in which the shaft or level occurs and must, therefore, be treated as a separate entity. As will be noted, most of the samples investigated are from hanging traps, that is, from the bottom of flows immediately overlying the flow being worked. In some cases a cross cut is run to reach another lode and this is also shown in Figure 2 to demonstrate roughly the positions of specimens Nos. 19 and 20. A conglomerate occurs where there was time for deposition between successive flows which need not be long. Forty-three samples in all were taken, twenty-three of which have been analyzed. The results are tabulated in the following order:

1. Basaltic and Felsitic flows.
2. Accessory minerals.

The flows are further subdivided into:

A. Early Keweenawan flows as represented by the Champion mine specimens.
B. Middle Keweenawan flows as represented by the Calumet and Hecla mine specimens.

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[4] Lane, A. C., The Geological Survey of Michigan, Ann. Report (1909).

A very full account of the geology of the district is given in "The Copper Deposits of Michigan," B. S. Butler and W. S. Burbank. Professional Paper 144, U. S. Geological Survey, Washington (1929).

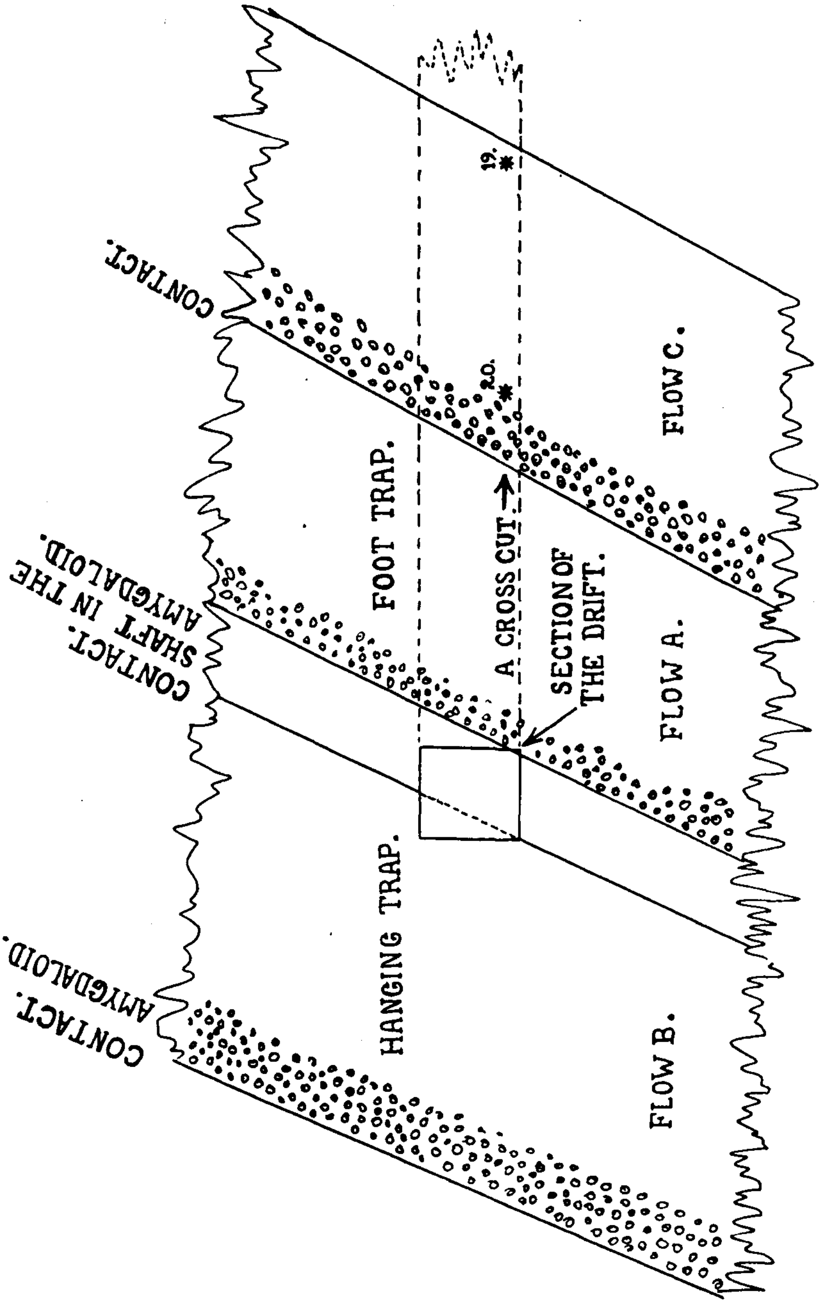
CONTACT.
FOOT TRAP.
19.
20.
FLOW C.
CONTACT. SHAFT IN THE AMYGDALOID.
A CROSS CUT →
SECTION OF THE DRIFT.
FLOW A.
CONTACT. AMYGDALOID.
HANGING TRAP.
FLOW B.

FIG. 2.—Enlarged cross-section of the flows.

C. Some flows intermediate between the middle and late flows.

D. Late Keweenawan flows as represented by the Quincy mine specimens and the Ashbed.

TABLE I.

1. A. THE CHAMPION MINE.

Specimen No. 1.—Shaft No. 4. Level No. 3. 310 feet vertically below the surface. A fine grained basalt five feet above the contact of the hanging trap.

0.33×10^{-13} gm. radium per gm. rock.

Specimen No. 3.—Shaft No. 1. Level No. 21. 2132 feet vertically below the surface. A fine grained basalt from the cross cut between the shaft and the drift, five feet above the contact of the hanging trap.

0.90×10^{-13} gm. radium per gm. rock.

Specimen No. 5.—Shaft No. 1. Level No. 38. 3728 feet vertically below the surface. A fine grained basalt from the cross cut between the shaft and the drift, five feet above the contact of the hanging trap.

2.62×10^{-13} gm. radium per gm. rock.

All the above specimens occur in the same flow, that immediately above the amygdaloidal lode being worked for copper in the Champion mine. The increase of activity with depth is worthy of note. Such a correlation also exists in the Calumet and Hecla district as shown below.

1. B. THE CALUMET AND HECLA MINES.

Specimen No. 9.—Centennial shaft. Level No. 1. 114 feet vertically below the surface. A medium-fine grained basalt three feet above the contact of the hanging trap to the Kearsarge lode.

0.71×10^{-13} gm. radium per gm. rock.

Specimen No. 12.—Shaft No. 12. Level No. 78. 4702 feet vertically below the surface. A medium-fine grained basalt three feet above the contact of the hanging trap to the Kearsarge lode.

4.16×10^{-13} gm. radium per gm. rock.

Specimen No. 13.—Shaft No. 12. Level No. 78. Depth as in previous specimen. The Kearsarge amygdaloid lode. Specimen sorted so that only original lava was obtained and the amygdaloidal material rejected.

2.45×10^{-13} gm. radium per gm. rock.

Specimens Nos. 19 and 20.—Shaft No. 12. Level No. 78. Depth as in previous specimen. These two specimens were taken from the bottom and near the top, respectively, of a thick ophite flow, the top of which is the Osceola amygdaloidal lode. In the cross cut from the Calumet and Hecla conglomerate to the Kearsarge lode. The "Draft mark" gives the distance along the horizontal cross cut, the angle of dip of this flow being 35°.

Specimen No. 19.—Draft mark 505 feet, or three feet above the bottom of the flow.

0.57×10^{-13} gm. radium per gm. rock.

Specimen No. 20.—Draft mark 275 feet or fifteen feet below the bottom of the amygdaloid and seventy-five feet along the cross cut from the top of the flow.

0.74×10^{-13} gm. radium per gm. rock.

Two points of interest are to be mentioned here. Comparing the results of specimen No. 12 and of specimen No. 19, both at approximately the same depth below the surface and from the bottom of basaltic flows, it is evident that the radioactive content may vary considerably from flow to flow. Moreover, comparing specimens Nos. 19 and 20, it is seen that the base of the flow is somewhat less active than near the top. This agrees with the idea that the more felspathic rocks are more radioactive since it has been pointed out that the upper parts of these lava flows tend to be more felspathic.[5]

That increase of basicity indicates a lower radium content is now generally accepted.

Specimen No. 17.—Shaft No. 12. Level No. 78. Depth as in previous specimen. A felsitic pebble from the Calumet and Hecla conglomerate taken about five feet from the bottom of the conglomerate.

7.00×10^{-13} gm. radium per gm. rock.

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[5] Powers, S., and Lane, A. C., Magmatic differentiation in effusive rocks. Bull. Amer. Inst. Mining Eng. (1916).

Lane, A. C., The Keweenaw series of Michigan. Mich. Geol. Survey, Ann. Report (1909), pp. 79–83, 658, 659, 739.

This specimen being felsitic follows the generalization mentioned above and is high in radium content compared to the basic flows.

1. C. SOME FLOWS INTERMEDIATE BETWEEN THE MIDDLE AND LATE FLOWS.

These specimens are from the surface with the exception of specimen No. 25, which was from the Calumet and Hecla collection.

Specimen No. 31.—The Greenstone. A specimen from the quarry at the Phoenix mine, medium-fine grain in structure and about four feet from the bottom of the flow.

0.88×10^{-13} gm. radium per gm. rock.

Specimen No. 29.—The Greenstone. A specimen from above the North Cliff mine. A little coarser grain than specimen No. 31, and about 100 feet up in the flow.

1.22×10^{-13} gm. radium per gm. rock.

Again a decrease is to be noted in the radium content towards the bottom of the flow although the relative positions of the specimens with regard to the whole flow is less certain here than in the case of specimens Nos. 19 and 20.

Specimen No. 25.—The "Mesnard Epidote" which Lane[6] describes as a somewhat altered volcanic ash at the horizon of the Chippewa felsite. Tamarack mine. In the cross cut 500 feet vertically below the surface.

4.71×10^{-13} gm. radium per gm. rock.

1. D. THE QUINCY MINE AND THE ASHBED.

The Quincy mine specimens were not taken personally. With these specimens it is impossible to correlate the radium content so closely since the Pewabic amygdaloid lodes at the Quincy mines consist of a group of thin flows and the two specimens examined although forming the hanging trap to the lode worked in the respective levels may not be from the same flow.[7]

Specimen No. 21.—Shaft No. 2. Level No. 7. 370 feet vertically below the surface. A medium grained basalt about two feet above the contact of the hanging trap.

1.18×10^{-13} gm. radium per gm. rock.

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[6] Lane, A. C., *loc. cit.*, pp. 409, 436.

[7] See "Copper Deposits of Michigan", pp. 178–181.

Specimen No. 23.—Shaft No. 2. Level No. 91. 6182 feet vertically below the surface. A medium grained basalt about two feet above the contact of the hanging trap.

$$0.70 \times 10^{-13} \text{ gm. radium per gm. rock.}$$

The Owl creek riverbed specimen constitutes the geologically most recent flow examined and was a surface specimen.

Specimen No. 28.—A fine grained basalt one foot above the contact of the hanging trap to the ashbed amygdaloid.

$$1.54 \times 10^{-13} \text{ gm. radium per gm. rock.}$$

The foregoing results, not including the felsitic rocks, give an average of 1.29×10^{-13} gm. radium per gm. of rock for the mean value of the plateau basalts as exemplified in the Keweenaw district. This result may be compared with the mean value of a very complete series by C. S. Piggot[8] for the volcanic Hawaiian lavas which showed on the average a radium content of 9.6×10^{-13} gm. per gm. of sample. Although the plateau type might be expected to show lower values, this difference seems rather large. Piggot has pointed out, however, that these lavas may exhibit abnormality since they do not differ appreciably from the mean values of a long list of granites examined by him. On the other hand, the plateau basalts of the Keweenaw Point may be abnormally low and it is proposed in the near future to examine some similar plateau basaltic structure exercising the same precautions as were taken in sampling and determining the Keweenawan rocks. Against an abnormality of the Keweenaw basalts, however, can be quoted the results of V. S. Dubey's[9] determinations made in Professor Mache's laboratory in Vienna on three basic lavas of the Gwalior series in India.

TABLE II.

Belaki-Bauri	1.7×10^{-13} gm. radium per gm. rock.
Santowa Temak	1.8×10^{-13} gm. radium per gm. rock.
Panier	2.3×10^{-13} gm. radium per gm. rock.
Mean	1.9×10^{-13}

On collecting together the data for the surface specimens, or those nearest to the surface in mines where no surface specimens were taken, an apparently definite correlation was found between the radium content and the time of laying down of the flow. Table III illustrates this correlation, the Champion mine hanging trap being the oldest determined and the following values falling in correct stratigraphical order, as reference to Figure 1 will show.

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[8] C. S. Piggot, Amer. Journal Science, 22, July (1931).

[9] V. S. Dubey, Nature, 126, 807 (1930).

TABLE III.

Specimen No. 1	Champion mine.	Level No. 3.	0.33×10^{-13}
Specimen No. 9	Centennial Shaft.	Level No. 1.	0.71
Specimen Nos. 31 and 29	Greenstone.	Surface.	0.88 – 1.22
Specimen No. 21	Quincy mine.	Level No. 7.	1.18
Specimen No. 28	Ashbed.	Surface.	1.54

Whether such a correlation happens to be a coincidence or not will be determined when more of the intervening flows have been examined. It must be mentioned that Joly[10] has found a similar relation between the old and late eruptions of Vesuvius, although these are of the volcanic type as compared to the plateau basalts.

TABLE IV.

2. THE ACCESSORY MINERALS.

The minerals examined are all closely connected with the copper and all contain copper either in the finely divided state or in chemical combination.

Specimen No. 37.—Copper Falls mine dump. Datolite. $H_2Ca_2B_2$-Si_2O_{10} from the Owl creek fissure. Containing microscopically disseminated native copper.

1.10×10^{-13} gm. radium per gm. mineral.

Specimen No. 40.—Ahmeek mine dump. Prehnite. $Al_2(SiO_4)_3$-. Ca_2H_2. Also containing disseminated copper.

0.18×10^{-13} gm. radium per gm. mineral.

Specimen No. 38.—Ahmeek mine. Algodonite. An arsenide of approximate constitution Cu_6As as shown by rough analysis, from the Mohawk fissure.

0.08×10^{-13} gm. radium per gm. mineral.

Specimen No. 38a.—Ahmeek mine. Domeykite. An arsenide of approximate constitution Cu_3As from the Mohawk fissure.

0.13×10^{-13} gm. radium per gm. mineral.

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[10] J. Joly, Phil. Mag., 18, 577 (1909).

Specimen No. 41.—Baltic mine dump. Shaft No. 2. A copper sulphide. Chalcocite Cu_2S from a fissure in the Baltic lode.

0.10×10^{-13} gm. radium per gm. mineral.

The determination of radium in the last four specimens above is accurate to only about 10% on account of the very small activity. All other results are accurate to about 3%.

The arsenides of the district have been studied in detail[11] and it seems that there is a definite order in the deposition of these minerals: Domeykite Cu_3As, Algodonite Cu_6As, Whitneyite Cu_9As and native copper. Specimens Nos. 38 and 38a are of interest in that the radium content is almost directly proportional to the amount of arsenic in the mineral. This would show that in this type of progressive mineralization and probably in all mineralization in the district, the radium, or more correctly the parent element uranium, and its disintegration products,[12] tends to separate from the mineralizing solution in the company of other minerals than the native copper, which is in accord with the chemical viewpoint in that it is not to be expected that the salts of uranium and its disintegration products will pass through the same chemical processes that precipitate the copper in the form of metal. An extrapolation of the results for the arsenides to zero arsenic content, although very rough, shows that the copper should have a much smaller activity than can possibly be measured at present. Such has been found to be the case, in fact the native copper of the Keweenaw Point was the first specimen from this district to be examined. Paneth and Koeck[13] found less than the limit of detection corresponding to $< 10^{-15}$ gm. per gm. of copper. This result has been checked in this research using 32.0 gms. of copper from the 91st level of the Quincy No. 2 Shaft.[14] The results of the determinations for two further specimens are given, the Duluth gabbro and a late granite. The Duluth gabbro, a hand specimen from out of the district, is of interest since it is considered by some to be the source of the copper in the mineralizing solutions and as underlying the Keweenawan rocks. The late granite shows the higher radium content associated with felspathic rocks as compared to the basic basalts. These two are included in the summarizing Table V below.

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[11] T. M. Broderick, Economic Geology, 24, 2 (1929). See also "The Copper Deposits of Michigan," pp. 56–58.

[12] All the minerals and rocks of this district are definitely old enough to have allowed sufficient time for the establishment of radioactive equilibrium.

[13] Paneth and Koeck, loc. cit.

[14] Only two substances in the course of a prolonged series of studies on a variety of rocks and minerals, etc., have so far been found to contain radium in a smaller amount than can be detected by modern apparatus. In addition to the Keweenaw copper an iron meteorite, Savik, Greenland, contained $<2 \times 10^{-15}$ gm. per gm.

TABLE V.

Specimen No.	Description.	Type.		Radium Content $\times 10^{13}$.	Calculated Uranium Content $\times 10^{7}$.
1	Champion mine.	Basalt.		0.33	0.97
3	Champion mine.	Basalt.		0.90	2.65
5	Champion mine.	Basalt.		2.62	7.71
9	Calumet and Hecla mines.	Basalt.		0.71	2.09
12	Calumet and Hecla mines.	Basalt.		4.16	12.23
13	Calumet and Hecla mines.	Basalt.		2.45	7.21
19	Calumet and Hecla mines.	Basalt.		0.57	1.68
20	Calumet and Hecla mines.	Basalt.		0.74	2.18
31	The Greenstone.	Basalt.		0.88	2.59
29	The Greenstone.	Basalt.		1.22	3.59
21	The Quincy mine.	Basalt.		1.18	3.47
23	The Quincy mine.	Basalt.	0.67		
			0.65		
			0.77	0.70	2.06
28	The Ashbed.	Basalt.		1.54	4.53
	Mean value.	Basalts.		1.29	3.79
17	Calumet and Hecla mines.	Felsitic.		7.00	20.59
25	Mesnard epidote.	Felsitic.		4.71	13.85
42	A late granite.	Granitic.		4.21	12.38
43	The Duluth Gabbro.	Gabbro.		0.69	2.03
37	Datolite.	Silicate.		1.10	3.24
40	Prehnite.	Silicate.		0.18	0.53
38	Algodonite. Arsenide.	Copper mineral.		0.08	0.24
38a	Domeykite. Arsenide.	Copper mineral.		0.13	0.38
41	Chalcocite. Sulphide.	Copper mineral.		0.10	0.29
	Mean value.	Copper minerals.		0.10	0.29

The corresponding content in uranium has been inserted in column 5, using the equilibrium value of 1 gm. U = 3.4×10^{-7} gm. Ra.

The author is especially indebted, to Dr. A. C. Lane, of Tufts College, Massachusetts, at whose instigation the work was commenced, to Professor Dr. F. Paneth, Koenigsberg, Germany, for placing the apparatus at his disposal and his help and guidance during the course of this research, and to the American Academy of Arts and Sciences for a liberal grant in support of the research.

SUMMARY.

1. The distribution of radium in the Keweenawan rocks and minerals has been determined by analyses of a series of specimens from known locations and positions in the copper mines and basaltic flows of Keweenaw Point, Michigan.

2. Evidence of a correlation between the radium content and both depth and stratigraphical arrangement has been found.

3. An average value for the radium content of "Plateau Basalt" as exemplified by the Keweenawan flows is given.

4. There is evidence of a regular order in the radium content of the accessory minerals in the copper lodes according to the method of precipitation from the mineralizing solutions. More data are required on this point.

BALTIMORE, MARYLAND.
KOENIGSBERG, I. PR. GERMANY.

RADIOACTIVITY MEASUREMENTS.

II. THE OCCURRENCE OF RADIUM, URANIUM AND POTASSIUM IN THE EARTH.

BY WM. D. URRY.[1]

In this paper it is intended to give the most probable distribution of uranium and radium from the centre of the earth's core to the surface. Since it is impossible to make measurements beyond the very outermost of shells any computation of this distribution must be by comparison with material at hand. Such a procedure has been carried out by H. S. Washington[2] for the distribution of the elements throughout the earth. It was thought of interest to follow the postulate of this author more especially since similar general distributions are now widely accepted and the average radium content of the proposed constituent minerals and core has been determined of late by accurate measurements. Although the details vary somewhat between the various authors, it is perhaps best to follow the composition as given by Washington. For the purpose of completeness, Washington's Table I is reproduced here in part.

TABLE I.

THICKNESS AND MASS OF THE EARTH SHELLS.

	Radial Thickness.	Mass. Metric Tons.
Central core	3400 Kms.	1.646×10^{21}
Lithosporic shell	700	0.513 "
Ferrosporic shell	700	1.360 "
Peridotitic shell	1540	2.417 "
Basaltic shell	40	0.065 "
Granitic shell	20	0.029 "
Whole earth	6400	6.030×10^{21}

The composition of the central core was assumed to be that of the average of 318 iron meteorites. The average radium content of 22 iron meteorites, all included in the above 318, as determined by

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[1] Contribution from the Research Laboratory of Physical Chemistry, Massachusetts Institute of Technology. No. 302.

[2] H. S. Washington, Amer. Journal Science, IX, May (1925).

Paneth and Koeck,[3] was found to be about 2.5 × 10^{-14} gm. radium per gm. of meteorite.[4]

Washington assumed that the material of the lithosporic shell is composed of equal weights of nickel-iron and olivine in analogy to the stony-iron or pallasitic meteorites. The ferrosporic shell he took to be comparable to the stony chondritic meteorites. The best radium determination for these meteorites is again probably that by Paneth and Koeck[5] on a composite sample of equal parts by weight of twenty chondritic meteorites supplied by Professor Dr. G. v. Hevesy.[6] The radium determination gave 12 × 10^{-14} gm. per gm. The average content of the pallasites can now be calculated from the chondritic average and the iron meteorite average to be about 7 × 10^{-14} gm. per gm.

For the composition of the peridotitic shell the average of twenty analyses of achondritic stony meteorites was calculated. There is no radium determination of a composite of these stony meteorites nor sufficient single analyses to give a mean value. For the lower basaltic shell the composition of Daly's[7] average basalt was used. Table II gives the mean values of radium determinations by several workers. The final value is derived by weighting the determinations in preference to averaging the means as given by each author. A mean value given by Joly is excluded since he has included with the basalts: gabbros, diabases and norites.

The granitic shell was calculated from the "Average Igneous Rock" analyses. Paneth and Koeck[8] have also determined the radium content of a composite sample of 282 rocks prepared by Hevesy.[9] The sample includes equal parts by weight of 133 granites, 9 quartz diorites, 35 diorites, 82 gabbros and norites and 23 ultrabasic rocks. The presence of the ultrabasic rocks will lower the mean value somewhat but on the other hand a certain percentage of this

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[3] F. Paneth and W. Koeck, Zeit. Physik. Chemie. Bodenstein-Festband 145 (1931).

[4] Bulletin No. 80, National Research Council, Washington, D. C., p. 416. A. Holmes gives 3.6 × 10^{-14} gm. per gm. including Toluca. Since the latest measurements on this meteorite are still in doubt, it has not been included in the average given above.

[5] Paneth and Koeck, *loc. cit.*

[6] G. v. Hevesy, E. Alexander and K. Wuerstlin, Z. Anorg. Ch. 194, 316 (1930).

[7] Daly, Igneous Rocks (1914), 27, No. 53.

[8] Paneth and Koeck, *loc. cit.*

[9] Hevesy, Alexander and Wuerstlin, *loc. cit.*

TABLE II.

Location.	Observer.	No. of specimens.	Ra. in gm. per gm. of rock.
Basalt. Ireland.	Strutt.[1]	1	55×10^{-14}
Basic rocks. Various.	Strutt.[2]	11	53 “
Basic rocks. (Archean.) Mysore State. India.	Smeeth and Watson.[3]	21	24 “
Basalts. England.	Strutt.[4]	4	37 “
Basalt. Deccan trap. India.	Dubey.[5]	1	45 “
Gwalior basalts. India.	Dubey.[6]	3	19 “
Keweenaw basalts. Michigan, U. S. A.	Urry.[7]	15	13 “
	Mean value.	56	28×10^{-14}

[1] Strutt, R. J., Proc. Roy. Soc. London, A. 80, 586 (1908).

[2] Strutt, R. J., Proc. Roy. Soc. London, A. 77, 472 (1906).

[3] Smeeth and Watson, Phil. Mag., 6, 35, 206 (1918).

[4] Strutt, R. J., Proc. Roy. Soc. London, A. 84, 377 (1910).

[5] Dubey, V. S., Bull. No. 80, National Research Council, Washington, D. C.

[6] Dubey, V. S., Nature, 126, 807 (1930).

[7] Urry, W. D. See the foregoing publication.

type will be present in the granitic layer. The determination gave 128×10^{-14} gm. per gm. rock. An average value for the radium content of igneous rocks from different parts of the world is given by Strutt as 170×10^{-14} gm. per gm.[10]

Piggot[11] has determined the radium content of a series of granites from Georgia to Greenland, including some gabbros and schists. The average of sixteen locations with about 100 determinations is given as 90×10^{-14} gm. per gm. With due regard to the above mentioned average values and on selection of the most probable results from a very complete table by G. S. Rogers,[12] the average content of the igneous rocks is taken as about 140×10^{-14}.

In figure 1 the radium content of the various shells of the earth as postulated by Washington is plotted against depth, the average content for any given shell being plotted against the depth to the middle point of that shell. On the same figure is plotted the potassium content as given by Washington also expressed in gm. per gm. Table III gives the complete data including the total radium, uranium and potassium content of each shell and the helium produced annually.

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[10] Strutt, R. J., Proc. Roy. Soc. London, A. 77, 472 (1906) and A. 78, 150 (1906). The above value is that corrected by Eve and McIntosh, Phil. Mag. 6, 14, 231 (1907) for the use of an erroneous constant.

[11] Piggot, C. S., Amer. Journal Science, 21, Jan. (1931).

[12] G. S. Rogers, U. S. Geological Survey, Professional Paper 121, "Helium bearing natural gas."

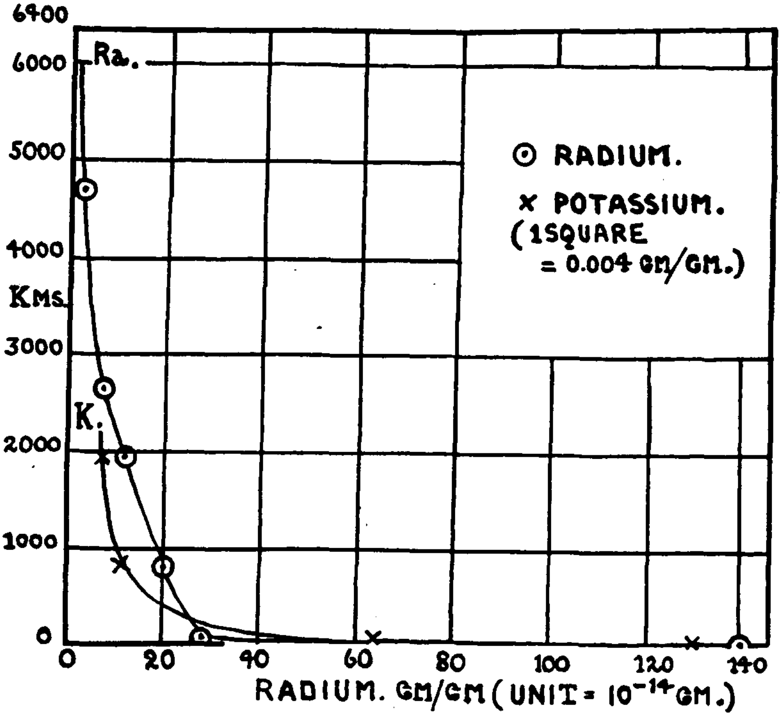

FIG. 1.—Variation of Radium and Potassium Content with depth.

By far the greatest concentration of the radium and potassium on reference to columns 3 and 8 is found to be in the uppermost thirty kilometers of the earth's surface. Unfortunately reliable thorium results sufficient in number to calculate mean values are not at hand.

That the radioactivity must fall off very appreciably with depth has been inferred in all calculations based on the hypothesis of a cooling earth. Since older computations of the average radioactivity of the outer shells have led to higher values than given here

TABLE III.

	Radial thickness.	Depth to middle of shell.	Radium.	Uranium.	Total Radium in shell.	Total Uranium in shell.	Helium produced annually.	Potassium.	Total Potassium in shell.
	Kms.	Kms.	Gm./gm.	Gm./gm.	Metric tons.		Cu. meters.	Gm./gm.	Met. tons.
Central Core.....	3400	4700	2.5×10^{-14}	0.74×10^{-7}	411×10^{5}	1233×10^{11}	14.30×10^{6}	....	....
Lithosporic Shell..	700	2650	7.0	2.06	359	1077	12.41	....	....
Ferrosporic Shell..	700	1950	12.0	3.53	1632	4896	56.78	0.0014	1903×10^{15}
Peridotitic Shell ..	1540	830	(20)	5.88	4834	14502	168.19	0.0022	5317
Basaltic Shell.....	40	40	28	8.24	182	546	6.33	0.0128	832
Granitic Shell	20	10	140	41.18	406	1218	14.13	0.0260	754
Whole Earth.....	6400				7824	23472	272.14		8806

it has often been necessary to postulate the existence of radioactive elements for only a few kilometers depth. Holmes[13] who has criticized such a complete removal of radioactive elements from the lower shells gives a value of 60 calories/sq. cm. as the annual heat loss by conduction to the surface and radiation into space. By taking most probable values for a thorium content of the three outer shells, values which however are not at all final, it is possible, in combination with the above values for radium and potassium, to calculate roughly the depth required to offset the above loss. Table IV sets out the various heat quantities required to be known.

TABLE IV.

Granitic shell.	20 kms. thick.	Uranium heating effect.	18.2 Cals/sq.cm.
		Potassium heating effect.	1.7 Cals/sq.cm.
Basaltic shell.	40 kms. thick.	Uranium heating effect.	7.3 Cals/sq.cm.
		Potassium heating effect.	1.7 Cals/sq.cm.
Total radium and potassium heating effect—both shells			28.9 Cals/sq.cm.
Taking a thorium content for the granitic shell of 10^{-5} gm. per gm. rock.		Thorium heating effect	12.9 Cals/sq.cm.
Taking a thorium content for the basaltic shell of 2×10^{-6} gm. per gm. rock.		Thorium heating effect	5.2 Cals/sq.cm.
		Total thorium heating effect of both shells	18.1 Cals/sq.cm.

Thus the sum total heating effect of the radioactivity in both shells of 47 cals/sq. cm. is 13 cals/sq. cm. less than the loss by radiation from the earth's surface. Assuming a thorium content of 1×10^{-6} gm. per gm. for the peridotitic shell, the total uranium, thorium and potassium heating effect per sq. cm. per cm. depth for this shell is 2×10^{-6} cals/sq. cm. and hence the remaining 13 cals would be supplied by a depth of 65 kilometers of the peridotitic shell. Even taking somewhat larger values for the thorium content the heat required to counterbalance the loss of 60 cals/sq. cm. annually would correspond to a depth reaching into the upper part of the peridotitic shell. This is in good accord with the postulates necessary in supporting Holmes's theory of convection currents in the substratum.[14]

Columns 4 and 6 give the uranium content of the earth calculated on the basis that in radioactive equilibrium the ratio of uranium to radium is 1 to 3.4 × 10^{-7}. The annual production of helium is calculated on the basis that 1 gm. uranium in equilibrium with all its disintegration products yields 1.16 × 10^{-7} cc. annually. A yearly loss of about 0.4% of the helium generated annually in the granitic shell would suffice in 1000 × 10^6 years to give the total helium in the atmosphere as calculated by Van Orstrand[15] to be about 58 × 10^{12} cubic meters. The average surface denudation since the close of the pre-cambrian times, even allowing that only a fraction of the helium is lost on denudation, is more than sufficient to produce the total atmospheric helium. One important determination, however, bearing on this question, namely, the distribution of helium between the atmosphere and the oceanic waters, will be made in the near future. It is thought that such a determination will account at least in part for the apparently large excess of helium freed in geological time over the amount existing in the atmosphere at present.

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[13] Holmes, A., Radioactivity and Earth Movements, Trans. Geol. Soc. Glasgow, XVIII, Part III (1928–29).

[14] Holmes, A., *loc. cit.*

SUMMARY.

1. A probable distribution of the radium and uranium throughout the earth is given basing the computations on Washington's proposed mineralogical and chemical structure and using the latest determinations for composite samples and analytical series.

2. The results show a considerably lower mean uranium content than has usually been adopted for the three outer shells and excludes the possibility of a complete concentration into the upper two shells, a conclusion that is in good agreement with Holmes' theory of convection currents in the substratum.

GENERAL REFERENCES.

[16] Holmes, A., Radioactivity and the earth's thermal history.
  Part I. The concentration of the radioactive elements in the earth's crust. Geol. Mag., 6, 2, 60 (1915).
  Part II. Radioactivity and the earth as a cooling body. Geol. Mag., 6, 2, 102 (1915).
  Part III. Radioactivity and Isostasy. Geol. Mag., 6, 3, 265 (1916).
  Part IV. A criticism of Parts I, II and III. Geol. Mag., 62, 504 (1925).
  Part V. The control of geological history by radioactivity. Geol. Mag. 62, 529 (1925).
[17] Jeffreys, H., The Earth's Thermal History, Geol. Mag., 64, 444 (1927).
  The Earth. 2d edition. Cambridge University Press (1929).

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[15] See "Helium bearing natural gas," same as footnote 12.

[18] Clarke, F. W., The evolution and disintegration of matter. U. S. Geological Survey, Professional Paper 132 D (1924).

[19] Humphreys, The Physics of the Air. Philadelphia (1920).

[20] Daly, R. A., Our mobile Earth. New York (1926).

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CONTRIBUTIONS FROM THE GRAY HERBARIUM OF HARVARD UNIVERSITY.

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CONTRIBUTIONS FROM THE GRAY HERBARIUM OF HARVARD UNIVERSITY,—No. CII.

1. STUDIES IN THE BROMELIACEAE.—IV.

By Lyman B. Smith.

Presented January 11, 1933. Received February 6, 1933.

The following paper places on scientific record the novelties to be included in the publication of the *Bromeliaceae* of Trinidad and also such new or noteworthy species as have come to the writer's attention in a general study of the family during the past year. The form used is that of previous papers in this series.

In this paper and the succeeding one, the authors are indebted to the following institutions for the opportunity to study their material or photographs thereof: the Herbarium of the Royal Botanic Gardens of Trinidad and Tobago (Trin), the Gray Herbarium of Harvard University (G), the New York Botanical Garden (NY), the United States National Museum (US), the Field Museum of Natural History (FM), the Royal Botanic Gardens at Kew (K), the Missouri Botanical Garden (Mo), the Riks Museum at Stockholm (S), the Biological Institute of São Paulo (SP), the Herbarium of the Botanical Museum at Berlin-Dahlem (B), the British Museum of Natural History (BM), the Conservatory of Botany at Geneva (Gen), the National Museum of Rio de Janeiro, the National Museum of Natural History of Buenos Aires (BA), the private herbarium of Cornelio Osten, Montevideo, Uruguay, the Dudley Herbarium of Stanford University, and the Brooklyn Botanic Garden (Bk).

Aechmea dichlamydea Bak. var. **trinitensis**, var. nov., spicis omnibus brevissime crasseque stipitatis. Pl. I, fig. 3.

TRINIDAD: Epiphytic, Broad Walk, Trinidad Botanic Garden, Port of Spain, 1890, *J. H. Hart in Herb. Trin. 4089* (Trin, TYPE, phot. G; NY); 1889, *Broadway 3627* (Trin); indefinite, 1867, *Prestoe* (K); *Herb. Trin. 916* (Trin).

Canistrum (Wittrockia) minutum (Mez), comb. nov. *Nidularium minutum* Mez in Fedde Rep. Spec. Nov. xvi. 4 (1919). Pl. I, figs. 4–5.

While at Alto da Serra, the type locality of *Nidularium minutum*, I collected what I took from its habit to be a close relative of *Nidularium Scheremetiewii*. However, a dissection of the fresh flowers revealed that each petal bore two finely divided scales a short distance above the base, thus making it a *Canistrum*. Quite evidently it had never been described under *Canistrum*, but its habit suggested that it might have been under *Nidularium* and its close correspondence to the description of *N. minutum* confirmed this surmise.

Canistrum minutum is most nearly related to *C. amazonicum*, from which it differs in its nearly symmetric evenly pale green sepals and narrower leaves.

BRAZIL: São Paulo: Estação Biologica, Alto da Serra, alt. 800–900 m., 1924, *D. Lemos in hb. S. P. 9480* (G, SP); 1929, *L. B. Smith, F. C. Hoehne & M. Kuhlmann 1829* (G); *J. King in hb. S. P. 24579* (G, SP).

Deuterocohnia Meziana O. Kuntze ex Mez emend. L. B. Smith, bracteis florigeris acutis vel obtusis: floribus distincte pedicellatis: sepalis suboblongis, late obtusis vel subtruncatis, asymmetricis. Mez in DC. Mon. Phan. ix. 465 (1896). *D. paraguariensis* Hassler in Ann. Conserv. & Jard. Bot. Genève, xx. 298 (1919). *D. divaricata* Mez in Fedde Rep. Spec. Nov. xvi. 9 (1919). Pl. I, figs. 1–2.

BRAZIL: Matto Grosso: Corumbá, 1892, *O. Kuntze* (NY, TYPE; phot. G); same, rupicolous on calcareous rocks, 1911, *Hoehne 3543* (Mus. Nac. Rio). PARAGUAY: calcareous banks, Sierra de Amambay, Cerro Margarita, northern Paraguay, 1908, *Rojas 11098* (Gen, TYPE of *D. paraguariensis*, phot. G; B (?), TYPE of *D. divaricata*); Colonia Risso, 1896, *Anisits 2419* (S); Itapucúmi, 1893, *Lindman A 2097* (S); Puerto Fonciere, 1916, *Rojas 2270* (Hb. Osten).

Owing to the fact that the type of *Deuterocohnia Meziana* consisted of young material, the sepals were erroneously described as acuminate, but a careful dissection shows them to be nothing of the sort. All the differences which are to be inferred from the descriptions of *D. Meziana* and *D. paraguariensis* are nullified either by mistakes in the description of the former of by a consideration of the difference of condition. The type of *D. Meziana* is young material poorly preserved, while that of *D. paraguariensis* is a mature specimen excellently preserved.

Guzmania plicatifolia, spec. nov., epiphyta vel terrestris, acaulis, florifera ad 9 dm. alta: foliis 8 dm. vel ultra longis, minutissime obscureque lepidotis; lamina lineari, 2 cm. lata, longe acuminata, valde plicata: scapo 8 mm. diametro, erecto, mox glabro; scapi bracteis foliaceis, erectis, dense imbricatis: inflorescentia laxe bipinnata, e spicis ad 15 composita, subcylindrica, 3 dm. longa; bracteis primariis ovato-lanceolatis, amplis, membranaceis, mox disruptis, infimis quam spicae multo longioribus: spicis breviter crasseque stipitatis, crasse ovoideis, 4–5 cm. longis, dense 12–15-floris; bracteis florigeris imbricatis, late ellipticis vel ovatis, acutis, 25 mm. longis, sepala bene superantibus, membranaceis, prominenti-venosis, subglabris: floribus solum perjuvenilibus vel fructiferis cognitis, sub-

sessilibus; sepalis liberis, anguste lanceolatis, acutis, 16 mm. longis, glabris, sulcatis: capsula tenuiter ellipsoidea, sepala subaequanti. Pl. II, figs. 4–5.

COSTA RICA: San José: forests, La Palma, alt. 1459 m., 1898, *Tonduz 12533* (US, type; phot. G); Cartago: Jocosal, south of Cartago, alt. 1500 m., 1920, *Lankester K 39* (K).

This plant is quickly distinguished from the great majority of species of *Guzmania* by its strongly plicate leaves, but resembles its near relative, *G. calamifolia* André, in this particular as well as in its very thin bracts. It differs from *G. calamifolia* in its elongate inflorescence, broad primary bracts, and short stout spikes.

Guzmania rosea, spec. nov., acaulis, 4–5 dm. alta: foliis rosulatis, ad 6 dm. longis, supra glabris, subtus dense minuteque brunneo-lepidotis, exterioribus ad vaginas atro-brunneas reductis; laminis foliorum interiorum ligulatis, 25–30 mm. latis, late acutis, ex sicco concoloribus: scapo erecto, glabro; scapi bracteis dense imbricatis, late ellipticis, apice acuminatis, infimis omnino supremis solum apice lepidotis: inflorescentia simplicissima, laxa, 25 cm. longa: bracteis florigeris orbicularibus, sepala involventibus, apiculatis, roseis, ad apicem extremum lepidotis, infimis sepala subaequantibus, alteris quam sepala bene brevioribus: floribus patentibus, ad 5 mm. pedicellatis; sepalis glabris, 38 mm. longis, ad 25 mm. connatis, late obtusis, paulo asymmetricis; petalis 50 mm. longis, alte conglutinatis, ex sicco aureis, obtusis; stylo gracili, petala subaequanti, stamina paulo superanti, ovario tenuiter pyramidato. Pl. II, fig. 6.

COLOMBIA: El Valle: on rocks, Poreto on the Rio Dagua, *Lehmann K 362* (K); Cauca: Timbiqui, 1903, *Lehmann K 355* (K, type; phot. G).

The simple inflorescence and large high-connate sepals of this species place it near *G. musaica* (Linden) Mez, from which it is immediately distinguishable by its lax elongate inflorescence and concolorous leaf-blades.

Tillandsia (§ **Allardtia**) **Archeri,** spec. nov., epiphyta, acaulis, 27 cm. alta: foliis rosulatis, 2 dm. longis, utrinque dense adpresseque lepidotis; vaginis magnis, castaneis, late ovatis; laminis anguste triangularibus, acuminatis, basi 15 mm. latis, subtus cinereis: scapo erecto, gracili, quam folia paulo breviori; scapi bracteis foliaceis, internodia multo superantibus: inflorescentia ovoidea, subdense bipinnata, 7 cm. longa, fulgide rubra; bracteis primariis late ovatis, lepidotis, infimis longe caudatis, quam spicae axillares bene longioribus: spicis lanceolatis, subsessilibus, 5–6-floris, 25–30 mm. longis; bracteis florigeris laxe imbricatis, rhachin vix obtegentibus, oblongo-ovatis, 12–15 mm. longis, quam sepala distincte longioribus, carinatis, apice incurvis, mox glabris, membranaceis, valde nervatis: floribus subsessilibus; sepalis oblongis, obtusis, 12 mm. longis, sparse lepidotis vel glabris, nervatis, posticis alte connatis; petalis oblongis, late obtusis, ca. 14 mm. longis; staminibus styloque inclusis. Pl. I, figs. 14–15.

COLOMBIA: ANTIOQUIA: Santa Elena, 1930, *W. A. Archer 1204* (US, TYPE; phot. G).

In the key in Mez's monograph this plant runs down to *T. compacta* from which it differs in its acuminate leaf-blades. In habit it more closely resembles *T. myriantha*, but differs in its highly connate posterior sepals. The spikes are suggestive of those of *T. rubella* and *T. oroyensis.*

Tillandsia flexuosa Sw. Authors have disagreed on the application of this name, but it seems best to use it in the sense of including the later *T. aloifolia* Hook. as a synonym. In the process of identifying and publishing on his West Indian collections, Swartz considered certain of his specimens to be identical with the plant figured by Jacquin in his Selectarum Stirpium Americanarum Historia t. 63, and doubtfully referred to *T. tenuifolia* L. Swartz recognized that Jacquin's plant was misdetermined and made the new name *T. flexuosa* for it. Unfortunately Swartz appears to have misapplied the name in identifying his own specimens, and this error has been used by some authors to give a different interpretation to the species and to reduce it to synonymy under *T. utriculata.* Such action would be justifiable had *T. flexuosa* been published as a new species, but it was obviously published as a new name and should take its significance from *T. tenuifolia* Jacq. which is generally admitted to be the same as *T. aloifolia* Hook.

Tillandsia (§ **Anoplophytum**) **latisepala**, spec. nov., caulescens, omnino argenteo-lepidotis: foliis dense quaquaverse dispositis, dense subpruinoseque lepidotis; vaginis inconspicuis; laminis patentibus, anguste triangularibus, 20–25 cm. longis, ad 17 mm. latis, ad apicem involutis: scapo gracili, erecto, leviter curvato, 15 cm. longo; scapi vaginis imbricatis, scapum omnino obtegentibus, ellipticis, longe caudatis: inflorescentia simplicissima, quaquaverse florigera, subdense ovoidea, 6 cm. longa: bracteis florigeris submembranaceis, latissime ovatis, sepala superantibus, infimis caudatis, 3 cm. longis, flores superantibus: floribus 22 mm. longis, subsessilibus; sepalis liberis, crasse coriaceis, 11 mm. longis, 9 mm. latis, apiculatis, dense pruinoseque albido-lepidotis; petalis ex sicco albis, angustis, obtusis; filamentis prope apicem transverse plicatis; stylo elongato, e corollae fauce emergenti, stamina paulo superanti. Pl. I, figs. 6–7.

PERU: CUZCO: Valle de Santa Ana, Huadquina, alt. 1500 m., 1931, *F. L. Herrera 3313* (US, TYPE; phot. G).

This species is easily distinguished from its near relatives by its very broad thick-coriaceous sepals.

Tillandsia xiphioides Ker-Gawl. The following specimen constitutes the first Bolivian record for the species:

BOLIVIA: POTOSI: prov. Nor-Chichas, on rocks near San Antonio, 1931, *M. Cardenas 93* (G).

Vriesia Alfarovii Mez. The discovery of additional material of this species necessitates a slight emendation of the original description and a large extension of range as follows:

Inflorescentia laxe paniculata; bracteis primariis eis scapi similibus, quam spicae axillares multo brevioribus; spicis lateralibus suberectis, ad 24 cm. longis, breviter stipitatis, basi a prophyllis 2 auctis, spica terminali basi a prophyllis multis scapi bracteis exacte similibus aucta.

COLOMBIA: EL VALLE: epiphytic, coastal thickets, Buenaventura, alt. 0–10 m., 1922, *Killip 11736* (G).

The Killip specimen is a very good match for the type material (*Tonduz 9883*, from Costa Rica) in the Gray Herbarium, but shows a larger proportion of the inflorescence. Had merely the terminal spike of the Killip specimen been collected, it would have appeared to be a simple inflorescence with a section of the scape, on account of its long sterile base. This, evidently, is just what happened in the case of the type material.

Vriesia Broadwayi, spec. nov., epiphyta, semimetralis: foliis 3 dm. longis; vaginis ellipticis, utrinque dense minuteque ferrugineo-lepidotis; laminis anguste lanceolatis, acuminatis, 2 cm. latis, supra glabris, subtus dissite lepidotis: scapo erecto, gracili; scapi bracteis dense imbricatis, dissite lepidotis, inferioribus foliaceis, superioribus late ellipticis, apiculatis, pallido-brunneis: inflorescentia simplici, racemosa, optime secunda, 7–9 cm. longa, laxe 5–7-flora, minutissime ferrugineo-lepidota; rhachi flexuosa; bracteis florigeris late ovatis, late acutis, brunneis, ad 22 mm. longis, quam sepala distincte brevioribus, cum floribus secunde versis: floribus crasse stipitatis; sepalis late ellipticis, rotundatis, 18 mm. longis, subcoriaceis, levissime nervatis, margine pallido excepto atro-brunneis; petalis ignotis: capsula subcylindrica, 35 mm. longa. Pl. I, figs. 9–10.

TRINIDAD: Heights of Aripo, 1922, *W. E. Broadway 9916* (Trin, TYPE; phot. G; K; NY), Mt. Tucuche, 1920, *N. L. Britton, Hazen & Mendelson 1258 in part* (G), 1929, *Broadway 7096* (K).

This species seems to be more nearly related to *V. oligantha* (Bak.) Mez than to any other known species. It differs in its finely ferruginous-lepidote narrower leaves and smaller sepals.

Vriesia Hoehneana, spec. nov., perconspicua, plerumque bimetralis vel ultra: foliis rosulatis, ad 5 dm. longis, dorso minutissime brunneo-lepidotis; vaginis magnis, late ellipticis, atro-castaneis; laminis ad 8 cm. latis, pallide viridibus, late acutis, apiculatis, venulis sinuosis transversalibus dense ornatis: scapo perelongato, basi ad 14 mm. diametro; scapi bracteis infimis foliaceis, quam internodia multo longioribus, supremis late lanceolatis, internodia subaequantibus: inflorescentia laxissime pauciramosa vel simplici; bracteis primariis eis scapi similibus, quam bases steriles spicarum multo brevioribus; spicis ad 28 cm. longis, 8–16-floris, in inflorescentia composita spicis lateralibus divergentibus, graciliter stipitatis, basi unibracteolatis vel nudis, spica terminali basi a bracteis multis scapi bracteis exacte similibus aucta; bracteis florigeris cum floribus secunde versis, ad 32 mm. longis, internodia subduplo aequantibus, suborbicularibus, latissime acutis vel apiculatis, ad apicem versus obtuse carinatis paulo incurvatisque, flavis, sublaevibus, minutissime obscureque brunneo-lepidotis: floribus bene secundis, ad 10 mm. stipitatis; sepalis ad 34 mm. longis, anguste ellipticis, utrinque sed praesertim intus minute brunneo-lepidotis; petalis citrinis, ad 5 cm. longis, basi ligulis binis magnis auctis; staminibus inclusis, filamentis ad apicem valde incrassatis. Pl. I, figs. 11–13.

BRAZIL: SÃO PAULO: in campo, Alto da Serra, Estrada do Vergueiro, 1920, *F. C. Hoehne 4724* (G, SP); Campo Grande, 1924, *F. C. Hoehne 9478* (G, SP); in campo, Estação Biologica, Alto da Serra, alt. 800–900 m., 1929, *L. B. Smith 1945* (G, TYPE).

This species is a native of the open rolling campo, and when seen in the field is unmistakable on account of its extremely elongate scape and few-branched or simple inflorescence. The spikes and flowers closely resemble those of *V. hieroglyphica* and *V. tessellata*, but these both have many-branched inflorescences with the sterile base of the terminal spike no more elongate than in the lateral ones. In the long scape-like base of the terminal spike *V. Hoehneana* resembles *V. Alfarovii*, but differs from that species in its much narrower sepals. The character of having the sepals strongly lepidote inside is a rather frequent one in the section *Xiphion* but has not been particularly noted. The fine irregular cross-veining of the leaves is the same type as found in *V. platynema* and figured by Gaudichaud (Atl. Bonite, t. 66 (1846)).

In Mez's monograph this species runs down to *V. Tweedieana*, but as shown further on in the present paper the name is quite evidently misapplied. Since no other name appears applicable to the specimens which Mez used in making his description of *V. Tweedieana*, and since their description is the only one which does not show distinct differences from the above-cited specimens, it becomes necessary to erect a new species based on the recent material from São Paulo but quite probably including some of the specimens placed under *V. Tweedieana* by Mez.

Hoehne has figured a small specimen of the species under the name of *V. tessellata* on page 97 of the Album da Secção de Botanica do Museu Paulista.

Vriesia Rodigasiana Morr. in Ill. Hort. xxix. 171, t. 467 (1882). *Tillandsia Tweedieana* Bak. in Journ. Bot. xxvi. 138 (1888); Brom. 224 (1889). *Vriesia Tweedieana* Mez in Mart. Fl. Bras. iii. pt. 3, 561 (1894).

Tillandsia Tweedieana was founded on *Tweedie 1342* and *Glaziou 16467*. The Tweedie specimen is conspecific with *Vriesia Rodigasiana*, while the Glaziou specimen later became the type of *Vriesia friburgensis* Mez. In his choice of name Baker evidently considers the Tweedie specimen as the type, and Mez further strengthens this position by removing the only other number of the original citation. Thus it seems best to avoid confusion and let *Vriesia Tweedieana* lapse into synonymy rather than to use it as antedating *V. friburgensis*.

Mez treated *V. Tweedieana* as a valid species under the section *Xiphion*, disregarding the fact that the type shows the stamens longer than the petals and that Baker so described them. The reason for this confusion is not yet solved, but it may lie with the Sello specimens, which Baker added to this species in his Handbook of Bromeliaceae and which Mez retained. Certainly there is much in Mez's description which does not fit the type, and which leads one to believe that he wrote from other specimens which he thought conspecific.

Vriesia tucumanensis Mez. This species should include *V. caldasiana* Mez as a synonym. When the two species were originally described by Mez in De Candolle's Monographiae, they were supposedly distinguished by *V. tucumanensis* having the stamens exserted, and by *V. caldasiana* having them included. In all other respects they appear very similar. Examination of type material of *V. caldasiana*

(*Mosén 4436*) from the Riks Museum at Stockholm shows that the stamens, contrary to the description, are exserted quite definitely at anthesis. Mez evidently saw immature material only.

Any objections to the equality of the two species on the ground of the wide geographic separation of their type localities are answered by the number of intermediate stations which have recently come to light.

BRAZIL: MINAS GERAES: epiphytic, banks of the Rio Verde, Caldas, 1876, *Mosén 4436* (S, ISOTYPE of *Vriesia caldasiana;* phot. G); Pouzo Alegre, 1929, *Hoehne 24550* (G, SP); PARANÁ: Ponta Grossa, 1903, *Dusén 2699* (S); Morretes, 1904, *Dusén 4357 in part* (S); RIO GRANDE DO SUL: Silveira Martins, Val Veneta, 1893, *Lindman A 1331* (S). PARAGUAY: Villa Rica, 1893, *Lindman A 1793* (S); 1919, *Jörgensen 4489* (Dudley Hb.). ARGENTINA: JUJUY: near Laguna de la Brea, 1901, *Fries 382* (S); FORMOSA: indefinite, 1919, *Jörgensen 2803* (G); TUCUMAN: Sierra de Tucuman, 1882, *F. Schultz* (B, TYPE; phot. G); Yerba Buena, Naranjal, Dept. Tafi, 1918, *Schreiter 738, 27/2331 in hb. BA* (G, BA, Hb. C. Osten).

In establishing the name of this species, *V. tucumanensis* has been selected rather than *V. caldasiana,* because it appears earlier in the same work and still more because it was the one to be described correctly as regards the stamens.

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2. THE BROMELIACEAE OF TRINIDAD AND TOBAGO.

By W. E. BROADWAY AND L. B. SMITH.

IN THE following treatment the authors feel that they have employed a very satisfactory means of combining and correlating the field and herbarium studies of tropical plants. One contributes a knowledge of local conditions, geography, climate and history, and the characters of the plants which are lost in herbarium specimens. The other supplies the information gained from the study of the wide range of specimens and literature to be found only in the larger herbaria, thus correlating local and general information. By keeping in close touch they are able to answer questions in a short time that would require years for either one alone.

The senior author's interest in the *Bromeliaceae* dates from his experiences as a student-gardener at Kew when he had charge of the living material to which the late J. G. Baker was constantly referring in the preparation of his "Handbook of the Bromeliaceae." Since then in coming to the West Indies and most particularly to Trinidad he has continued to collect and study the family, discovering a number of rare or new species in the process. The junior author's article on "The *Bromeliaceae* of British Guiana."[1] suggested the present treatment as a natural sequel.

The island of Trinidad ranks very high in tropical America for the thoroughness of its botanical exploration. Few of the West Indies can compare with it in this regard and Gleason in his survey "The progress of botanical exploration in tropical South America"[2] rates it as one of twelve restricted regions which are considered as adequately known. Consequently it is not surprising that the present article required the description of but two novelties, although a number of previously described species have been added to the flora. Tobago is distinctly less known than Trinidad and has only about a third as many recorded species of *Bromeliaceae*. However, the much more uniform topography and smaller area of Tobago make it seem unlikely that further collecting will greatly affect this difference between the two island floras.

The two striking regional features of the Trinidad *Bromeliaceae* are the strong development of the genus *Bromelia* on the Bocas Islands and Patos between the northwest corner of Trinidad and Venezuela, and the localizing of the rare *Glomeropitcairnia erectiflora* and most of the Trinidad endemics, *Vriesia Broadwayi*, *V. stenostachya*, *Gravisia aripensis* and *Aechmea porteoides*, in the rain-forest of the mountainous north coast. The two remaining Trinidad endemics, *Tillandsia subulifera* and *Aechmea dichlamydea* var. *trinitensis* are still too imperfectly known to be said to characterize any particular region, and as yet no other strong local tendencies have been observed in the non-endemic species.

Slightly less than half of the species of Trinidad and Tobago reappear in Venezuela, but this is explained largely by the relative lack of botanical exploration in the latter country. There are a large number of species which seem to have come south from the other Antilles and about an equal amount from Brazil and the Guianas.

The great majority of the species in this treatment have already found favor in cultivation. Several were grown over one hundred years ago, and the introduction of *Bromelia chrysantha* in Europe dates before 1797. So far as known the following species have not yet been tried by the horticulturists: *Glomeropitcairnia erectiflora*, *Tillandsia subulifera*, *triticea*, *micrantha*, *subimbricata*, *rubra*, *didistichoides* and *complanata*, *Vriesia stenostachya*, *macrostachya*, *Broadwayi*, *albi-*

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[1] Contrib. Gray Herb. lxxxix. 46 (1930).
[2] Bull. Torr. Bot. Club, lix. 21 (1932).

flora and *procera*, *Guzmania megastachya*, *Catopsis Berteroniana* and *sessiliflora*, *Hohenbergia oligosphaera* and *Aechmea porteoides*, although their attractive or bizarre appearance, as indicated in their descriptions, strongly recommends them. For the most part they are very rare or local species and have thus escaped attention, but it seems strange that no record can be found of the cultivation of such a widespread and striking species as *Tillandsia rubra*.

The generic sequence of Harms[1] has been adopted for the present work since it shows better than any other system the natural development from simple to complex forms. However, it has seemed best to retain the broader generic limits indicated by Mez in such cases as *Bromelia*, *Vriesia* and *Guzmania*.

The general acknowledgements and the abbreviations of herbaria cited are to be found on page 145. The authors are particularly indebted to Prof. Gunnar Samuelsson of the Riks Museum at Stockholm, to Mr. J. E. Dandy of the British Museum of Natural History and to Mr. N. Y. Sandwith of the Royal Botanic Gardens at Kew for their aid in settling troublesome questions on the identity of types.

BROMELIACEAE J. St.-Hil. Herbs or rarely shrubby perennials, mostly epiphytic. Leaves spirally arranged, usually basal, dilated-sheathing below, simple, entire or spinulose-serrate, bearing peltate scales, at least when young, and these serving to hold moisture. Inflorescence simple or paniculate, of spikes or racemes, usually bearing brightly colored conspicuous bracts. Flowers perfect in all the species of Trinidad and Tobago. Perianth heterochlamydeous, the segments free or variously joined. Stamens 6, filaments free or joined to the petals or to each other. Style 3-parted. Ovary inferior to superior, 3-celled. Fruit capsular or baccate. Seeds naked, winged, or plumose. Embryo small, situated at the base of the copious mealy endosperm. About 50 genera and 1500 species, strictly confined to tropical and subtropical America.—Expos. Famil. i. 122–5, t. 19 (1805); Beer, Fam. Brom. 1–271 (1857); Benth. & Hook. f. Gen. iii. 657–670 (1883); Wittm. in Engler & Prantl, Nat. Pflanzenf. ii. Abt. 4, 32–59 (1887–8); Bak. Handb. Brom. 1–243 (1889); Mez in Mart. Fl. Bras. iii. pt. 3, 173–634 (1891–4); Mez in DC. Mon. Phan. ix. 1–990 (1896); Harms in Engler & Prantl, Nat. Pflanzenf. ed. 2, xv a. 65–159 (1930).

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[1] Harms in Engl. & Prantl, Nat. Pflanzenf. 2 Aufl., xv a. 65 (1930).

KEY TO GENERA.

1. Ovary wholly or in part superior: fruit a capsule: seeds variously appendaged.
2. Seeds with an entire caudate appendage at each end, not plumose: leaves often somewhat spinulose-serrate: ovary only part superior............Subfamily *Pitcairnioideae*. 1. *Pitcairnia*.
2. Seeds plumose: leaves always entire..........Subfamily *Tillandsioideae*.
3. Seeds plumose at both ends: ovary only half superior. Tribe *Glomeropitcairnieae*. 2. *Glomeropitcairnia*.
3. Seeds plumose only at base or apex: ovary wholly or almost wholly superior........................Tribe *Tillandsieae*.
4. Plume of the mature seed straight: sepals symmetric or if asymmetric then the spikes distichous-flowered.
5. Petals free: inflorescence of one or more distichous-flowered spikes, if rarely polystichous then simple and the leaves long-acuminate.
6. Petals naked..............................3. *Tillandsia*.[1]
6. Petals each bearing 2 scales at base..............4. *Vriesia*.[1]
5. Petals joined or closely agglutinated: inflorescence always of polystichous-flowered spikes...........5. *Guzmania*
4. Plume of the mature seed folded over: sepals strongly asymmetric: inflorescence of polystichous-flowered spikes....................................6. *Catopsis*.
1. Ovary wholly inferior: fruit baccate, fleshy to coriaceous: seeds naked: leaves usually spinulose-serrate..Subfamily *Bromelioideae*.
7. Pollen grains entire, bearing neither pores nor longitudinal folds: margins of the petals free, petals connected by the filaments.....................................7. *Bromelia*.
7. Pollen grains variously sculptured: petals not connected by the filaments.
8. Pollen grains bearing pores: stamens usually included at anthesis: sepals mostly mucronate.
9. Petals naked.
10. Ovules few in each cell: inflorescence loosely paniculate: flowers not over 5 mm. long..........8. *Araeococcus*.
10. Ovules many in each cell.
11. Flowers in a number of very dense cone-like spikes. 9. *Hohenbergia*.[2]
11. Flowers in elongate lax spikes in an open inflorescence. 10. *Wittmackia*.
9. Petals each bearing 2 scales on the inner surface.
12. Fruits and bracts always remaining distinct.
13. Pollen grains bearing many (more than 5) pores: inflorescence compound: flowers in dense fascicles with only the petals exserted from the bracts..........................11. *Gravisia*.
13. Pollen grains bearing either 2 or 4 pores.........12. *Aechmea*.

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[1] See artificial key on page 159 for the identification of specimens without good flowers.

[2] Without good flowers *Hohenbergia oligosphaera* is apt to be confused with *Gravisia aquilega*, but it may be distinguished by its floral bracts which are suborbicular to broadly ovate and abruptly contracted to an acuminate apex (Pl. I, fig. 8) instead of ovate, abruptly acute, and mucronate.

12. Fruits and bracts of the inflorescence coalescing at maturity to form a compound fruit.......13. *Ananas.*
8. Pollen grains bearing longitudinal folds: stamens exserted at anthesis: sepals obtuse or acute, never mucronate: flowers 6–8 cm. long........................14. *Billbergia.*

1. **PITCAIRNIA** L'Hérit.

Plants usually stemless. Leaves fasciculate, entire or spinulose-serrate, sheath small, often bulbous-thickened, blade linear to lanceolate with definite petiole, sometimes dimorphic with some blades reduced to horny spinulose-serrate spines. Inflorescence simple or compound, racemose. Sepals free, acute but never mucronate. Petals free, usually zygomorphic, naked or with a single triangular scale at base. Stamens equaling or shorter than the petals, anthers linear, versatile. Ovary usually superior for most of its length, style filiform, equaling the petals. Ovules many. Capsule septicidal or rarely indehiscent. Seeds appendaged at either end or rarely with a dorsal wing.—Sert. Angl. 7, t. 11 (1788).

About 170 species. Terrestrial or saxicolous, rarely epiphytic. Mexico, Central America, the Antilles and South America south to northern Argentina.

1. **P. integrifolia** Ker-Gawl. Half a meter high: leaves all alike, 7 dm. long, 13–18 mm. broad, white-lepidote beneath, strongly constricted between the sheath and the blade, the outermost ones slightly serrulate near the base: scape slender, straight, its bracts triangular, with long filiform tips: inflorescence laxly bipinnate, pyramidal, sparingly furfuraceous-lepidote to glabrous: floral bracts lanceolate, acute, mostly shorter than the slender flower-pedicels: flowers 35–40 mm. long; sepals narrowly triangular, acuminate, 10–12 mm. long; petals yellow at base and bearing a scale, the remainder bright red; ovary about 2/3 superior: capsule broadly fusiform.—Ker-Gawl. in Bot. Mag. xxxvi. t. 1462 (1812).

TRINIDAD: Botanic Gardens, St. Ann's, 1930, *Broadway 7434* (G); Fort George Hill, Cocorite, 1903, *J. R. Johnston 109* (G); cliff, Carenage, *Britton & Hazen 233* (G, NY); coastal rocks, Chaguaramas, 1921, *N. L. Britton 2715* (NY); Hort. Trin., 1890, *Herb. Trin. 4362* (Trin); Monos Island, 1917, *R. O. Williams* (Trin); Tetron Bay, 1927, *Broadway 6776* (Mo); 1928, *Broadway 6887* (Mo).

[Endemic? The source of the type is obscure; the only definite records are from Trinidad.]

2. **GLOMEROPITCAIRNIA** Mez.

Large stemless herbs, with rosulate, linear, entire leaves. Scape elongate, erect. Inflorescence of sessile or short-stipitate heads or short spikes, bipinnate or tripinnate. Flowers perfect. Sepals free, subsymmetrical. Petals free, bearing one or two scales at base. Stamens included. Ovary about half inferior. Capsule imperfectly dehiscent. Seeds with pappiform coma at both ends.—Bull. Herb. Boiss. ser. 2, v. 232 (1905).

Two species. Lesser Antilles, Trinidad, Island of Margarita (to Venezuela).

1. **G. erectiflora** Mez. Plant 1–2 m. high: leaves many, 7 dm. long, 8 cm. broad, acute but soon rounded by withering: scape stout, much exceeding the leaves: inflorescence bipinnate with erect or ascending spikes, interrupted-thyrsoid, red, glabrous; primary bracts suberect, coriaceous, elliptic, rounded-apiculate, shorter than the axillary spikes: spikes erect or nearly so, subovoid, the lower short-stipitate; floral bracts elliptic, rounded, mucronate, ecarinate, shorter than the sepals: flowers sessile, erect; sepals sublanceolate, carinate, 21–28 mm. long.—Bull. Herb. Boiss. ser. 2, v. 233 (1905); L. B. Smith in Contrib. Gray Herb. xcv. 42, t. x, figs. 6–7 (1931). Pl. II, figs. 11–12.

TRINIDAD: Mt. Tucuche, summit, 1893, *Broadway 5250* (Trin, G, NY); 1925, *Broadway 5660* (FM); 1929, *Broadway 5660, 7129* (K); 1931, *Broadway 7461* (G).

[Island of Margarita.]

Epiphytic.

3. TILLANDSIA L.

Caulescent or acaulescent herbs, habit very variable. Leaves rosulate or fasciculate or evenly distributed along a stem, polystichous or distichous, entire, usually subtriangular or ligulate. Scape usually distinct. Inflorescence various, usually of distichous spikes, if spikes polystichous then single, rarely reduced to a single flower. Flowers perfect. Sepals usually symmetric, free, or equally or posteriorly joined. Petals free, naked. Stamens of various lengths relative to the petals and to the pistil. Ovary superior, glabrous. Ovules usually many and caudate. Capsule septicidal. Seeds erect, narrowly cylindric or fusiform, coma white, straight.—Sp. Pl. 286 (1753).

About 350 species. Tropical and subtropical America, from the southern United States, Mexico and the Antilles, to central Argentina and Chile.

KEY TO SPECIES.

1. Stamens exserted, longer than the petals................§ *Platystachys.*
    2. Floral bracts remote, equaling or shorter than the internodes: spikes elongate.

3. Flowers and bracts erect, appressed to the rhachis....1. *T. utriculata.*

3. Flowers and bracts spreading........................2. *T. flexuosa.*

2. Floral bracts usually more or less imbricate, always distinctly longer than the internodes: spikes not elongate.

4. Leaves fasciculate with filiform blades: inflorescence of several spikes crowded into a dense head or reduced to a single polystichous-flowered spike..............3. *T. juncea.*

4. Leaves rosulate: inflorescence of distinct spikes or of a single distichous-flowered spike.

5. Floral bracts densely imbricate, wholly concealing the rhachis, glabrous or nearly so............4. *T. fasciculata.*

5. Floral bracts rather lax, concealing the rhachis little or not at all.

6. Floral bracts equaling or exceeding the sepals.

7. Floral bracts subglabrous: leaf-sheaths scarcely or not at all inflated.......................5. *T. canescens.*

7. Floral bracts densely lepidote: leaf-sheaths strongly inflated..............................6. *T. bulbosa.*

6. Floral bracts distinctly shorter than the sepals, very lax. 7. *T. subulifera.*

1. Stamens included, shorter than the petals.

8. Sepals strongly asymmetric, oblong or obovate, free. § *Pseudo-Catopsis.*

9. Inflorescence bipinnate, spikes 8 cm. long or more: floral bracts 7 mm. long.........................8. *T. triticea.*

9. Inflorescence tripinnate or if bipinnate the spikes not over 6 cm. long: floral bracts scarcely over 4 mm. long. 9. *T. micrantha.*

8. Sepals symmetric, ovate or lanceolate.

10. Style elongate, equaling or exceeding the stamens: inflorescence always of distichous spikes..............§ *Allardtia.*

11. Plants 1–2 m. high: inflorescence of many pinnately arranged spikes.

12. Plant slender: spikes linear in outline, 12 mm. wide. 10. *T. subimbricata.*

12. Plant stout, often 2 m. high: spikes lanceolate in outline, 50 mm. wide.........................11. *T. rubra.*

11. Plants much smaller: inflorescence simple or of digitately arranged spikes.

13. Inflorescence single, terminal, digitate.....12. *T. didistichoides.*

13. Inflorescences several, each a single spike on a long slender scape arising from the axil of a rosette-leaf...........................13. *T. complanata.*

10. Style shorter than the stamens or else the inflorescence simple and polystichous.

14. Stamens emerging from the throat of the corolla: filaments transversely plicate..................§ *Anoplophytum.*

15. Inflorescence simple and polystichous.

16. Sepals equally subfree........................14. *T. stricta.*

16. Sepals much connate posteriorly.............15. *T. pulchella.*

15. Inflorescence digitate-compound: spikes distichous-flowered..........................16. *T. Gardneri.*

14. Stamens deeply included.

17. Petal-blades broad and conspicuous, large......§ *Phytarrhiza.*

18. Spike lax: floral bracts spreading, not at all imbricate. 17. *T. monadelpha.*

18. Spikes dense: floral bracts imbricate and concealing the rhachis.
  19. Floral bracts green or light rose, drying light: petal-blades elliptic................18. *T. anceps.*
  19. Floral bracts deep rose, drying dark: petal-blades large, suborbicular............19. *T. Morreniana.*
17. Petal-blades narrow, small: leaves subulate, distichous. § *Diaphoranthema.*
20. *T. usneoides.*

Artificial Key to the Species of Tillandsia and Vriesia for the Identification of Material without Good Flowers.

1. Spikes with flowers distichous or secund, or else the inflorescence reduced to a single flower.
  2. Sepals asymmetric, oblong or obovate, free, not over 10 mm. long.
    3. Inflorescence bipinnate, spikes 8 cm. long or more: floral bracts 7 mm. long..........................8. *T. triticea.*
    3. Inflorescence tripinnate or if bipinnate the spikes not over 6 cm. long: floral bracts scarcely over 4 mm. long. 9. *T. micrantha.*
  2. Sepals symmetric, ovate or lanceolate.
    4. Flowers strongly secund.
      5. Inflorescence simple, few-flowered: floral bracts broadly acute, shorter than the sepals............6. *V. Broadwayi.*
      5. Inflorescence usually compound: floral bracts acuminate, longer than the sepals...................9. *V. paniculata.*
    4. Flowers not at all secund.
      6. Plant moss-like: inflorescence reduced to a single flower which soon becomes pseudo-axillary.......20. *T. usneoides.*
      6. Plant not at all moss-like: inflorescence several- to many-flowered.
        7. Flowers spreading at anthesis and after.
          8. Inflorescence pendulous, simple: floral bracts shorter than the sepals.......................3. *V. simplex.*
          8. Inflorescence erect.
            9. Leaf-blades lorate: plants 1 m. or more high: scape-bracts foliaceous.
              10. Inflorescence simple, many-flowered: rhachis very stout.................5. *V. macrostachya.*
              10. Inflorescence compound: rhachis slender....8. *V. procera.*
            9. Leaf-blades narrowly triangular: upper scape-bracts not at all foliaceous.
              11. Floral bracts equaling or shorter than the internodes: inflorescence usually compound: sepals 22 mm. long............2. *T. flexuosa.*
              11. Floral bracts distinctly longer than the internodes: inflorescence simple: sepals 13 mm. long..................17. *T. monadelpha.*
        7. Flowers spreading only in so far as they are forced to make room for those directly above them.
          12. Floral bracts equaling or shorter than the internodes: inflorescence and spikes very lax...1. *T. utriculata.*
          12. Floral bracts distinctly longer than the internodes.

13. Inflorescence simple or digitately compound from a few spikes.

14. Floral bracts equaling or exceeding the sepals.

15. Inflorescences several, each a single spike on a long slender scape arising from the axil of a rosette-leaf......13. *T. complanata.*

15. Inflorescence single, terminal.

16. Leaf-blades lorate.

17. Floral bracts up to 7 cm. long. 2. *V. longibracteata.*

17. Floral bracts 35 mm. long....4. *V. chrysostachys.*

16. Leaf-blades narrowly triangular to filiform.

18. Leaf-sheaths broadly ovate, conspicuous, several times broader than the blades.

19. Sepals much connate posteriorly.

20. Floral bracts subglabrous: leaf-sheaths scarcely or not at all inflated........5. *T. canescens.*

20. Floral bracts densely lepidote: leaf-sheaths strongly inflated. 6. *T. bulbosa.*

19. Sepals equally sub-free ....12. *T. didistichoides.*

18. Leaf-sheaths inconspicuous, rarely more than twice as broad as the blades.

21. Spikes crowded into a dense head.

22. Leaf-blades filiform: leaves ferruginous at base........3. *T. juncea.*

22. Leaf-blades narrowly triangular: leaves cinereous throughout. 16. *T. Gardneri.*

21. Spikes distinct or only a single spike.

23. Posterior sepals much connate: upper scape-bracts filiform-caudate: leaf-sheaths evenly ferruginous........4. *T. fasciculata.*

23. Posterior sepals free: upper scape-bracts acute: leaf-sheaths longitudinally red and green striped.

24. Floral bracts green or light rose, drying light........18. *T. anceps.*

24. Floral bracts deep rose, drying dark.........19. *T. Morreniana.*

14. Floral bracts distinctly shorter than the sepals, very lax...................7. *T. subulifera.*

13. Inflorescence amply paniculate from a number of pinnately arranged spikes.

25. Spikes 12 mm. wide, narrowly linear: leaf-blades narrowly triangular.......10. *T. subimbricata.*

25. Spikes much broader: leaf-blades lorate.

26. Floral bracts twice as long as the sepals. 1. *V. stenostachya.*

26. Floral bracts but slightly exceeding the sepals.

27. Spikes narrowly oblong, suberect....7. *V. albiflora.*

27. Spikes lanceolate, spreading or reflexed.11. *T. rubra.*

1. Spike polystichous-flowered: inflorescence simple.
28. Floral bracts coriaceous . . . . 3. *T. juncea.*
28. Floral bracts membranaceous.
29. Sepals equally sub-free . . . . 14. *T. stricta.*
29. Sepals much connate posteriorly . . . . 15. *T. pulchella.*

1. **T. utriculata** L. One to two meters high: leaves densely rosulate, minutely punctulate-lepidote; sheaths large, ovate; blades narrowly triangular, subulate-acuminate, 50 mm. broad at base: scape erect, its bracts ovate, caudate to obtuse, rather remote: inflorescence lax, amply paniculate or rarely simple; primary bracts small, resembling the upper scape-bracts; spikes laxly 6–14-flowered, elongate, ascending, bearing several sterile bracts at base; floral bracts remote, ovate, acute or obtuse, glabrous, nerved, much shorter than the sepals: flowers erect, up to 40 mm. long; sepals equally subfree, obtuse, 18 mm. long, glabrous, nerved, ecarinate; petals white, tubular, erect, much exceeding the sepals; stamens and style slender, much exceeding the petals: capsule 35 mm. long, erect.—Sp. Pl. 286 (1753).

TRINIDAD: Tabaquite, near El Paraiso, 1917, *Broadway 8950* (Trin); Hort. Trin., 1890, *Broadway 4407* (Trin); St. Ann's on tall Saman trees, 1923, *Broadway 5165* (FM, Mo); Arima, 1906, *Dannouse* (Trin); Chaguanas, 1844, *Herb. Trin. 1750 in part* (Trin); indefinite, *Broadway 2380, 2631b, Fendler 815* (Mez!); *Herb. Trin. 2010* (Trin); 1921, *L. H. & E. Z. Bailey* (NY). TOBAGO: Government House grounds, 1910, *Broadway 3400* (FM, Mo), *3640* (FM, NY); indefinite, *Broadway 3056* (Mez!).

[Florida, Mexico, Antilles, Venezuela.]

Epiphytic.

2. **T. flexuosa** Sw. (Cf. p. 148). Stemless, 1 m. high or often less: leaves densely rosulate, 4–5 dm. long, sheaths large, ovate, forming an ellipsoid pseudo-bulb, blade narrowly triangular, 25 mm. broad, often transversely banded: scape slender, erect, glabrous, its bracts elliptic: inflorescence simple or laxly bipinnate: primary bracts lanceolate, small: spikes lax; rhachis slender, flexuous, strongly flattened opposite the flowers; floral bracts elliptic-acute, nerved, shorter than the sepals: flowers remote, spreading, short-stipitate, 45–55 mm. long, white, rose, or purple; sepals free, obtuse, 22 mm. long; stamens and style exserted: capsule slenderly cylindric, 5–9 cm. long.—Prodr. 56 (1788). *T. aloifolia* Hook. Exot. Fl. iii. t. 205 (1827).

TRINIDAD: San Fernando, 1873, *Herb. Trin. 2011* (Trin); Gasparee, 1861, *Herb. Trin. 846* (Trin); Lapeyrouse Cemetery, Port of Spain, 1930, *Broadway* (G); San Juan, 1927, *Broadway 6782* (S); Chacachacare, 1921, *Broadway 10571* (Trin); Hort. Trin., 1888, *Broadway 2423* (Trin); same, 1890, *Broadway 4406* (Trin); Erin, 1915, *Broadway 7337* (Trin); L'Islet, near Chatham, 1916, *Broadway 8649* (Trin); Icacos, near St. Quinton, 1915, *Broadway 7372* (Trin); Patos, 1921, *Broadway* (Trin); the Savannah, Port of Spain, 1906, *Hart*

(Trin); indefinite, *Herb. Trin. 609* (Trin); 1919, *Broadway* (Trin, G, NY); 1915, *Broadway* (NY); *De Schach* (Mez!); *Broadway 2805, 3232* (Mez!). TOBAGO: Calder Hall road on hedge shrubs, 1914, *Broadway 4746* (G, Mo); indefinite, *Eggers 5667* (Mez!); *Broadway 4804* (Mez!).

[Florida, the Antilles, Central America, Colombia, Venezuela, British and Dutch Guiana.]

Epiphytic.

3. **T. juncea** (R. & P.) LeC. Stemless, up to 35 cm. high, sometimes propagating by stolons: leaves densely fasciculate, subulate; sheaths triangular, ferruginous-tomentose-lepidote; blades erect, filiform, channelled: scape erect, its bracts ovate, long-caudate, densely imbricate: inflorescence short, compact, of a few densely massed distichous spikes, or sometimes reduced to a single polystichous spike; primary bracts like the scape-bracts, mostly shorter than the axillary spikes: spikes erect, sessile, lanceolate in outline, stout; floral bracts ovate, densely imbricate, carinate, even, lepidote, exceeding the sepals, coriaceous: flowers erect, 40 mm. long; sepals lanceolate, glabrous, even, about 21 mm. long, posterior ones connate for 1/3 their length; petals tubular-erect, violet; stamens and style exserted: capsule 30 mm. long, cylindric.—LeC. in Ann. Lyc. Nat. Hist. N. Y. ii. 130 (1828). *Bonapartea juncea* R. & P., Fl. Peruv. iii. 38, t. 262 (1802).

TRINIDAD: Queen's Park Savannah, Port of Spain, 1931, *Broadway* (G); Caura, Valley of Tumbasson, 1928, *F. C. Buthn 11976* (Trin); St. Ann's, 1894, *Broadway 5501* (Trin); Arima, 1906, *Dannouse* (Trin, NY); indefinite, *Fendler 825*, *Broadway 2824* (Mez!).

[Antilles and Mexico to Venezuela and Peru.]

Epiphytic.

4. **T. fasciculata** Sw. Very variable, the Trinidad specimens belonging to the typical form: leaves rosulate, 2–6 dm. long, densely and minutely pale-appressed-lepidote; sheaths ovate, castaneous; blades narrowly triangular, acuminate, involute: scape stout, erect, its bracts leaf-like, densely imbricate: inflorescence (in the Trinidad specimens) of 1–5 large subsessile spikes digitately arranged; primary bracts like the upper scape bracts: spikes lanceolate in outline, thick, somewhat compressed; floral bracts ovate, acute, 35–50 mm. long, exceeding the sepals, glabrous or nearly so, carinate at apex, densely imbricate: flowers erect, subsessile; sepals lanceolate, mucronate-acute, glabrous, 27 mm. long, carinate, much joined posteriorly; petals violet, tubular-erect; stamens and style exserted.—Prodr. 56 (1788).

TRINIDAD: St. Ann's, 1907, *Broadway 2653* (FM); 1923, *5166* (FM, Mo); Quare River forests, 1931, *Broadway* (G); near no. 6 bridge, Cumuto Road, Cumuto, 1931, *Broadway 7716* (G); Tabaquite, 1918, *Broadway* (NY); Arima,

1902 and 1906, *Dannouse* (Trin); the Savannah, Port of Spain, 1906, *A. Fredholm* (Trin); San Fernando, 1873, *Herb. Trin. 2006* (Trin); indefinite, *Broadway 3234* (Mez!). TOBAGO: Mason Hall River, 1913, *Broadway* (NY); Adelphi, 1913, *Broadway* (G); indefinite, *Broadway 4285* (Mez!).
[Florida, the Antilles and Mexico to Dutch Guiana and Colombia.]

Epiphytic, saxicolous, or terrestrial.

5. **T. canescens** Sw. Stemless, about 2 dm. high: leaves utriculate-rosulate, equaling or exceeding the inflorescence, densely appressed-lepidote; sheaths conspicuous, broadly ovate, castaneous; blades narrowly triangular, involute-subulate, cinereous: scape erect or ascending, its bracts broadly ovate, inflated, long-caudate: inflorescence of a few digitately arranged spikes; primary bracts like the upper scape-bracts, shorter than the spikes: spikes lanceolate in outline, compressed, few-flowered; floral bracts ovate, strongly carinate, glabrous, strongly nerved, reddish, equaling or exceeding the sepals: flowers erect; sepals much connate posteriorly, 14–17 mm. long, elliptic, carinate, glabrous, even; petals violet, twice as long as the sepals; stamens and style exserted: capsule 28 mm. long, cylindric.—Sw. Prodr. 57 (1788).

TRINIDAD: indefinite, *De Schach* (Baker!).
[Antilles.]

Epiphytic.

6. **T. bulbosa** Hook. Stemless, 3 dm. high or usually much less: leaves inflated, bulbous-rosulate at base, blade involute-subulate: scape erect, its bracts elliptic with long blades exceeding the inflorescence: inflorescence simple or digitate from a few spikes: spikes lanceolate, densely 5–8-flowered, compressed, 5 cm. long; floral bracts ovate-acute, 15 mm. long, exceeding the sepals, densely lepidote, coriaceous: flowers erect, 3 cm. long, sessile; sepals glabrous, joined posteriorly; petals violet; stamens and style exserted: capsule cylindric, 4 cm. long.—Exot. Fl. iii. t. 173 (1827).

TRINIDAD: Tabaquite, 1918, *Broadway* (G, NY); Tamana, 1915, *Broadway 7895* (Trin); Moruga, Edward Trace, 1916, *Broadway 8205* (Trin); Quare River forests, 1931, *Broadway* (G); indefinite, *Fendler 823, 824* (Mez!); *Broadway 2196, 3237* (Mez!). TOBAGO: indefinite, *Eggers 5570* (Mez!); *Broadway 4287* (NY).
[Mexico and the Antilles to Colombia and eastern Brazil.]

Epiphytic.

7. **T. subulifera** Mez. Stemless, 15 cm. high: leaves few in an elongate pseudo-bulb, concolorous, 13 cm. long; blades linear, 5 mm. wide, angular-folded, subulate, erect, rigid, appressed-canescent-lepidote: scape erect, its bracts coriaceous, appressed-lepidote, the

upper ones not at all caudate: inflorescence erect, simple, distichous, 4-flowered, about equaling the leaves, 50 mm. long; axis slender, geniculate; floral bracts thickly coriaceous, strongly nerved, carinate, incurved, 20 mm. long, shorter than the sepals, densely appressed-canescent-lepidote: flowers erect or nearly so, 32 mm. long, yellow or white in the dried specimen; sepals 22 mm. long, narrowly obtuse, free, coriaceous, lepidote like the bracts; petals tubular-erect, stamens and style exserted: capsule 6 cm. long.—Mez in Fedde Rep. Spec. Nov. xvi. 74 (1919).

TRINIDAD: by Longdenville, *Broadway 4200* (B, TYPE).

8. **T. triticea** Burch. Stemless, 4–7 dm. high: leaves 3 dm. long, densely and minutely appressed-lepidote, usually dark-spotted, 3 cm. broad above the broadly ovate dark brown sheath, abruptly acute, apiculate: scape erect, glabrous, its bracts narrowly ovate, acute, exceeding the internodes: inflorescence distichous, laxly bipinnate, glabrous, 2–3 dm. long; primary bracts lanceolate, much shorter than the spikes: spikes linear, 20–30-flowered, 8 cm. long or more, spreading, rhachis slender, flexuous; floral bracts broadly ovate, acute, 7 mm. long, equaling the sepals, not imbricate, glabrous, even: flowers sessile, suberect; sepals obovate, asymmetric; petals yellowish; stamens included: capsule narrowly cylindric.—Burch. in Journ. Bot. xxvi. 42 (1888).

TRINIDAD: Mt. Tucuche, 1920, *Britton, Coker & Rowland 1323 in part* (NY).

[British Guiana, Brazil, Bolivia, Peru.]

Epiphytic.

9. **T. micrantha** Bak. Stemless, 4 dm. high; leaves 3–4 dm. long; sheaths large, elliptic, brown; blade lingulate, rounded-apiculate, 3–4 cm. broad, often irregularly purple-spotted: scape slender, erect or ascending, its bracts ovate-lanceolate, acuminate to rounded-apiculate, shorter than or equaling the internodes: inflorescence lax, tripinnate or bipinnate with the primary branches not over 6 cm. long; axis flexuous; primary bracts like the upper scape-bracts: spikes laxly 10–25-flowered, 5–10 mm. broad; floral bracts ovate-acute, convex, ecarinate, 4–5 mm. long, about equaling the sepals, not imbricate, sparsely punctulate-lepidote: flowers suberect, sessile, 2–4 mm. apart, not touching at maturity, much exceeding the internodes; sepals obovate, sparsely punctulate-lepidote or glabrous, even; petals white, slightly exceeding the sepals; stamens deeply included, exceeding the pistil, anthers ovoid, mucronate.—Bak. in Journ. Bot. xxv. 303 (1887),

not *T. micrantha* Bak. Bull. Torr. Bot. Club, xxix. 698 (1902), published as a new species in spite of the previous use of the name for a perfectly valid species; L. B. Smith in Contrib. Gray Herb. xcv. 45, t. ix, figs. 5–9 (1931).

TRINIDAD: Mt. Tucuche, 1920, *Britton, Coker & Rowland 1323 in part* (G); 1922, *Freeman & Williams 10088* (Trin); Heights of Aripo, 1922, *Broadway 9912* (Trin, NY); Mt. Tamana, 1925, *Broadway* (FM); indefinite, 1877–80. *Fendler 818* (K, TYPE; phot. G).
[Venezuela.]

Epiphytic.

10. **T. subimbricata** Bak. Stemless, 1 m. high: leaves up to 8 dm. long, 40 mm. wide above the broadly ovate sheath, densely punctulate-lepidote; blade narrowly triangular, subulate-acuminate: scape stout, erect, its bracts triangular-elliptic, long-caudate, imbricate: inflorescence laxly tripinnate, ample; primary bracts like the upper scape-bracts, much shorter than the axillary branches; branches of the inflorescence curved-ascending: spikes up to 20-flowered, 15 cm. long, 12 mm. wide; floral bracts narrowly ovate, inrolled at apex, 18–20 mm. long, glabrous, strongly nerved, exceeding the sepals: flowers erect or nearly so; sepals equally sub-free, narrowly elliptic, obtuse, 11–13 mm. long, glabrous, strongly nerved; petals blue or lilac, 25 mm. long; stamens included, equaling the style.—Bak. in Journ. Bot. xxv. 304 (1887); L. B. Smith in Contrib. Gray Herb. lxxxix. 15, t. iv, fig. 1 (1930). *T. orthorhachis* Mez & C. F. Bak. in Bull. Torr. Bot. Club, xxx. 435 (1903).

TRINIDAD: Marabella, 1868, *Herb. Trin. 2007* (Trin); woods near Wellington Estate, 1867, *Herb. Trin. 773* (Trin); Moruga, 1916, *Broadway 7600* (Trin, NY); Caroni Reclamation Lands, 1923, *Broadway 10231* (Trin, NY); St. Ann's, 1931, *Broadway* (G); indefinite, *Herb. Trin. 2005* (Trin, US, phot. G); *Fendler 816* (K, TYPE; BM).
[Jamaica, Cuba, Nicaragua, Panama, Colombia.]

Epiphytic.

11. **T. rubra** R. & P. Stemless, often 2 m. high: leaves densely rosulate; sheaths subovate, inconspicuous; blade linear, up to 8 cm. broad, triangular-acute: scape stout, erect, its bracts leaf-like, imbricate: inflorescence amply paniculate; primary bracts like the upper scape-bracts, much shorter than the axillary spikes: spikes lanceolate, acute, complanate, densely 6–28-flowered, 18–30 cm. long, 4–5 cm. wide, spreading or recurved, more or less stipitate with sterile bracts at base; floral bracts narrowly obovate, keeled at apex, glabrous, even, about equalling the sepals: flowers erect, short-stipitate, sepals equally sub-free, 35–45 mm. long, oblong-lanceolate, carinate; petals

5–6 cm. long, slightly exceeding the stamens.—Fl. Peruv. iii. 40, t. 266 (1802). *Tillandsia paniculata* Ch. & Schdl. in Linnaea, vi. 54 (1831), not L. (1762).

TRINIDAD: Morne Bleu, 1922, *Freeman 10570* (Trin); Maracas road to the waterfall, 1924, *Broadway 5363* (FM, Mo, K); Maracas, on the face of the waterfall, 1924, *Broadway 5366 in part* (Mo); Blanchisseuse road, St. Patrick Estate, 1930, *Broadway 7462* (G, S); Ortinola Estate, Maracas, 1931, *Broadway 7459* (G).

[Mexico and the Antilles to Venezuela, Peru and Bolivia.]

Epiphytic.

12. **T. didistichoides** Mez. Stemless, about 2 dm. high: leaves densely bulbose-rosulate, densely appressed-lepidote; sheaths broadly ovate, conspicuous, deep castaneous; blades narrowly triangular, involute-subulate: scape slender, decurved, its bracts broadly ovate, imbricate, inflated, caudate: inflorescence bipinnate; primary bracts like the upper scape-bracts, much shorter than the spikes, erect, concealing the main axis of the inflorescence: spikes rather densely 8–10-flowered, linear-lanceolate in outline, strongly compressed, 4–5 cm. long, 10–15 mm. wide, long-stipitate with several sterile bracts at base; floral bracts elliptic, acute, 15 mm. long, much exceeding the sepals, carinate, strongly nerved, obscurely lepidote, bright pink: flowers erect, 20 mm. long; sepals free, 10 mm. long, elliptic, acute, glabrous, strongly nerved; petals lilac; stamens slightly shorter than the petals.—Mez in DC. Mon. Phan. ix. 778 (1896). Pl. II, figs. 1–3.

TRINIDAD: indefinite, *Fendler 827* (BM, TYPE; K, phot. G); Saut d'Eau, north coast, 1930, *Broadway 7433* (G); Las Lapas road off the Blanchisseuse road, 1931, *Broadway 7457* (G, sterile specimen); Mt. Tucuche, near 12-mile post, 1931, *Broadway 7458* (G).

[Venezuela.]

Epiphytic.

13. **T. complanata** Benth. Stemless: leaves rosulate, minutely appressed-lepidote, often exceeding the inflorescences, 3–4 dm. long, 4–5 cm. broad above the broadly ovate sheath, often purple-tinged, ligulate, apex broadly acute to rounded-apiculate, scapes several to many from a single leaf-rosette, slender, each bearing a single spike; scape-bracts narrowly elliptic, acute or apiculate, mostly exceeding the internodes: spikes slender, 6–24-flowered, 4–8 cm. long, 15 mm. broad; floral bracts imbricate, elliptic, obtuse, ecarinate, 15 mm. long, even or finely striate-nerved, exceeding the sepals; flowers sessile, 22 mm. long; sepals much joined posteriorly, glabrous, nerved; petals blue or violet; stamens and style included: capsule cylindric, 4 cm. long.—Bot. Sulph. 173 (1845).

TRINIDAD: Heights of Aripo, 1922, *Broadway 10568* (Trin, K); Quare road via Valencia, near the reservoir, 1927, *Broadway 6780* (K); Las Lapas road off the Blanchisseuse road, 1931, *Broadway 7456* (G).
[Antilles, British Guiana, Venezuela, Colombia, Ecuador, Peru, Bolivia.]

Epiphytic.

14. **T. stricta** Soland. Stemless or short-stemmed, 2 dm. or less high: leaves densely rosulate, 14 cm. long, the outer ones strongly recurved, sheath small, membranaceous, blade narrowly triangular, 8 mm. broad, filiform-acuminate, densely appressed-lepidote: scape slender, glabrous, its bracts ovate, long-caudate, membranaceous, imbricate: inflorescence simple, polystichous, short-cylindric, 35 mm. wide; floral bracts broadly ovate, exceeding the sepals, glabrous or lepidote toward apex, inflated, nerved, roseate, the lower ones long-caudate: flowers suberect, 2 cm. long, sessile; sepals lance-oblong, compressed, keeled, nerved, equally short-connate; petals blue changing to red, exceeding the pistil; stamens shorter than the pistil: capsule cylindric, 38 mm. long.—Soland. in Bot. Mag. xxxvii. t. 1529 (1813).

TRINIDAD: Caroni Reclamation Lands, 1923, *Broadway 10232* (Trin, NY).
[Venezuela, British and Dutch Guiana, Brazil.]

Epiphytic.

15. **T. pulchella** Hook. Plant usually caulescent, often branching, rather variable: leaves densely polystichous-ranked, narrowly triangular, subulate-acuminate, channelled, densely and minutely appressed-lepidote, 4–7 mm. wide above the inconspicuous sheath: scape slender, short, largely concealed by the leaves, its bracts membranaceous, elliptic, caudate, rose: inflorescence of a single short 4–14-flowered polystichous spike; floral bracts like the scape-bracts but less caudate, much exceeding the sepals, punctulate-lepidote, strongly nerved: flowers erect; sepals almost completely connate posteriorly, lanceolate, acute, 10 mm. long; petals blue or white, 20 mm. long, broadly obtuse; stamens included, shorter than the style: capsule cylindric, 15 mm. long.—Hook. Exot. Fl. iii. t. 154 (1827).

TRINIDAD: indefinite, *De Schach* (Baker!).
[Antilles, Venezuela, Guiana, Brazil, Bolivia, Argentina.]

Epiphytic, often in large masses. Originally described from Trinidad.

16. **T. Gardneri** Lindl. Stemless: leaves densely rosulate, narrowly triangular, 20 cm. long, cinerous, densely tomentose-lepidote: inflorescence of a few short densely massed spikes: spikes ovate, acute, 4–6-flowered; floral bracts ovate, acute, 14 mm. long, exceeding the

sepals, carinate, densely lepidote: flowers erect, 15 mm. long, sessile; sepals ovate, acute, 10 mm. long, coriaceous, even; petals rose; stamens included, equaling or exceeding the style: capsule cylindric, 37 mm. long.—Lindl. in Bot. Reg. xxviii. sub t. 63 (1842).

TRINIDAD: Port of Spain, 1920, *N. L. Britton 1181* (G); Hort. Trin., 1895, *W. Lunt 5999* (Trin); Valley of Tumbasson, Caura, 1928, *F. C. Buthn 11978* (Trin); St. Ann's, 1916, *Broadway 8168* (Trin); small savannah near Government House kitchen garden, 1907, *Broadway* (Trin); indefinite, *Fendler 822* (Baker!), *Broadway 2631* (Mez!).
[Venezuela, Brazil.]

Epiphytic or saxicolous.

17. **T. monadelpha** (Morr.) Bak. Stemless, 35 cm. high: leaves densely rosulate, 2 dm. long, obscurely punctulate-lepidote; sheath ovate; blade very narrowly triangular, 10–15 mm. broad: scape erect, slender, glabrous, its bracts elliptic-lanceolate, lepidote at apex, imbricate: inflorescence simple, distichous, oblong-linear, about 22-flowered, appearing lax by the spreading of the flowers, 13 cm. long; rhachis straight, angled, thickened at base of each flower; floral bracts ovate-acute, spreading, 17 mm. long, equaling the sepals, keeled, nerved, sparsely lepidote, soon glabrous: flowers sessile, spreading, 30 mm. long; sepals equally short-connate, glabrous, keeled; petals white, blade ovate, reflexed; stamens deeply included, exceeding the style, filaments connate: capsule narrowly cylindric, 4–7 cm. long.—Bak. in Journ. Bot. xxv. 281 (1887). *Phytarrhiza monadelpha* Morr. in Belg. Hort. xxxii. 168, t. 7 (1882). *Tillandsia graminifolia* Bak. in Journ. Bot. xxv. 281 (1887). *T. monobotrya* Mez in Fedde Rep. Spec. Nov. xvi. 77 (1919).

TRINIDAD: Chaguanas, 1868, *Finlay 2004* (Trin, NY); 1889, *Broadway 3741* (Trin); Arima, 1906, *Dannouse* (Trin); Tamana forests, 1915, *Broadway 7808*, *7818* (Trin); vicinity of Tabaquite, 1921, *N. L. Britton, Freeman & Nowell 2574* (NY); indefinite, *Fendler 828* (Baker!); *Broadway 2807* (Mez!).
[Guiana, Central America, Colombia, Ecuador.]

Epiphytic.

18. **T. anceps** Lodd. Stemless, 3 dm. high: leaves rosulate, 15–20 cm. long, minutely punctulate-lepidote; sheaths triangular-ovate, longitudinally red-striate; blades narrowly triangular, acuminate, 7 mm. broad: scape stout, erect, short and largely concealed by the leaves, its bracts broadly ovate, acute or acuminate, glabrous, even, densely imbricate, gradually passing into the floral bracts: inflorescence of a single stout compressed densely distichous spike, 10–20-flowered, up to 16 cm. long, 55 mm. wide; floral bracts triangular-acute,

up to 40 mm. long, much exceeding the sepals, strongly carinate, glabrous, even, pale rose with greenish margins: flowers erect, short-stipitate; sepals equally sub-free, 30 mm. long, lanceolate, acute, carinate toward apex; petals more than twice as long as the sepals, claw linear, white, blade lance-elliptic, acute, blue, spreading; stamens deeply included, exceeding the style.—Lodd. Bot. Cab. viii. t. 771 (1823). *T. lineatifolia* Mez in DC. Mon. ix. 686 (1896).

TRINIDAD: Tamana forests, 1915, *Broadway 7809* (Trin); Mt. Tamana, 1925, *Broadway* (FM).
[Honduras, Costa Rica, Panama, Venezuela, British Guiana.]

Epiphytic. Originally collected in Trinidad by Adam. Introduced in European cultivation by Loddiges.

19. **T. Morreniana** Regel. Habit almost identical with that of *T. anceps* and rather difficult to distinguish from it without good flowers: inflorescence rather more dense than in *T. anceps* and the floral bracts deep rose drying to a blackish brown: petals with sub-orbicular blade, blue, up to 60 mm. long.—Regel in Gartenfl. xix. 41 (1870). *T. Lindeni* Morr. in Belg. Hort. xix. 321, t. 18 (1869); not *T. Lindeniana* Regel in Gartenfl. xviii. 193 (1869).

TRINIDAD: Hort. Trin., *Herb. Trin. 4392* (Trin).
[Peru.]

Introduced in cultivation.

20. **T. usneoides** L. Long-filiform-caulescent, flexuous, often densely massed, gray to somewhat ferruginous, densely tomentose-lepidote throughout: leaves distichous; sheath elliptic; blade linear, less than 1 mm. broad, recurving: flowers solitary, soon becoming axillary by the elongation of the stem, short-pedicelled, yellowish green or blue, 10–12 mm. long; sepals narrowly ovate, equally connate at base, 7 mm. long; petals small, blade narrow, recurved; stamens deeply included, exceeding the style: capsule narrowly cylindric, abruptly short-beaked.—Sp. Pl. ed. 2. 411 (1762).

TRINIDAD: indefinite, *Herb. Trin. 822* (Trin); *Fendler 813* (Mez!).
TOBAGO: indefinite, *Broadway 2492* (Mez!).
[Southern United States to central Argentina and Chile.]

Epiphytic.

Species of Uncertain Position.

T. compressa Bert. So far all Trinidad specimens labelled as this species, on examination, have proved to be *T. fasciculata*.

4. **VRIESIA** Lindl.

Large showy herbs. Leaves densely rosulate, often banded or mottled, entire. Inflorescence various, spikes usually distichous-flowered. Floral bracts conspicuous, enfolding or concealing the bases of the flowers. Flowers with short stout pedicels, fugacious. Sepals free. Petals free, ligulate, scales usually two on the inside of each petal and entire. Ovary nearly or quite superior, style elongate. Ovules many, usually caudate. Seeds fusiform, with a long coma.—Lindl. in Bot. Reg. xxix. t. 10 (1843).

About 100 species. Epiphytic, rarely terrestrial. Chiefly Brazilian with outlying species from Mexico and the Antilles to Argentina.

KEY TO SPECIES.[1]

1. Petals not flaccid, erect or nearly so................Subgenus *Euvriesia.*
2. Stamens exserted at anthesis............................§ *Genuinae.*
3. Inflorescence compound: spikes dense............1. *V. stenostachya.*
3. Inflorescence simple.
4. Inflorescence erect: floral bracts imbricate, much exceeding the sepals..........................2. *V. longibracteata.*
4. Inflorescence pendulous: floral bracts remote, spreading, shorter than the sepals....................3. *V. simplex.*
2. Stamens included at anthesis............................§ *Xiphion.*
5. Inflorescence simple or rarely forked.
6. Inflorescence dense: flowers not secund........4. *V. chrysostachys.*
6. Inflorescence lax.
7. Flowers and floral bracts not at all secund: inflorescence many-flowered....................5. *V. macrostachya.*
7. Flowers and floral bracts secund: inflorescence few-flowered............................6. *V. Broadwayi.*
5. Inflorescence definitely paniculate.
8. Floral bracts erect or nearly so: spikes rather dense..7. *V. albiflora.*
8. Floral bracts spreading: spikes lax..................8. *V. procera.*
1. Petals flaccid, pendant..........................Subgenus *Alcantarea.*
9. *V. paniculata.*

1. **V. stenostachya** (Bak.) Mez. Leaves 7 dm. long, 6–8 cm. wide, linear: inflorescence compound; spikes 5 dm. long, bearing many sterile bracts at base; floral bracts 55 mm. long, rose: flowers with pedicels 2–4 mm. long; sepals narrowly elliptic, obtuse, 26 mm. long; petals yellow (?).—Mez in DC. Mon. Phan. ix. 564 (1896). *Tillandsia stenostachya* Bak. in Journ. Bot. xxvi. 109 (1888).

TRINIDAD: by Maracas Falls, 1845, *Crueger* (K, TYPE); 1925, *Broadway 5560* (FM, phot. G, Mo); wet cliff at waterfall, Mt. Tucuche, 1920, *Britton, Hazen & Mendelson 1317* (NY, phot. G); *Freeman 9068* (Trin); 1924, *Broadway 5366 in part* (NY).
[Endemic.]

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[1] For key to use in identifying imperfect material see p. 159.

2. **V. longibracteata** (Bak.) Mez. Stemless, 1 m. high or less: leaves 4 cm. or more broad, linear, abruptly acute, concolorous in all Trinidad material: scape erect, its bracts broadly ovate, acute or apiculate, densely imbricate: inflorescence simple, narrowly lanceolate, densely many-flowered; floral bracts up to 7 cm. long, lanceolate, acute, red at anthesis, later turning green, more or less incurved: flowers yellow, rarely known to emerge from the bracts in the Trinidad specimens (Broadway!), pedicels 7 mm. long; sepals narrowly elliptic, 25 mm. long.—Mez in DC. Mon. Phan. ix. 568 (1896), as *longebracteata*. *Tillandsia longibracteata* Bak. in Journ. Bot. xxvi. 81 (1888).

TRINIDAD: near summit of Mt. Tucuche, 1920, *N. L. Britton, Hazen & Mendelson 1363* (US, NY); Heights of Aripo, 1922, *Broadway 9913* (Trin, G, NY); *9918* (Trin, NY); Blanchisseuse road, 1925, *Broadway 5669* (FM, Mo); 1928, *Broadway* (K); Mora forest, 1927, *Broadway* (Mo); Morne Bleu, 1922, *Freeman 10569* (Trin); Valencia road, 1931, *Broadway 7584* (G); indefinite, *Fendler 830, 831* (Baker!); *Crueger* (Mez!); *Herb. Trin. 2008 in part* (Trin). TOBAGO: Mason Hall, 1913, *Broadway* (US).
[Venezuela, British Guiana.]

Epiphytic and terrestrial.

3. **V. simplex** (Vell.) Beer. Leaves rosulate, up to 5 dm. long; sheaths broadly elliptic, often with a strong purple tinge, minutely punctulate-lepidote; blades ligulate, up to 35 mm. broad, rounded-apiculate or abruptly acute: scape slender, strongly decurved, its bracts elliptic, varying from distinctly shorter than the internodes to slightly imbricate: inflorescence simple, pendulous, very lax, much exceeding the leaves if extended; rhachis slender, flexuous; floral bracts broadly ovate, 3–5 cm. long, red with yellow apex, much exceeded by the sepals: flowers remote, spreading, 4–8 cm. long, long-pedicelled; sepals oblong, yellow, 30–35 mm. long; petals yellow with green tips; stamens exserted.—Beer, Brom. 97 (1857). *Tillandsia simplex* Vell. Fl. Flum. 133 (1825), Icones, iii. t. 130 (1827). *Vriesia scalaris* Morr. in Belg. Hort. xxix. 301 (1879).

TRINIDAD: Tamana, *Herb. Trin. 1749* (Trin).
[Brazil, Venezuela, Colombia.]

Epiphytic.

4. **V. chrysostachys** Morr. Stemless, about 5 dm. high: leaves rosulate, ligulate, acuminate, concolorous, 40–48 mm. wide: scape erect, its bracts densely imbricate: inflorescence simple or of 2–3 erect spikes; spikes very narrow, 30 cm. long, 2 cm. broad; floral bracts 35 mm. long, distichous, yellow, coriaceous, exceeding the flowers, densely imbricate, slightly if at all incurved: flowers yellow,

25 mm. long, subsessile; sepals ovate-lanceolate, acute, 15 mm. long, apex cucullate-incurved; petals 24 mm. long, their scales entire, acute. —Morr. in Belg. Hort. xxxi. 87 (1881). *Tillandsia chrysostachys* Bak. in Bot. Mag. cxii. t. 6906 (1886). *Tillandsia trinitensis* Bak. Brom. 211 (1889).

TRINIDAD: *Fendler 820* (Mez!).
[Peru.]

5. **V. macrostachya** (Bello) Mez. One meter or more high: leaves 1 m. long; sheath large, dark brown; blade ligulate, 8 cm. broad, rounded-apiculate, concolorous: scape very stout, erect, its bracts ovate-elliptic, acute, densely imbricate: inflorescence simple, 5–7 dm. long, many-flowered; rhachis over 1 cm. thick at base; floral bracts broadly ovate, 6 cm. long, about equaling the sepals, ecarinate, coriaceous: flowers spreading with stout pedicels 13 mm. long; sepals ovate, obtuse, 42 mm. long, 17 mm. wide: coma whitish.—Mez in DC. Mon. Phan. ix. 601 (1896). *Caraguata macrostachya* Bello in Anal. Soc. Esp. Hist. Nat. xii. 122 (1883).

TRINIDAD: Las Lapas, off the Blanchisseuse road, 1927, *Broadway 7361* (G).
[Cuba, Porto Rico, Venezuela.]

Epiphytic.

6. **V. Broadwayi** L. B. Smith (see p. 149). Five dm. high: leaves 3 dm. long; sheath elliptic, densely and minutely ferruginous-lepidote throughout; blade narrowly lanceolate, acuminate, 2 cm. wide, glabrous above, scantly lepidote below: scape erect, slender; scape-bracts densely imbricate, sparsely lepidote, the lower foliaceous, the upper broadly elliptic, apiculate, pale brown: inflorescence simple, racemose, strongly secund, 7–9 cm. long, laxly 5–7-flowered, minutely ferruginous-lepidote; rhachis flexuous; floral bracts broadly ovate, abruptly acute, brown, up to 22 mm. long, distinctly shorter than the sepals, secund with the flowers: flowers with stout pedicels; sepals broadly elliptic, obtuse, 18 mm. long, subcoriaceous, faintly nerved, dark brown except for the pale margin; petals not known: capsule subcylindric, 35 mm. long. Pl. I, figs. 9–10.

TRINIDAD: Heights of Aripo, 1922, *Broadway 9916* (Trin, TYPE; phot. G; K; NY); Mt. Tucuche, 1920, *N. L. Britton, Hazen & Mendelson 1258 in part* (G); 1929, *Broadway 7096* (K).
[Endemic.]

Epiphytic.

7. **V. albiflora** Ule. Over 1 m. high: leaves 6 dm. long, blade ligulate, 7 cm. wide, flat, rounded-apiculate, subglabrous: scape erect,

stout, its bracts elliptic, acute, imbricate: inflorescence laxly bipinnate; primary bracts lanceolate, much shorter than the spikes: spikes suberect, linear, 25 cm. long, 2–3 cm. broad, compressed, prophyllate; floral bracts lanceolate, 25 mm. long, equaling the sepals, keeled, glabrous, even, imbricate: flowers erect; sepals narrowly elliptic, acute, coriaceous; petals ligulate, bearing 2 scales 10 mm. from the base, white; stamens and style included: capsule short, equaling the sepals.—Ule in Verhandl. Bot. Ver. Brandenb. xlviii. 141 (1907). *Tillandsia rhododactyla* Mez, Fedde Rep. Spec. Nov. xvi. 76 (1919).

TRINIDAD: near Tamana, 1868, *Herb. Trin. 2008 in part* (Trin); Tamana forest, 1915, *Broadway 7806* (Trin, NY).
[Guiana, Brazil, Peru.]

Epiphytic.

8. **V. procera** (Mart.) Wittm. Fully 1 m. or more high: leaves 4 dm. long, sublinear, 6 cm. wide above the large ovate sheath, rounded-apiculate, glabrate, green: scape-bracts leaf-like, mostly exceeding the internodes: inflorescence laxly paniculate, its branches flexuous, slender; floral bracts 28 mm. long, shorter than the sepals, elliptic, acute, ecarinate, enfolding the base of the flower: flowers spreading, remote, 35 mm. long; sepals oblong, rounded; petals yellow, scales lanceolate: coma brownish.—Wittm. in Engl. Bot. Jahrb. xiii. Beibl. 29, 21 (1891), as *Vriesea*. *Tillandsia procera* Mart. in R. & S. Syst. vii. 1224 (1830). *Vriesia gracilis* Gaud. Voy. Bonite Atl. t. 67 (1846). *Tillandsia viscidula* Britton, Bull. Torr. Bot. Club, xlviii. 328 (1922).

TRINIDAD: forest, four miles east of Arima, 1920, *N. L. Britton, Hazen & Mendelson 596* (Trin); Moruga, 1921, *N. L. Britton & Broadway 2430* (NY, TYPE of *Tillandsia viscidula;* Trin); Usine St. Madeleine, San Fernando, 1925, *Broadway 5623* (FM, K); Maracas road to the Bay, on the ridge, 1926, *Broadway 6135* (S, K); near no. 6 bridge, Cumuto road, Cumuto, 1931, *Broadway 7717* (G); indefinite, *Broadway* (Mo); *2774* (Mez!).
[British and Dutch Guiana, Brazil.]

Epiphytic.

9. **V. paniculata** (L.) Mez. Stemless, 1 m. high or less: leaves rosulate, linear-lanceolate, acute or acuminate, up to 60 cm. long, 4–6 cm. wide: scape erect, its bracts leaf-like, densely imbricate: inflorescence usually compound and few-branched; spikes laxly few-flowered, erect; floral bracts ovate-acuminate, exceeding the sepals, ecarinate, straight, partially enfolding the flowers, not at all imbricate, submembranaceous: flowers 6–9 cm. long, with stout pedicels 6 mm. long; sepals 35 mm. long, elliptic, acuminate, subpungent; petals long, flaccid, pendent, white.—Mez in DC. Mon. Phan. ix. 614 (1896), as *Vriesea panniculata*. *Renealmia paniculata* L. Sp. Pl. 286 (1753).

TRINIDAD: indefinite, *Broadway 3235* (Mez!).
[Antilles, Panama, Colombia.]

5. GUZMANIA R. & P.

Leaves densely rosulate, entire, sheaths conspicuous. Inflorescence various, spikes always polystichous. Flowers perfect. Sepals usually somewhat connate. Petals connate or closely conglutinated, naked, yellow or white. Stamens usually included, filaments more or less fused to the petals. Ovary wholly superior, pyramidal, ellipsoid or ovoid, glabrous. Ovules many, densely glomerate. Capsule septicidal. Seeds with a long straight usually brownish coma.—Fl. Peruv. iii. 37, t. 261 (1802).

About 80 species. Chiefly Andean with outlying species in Central America, the Antilles, Venezuela, Guiana and the Amazon Basin.

Key to Species.

1. Inflorescence simple.
  2. Inflorescence short, corymbiform .... 1. *G. lingulata.*
  2. Inflorescence cylindric, bracts membranous .... 2. *G. monostachia.*
1. Inflorescence compound .... 3. *G. megastachya*

1. **G. lingulata** (L.) Mez. Stemless, 3 dm. high: leaves 3–4 dm. long, obscurely punctulate-lepidote, blade lingulate, acute, 3–4 cm. wide: scape stout, erect, often short, its bracts leaf-like, densely imbricate: inflorescence short, corymbiform, 7 cm. broad; outer bracts lanceolate, forming an involucre which wholly conceals the flowers, red-tinged; floral bracts shorter than the flowers, linear, cucullate: flowers erect, short-pedicelled, 45 mm. long, sepals free, 17 mm. long, narrowly linear, rounded, glabrous; petals linear, cucullate, white; ovary ellipsoid: capsule subellipsoid, 3 cm. long, short-beaked; coma reddish brown.—Mez in DC. Mon. Phan. ix. 899 (1896). *Tillandsia lingulata* L. Sp. Pl. 286 (1753). *Caraguata lingulata* Lindl. in Bot. Reg. xiii. sub. t. 1068 (1827).

TRINIDAD: vicinity of Valencia, 1921, *Britton, N. L. & E. G. 2089* (G, NY); Quare River, 1922, *Williams 10681* (Trin); Long stretch, via Valencia, 22–23 mile posts, 1928, *Broadway* (Mo); indefinite, *Fendler 836* (Mez!). TOBAGO: indefinite, 1918, *Williams* (Trin); *Eggers 5799, 5861b* (Mez!).
[Central America, Antilles, Colombia, Venezuela, Guiana, Brazil.]

Epiphytic.

2. **G. monostachia** (L.) Rusby. Stemless, 2–4 dm. high: leaves rosulate, obscurely punctulate-lepidote, soon glabrous; sheaths broadly ovate, brownish; blades linear, 20 mm. wide, acute, yellowish green: scape erect, its bracts ovate, acute, imbricate, pale green: inflorescence of a single elongate polystichous spike, cylindric, 8–15 cm.

long, sterile toward apex; floral bracts imbricate, ovate, acute, membranous, the fertile ones pale green with conspicuous brown longitudinal stripes, about equaling the flowers, the sterile ones bright red: flowers erect, white, 23–29 mm. long; sepals joined equally for about 1/4 their length, 18 mm. long, obovate, broadly rounded, even, coriaceous; petals fused for most of their length, lobes elliptic, broadly rounded; stamens included, filaments connate with the petals for most of their length: capsule cylindric, 2–3 cm. long.—Rusby ex Mez in DC. Mon. Phan. ix. 905 (1896), as *G. monostachya*. *Tillandsia monostachia* L. Sp. Pl. ed. 2. 410 (1762). *Guzmania tricolor* R. & P. Fl. Peruv. iii. 38, t. 261 (1802).

TRINIDAD: San Fernando, 1868, *Herb. Trin. 827* (Trin); Hort. Trin., 1890, *Broadway 4090* (Trin); Arima, 1904, *Dannouse* (Trin); Maracas road to the waterfalls, 1927, *Broadway 6598* (K); St. Augustine, 1929, *Beattie 12275* (Trin); St. Ann's, 1931, *Broadway* (G); Valencia road, near the Oropuche River, 1931, *Broadway 7585* (G); indefinite, *Crueger 3604* (Trin); *Fendler 821, 829; Ryan; Broadway 2676, 2729, 3233* (Mez!). TOBAGO: Mason Hall, riverside, 1913, *Broadway* (Trin, G, NY); *4688* (G); indefinite, *Eggers 5708, 5936; Meyer* (Mez!).
[Antilles, Venezuela, Colombia, Ecuador, Peru.]

Epiphytic.

3. **G. megastachya** (Bak.) Mez. Stemless, 7 dm. or more high: leaves 5 dm. long, obscurely punctulate-lepidote, linear, 4 cm. wide, acute: scape erect, stout, its bracts elliptic-lanceolate, acute, densely imbricate: inflorescence many-flowered, elongate, subcylindric, about 2 dm. long; branches much abbreviated; primary bracts broadly ovate, acute, 55–80 mm. long, exceeding the axillary spikes: spikes subsessile, polystichous-flowered; floral bracts subtriangular, 40 mm. long, equaling or exceeding the sepals, strongly carinate, submembranaceous: flowers 62 mm. long, subsessile; sepals free, linear, acute, strongly nerved; petals conglutinated into a narrow tube for most of their length, lobes narrowly elliptic, rounded; stamens included: capsule 37 mm. long, subclavate.—Mez in DC. Mon. Phan. ix. 928 (1896). *Tillandsia foliosa* Griseb. Fl. W. Ind. 597 (1864), not Mart. & Gal. (1843). *T. megastachya* Bak. in Journ. Bot. xxvi. 46 (1888). *T. magna* Bak. Brom. 210 (1889).

TOBAGO: indefinite, *Eggers 6936* (Mez!).
[Lesser Antilles.]

6. CATOPSIS Griseb.

Stemless herbs. Leaves densely utriculate-rosulate, entire, minutely appressed-lepidote, green, sheath large, ovate. Scape conspicuous.

Inflorescence usually bipinnate, rarely simple or tripinnate, exceeding the leaves, its branches polystichous-flowering. Flowers small or minute, sessile, perfect in the Trinidad species. Sepals free, usually obtuse, strongly asymmetric, glabrous. Petals free, naked. Stamens included, anthers ovate or elliptic. Ovary superior, broadly ovoid or ellipsoid in the perfect or fertile flowers, style shorter than the ovary or absent. Ovules few–many, long-caudate. Capsule septicidal. Seeds with coma folded over, their bases projecting from the capsule.—Griseb. in Goett. Nachr. 1864. 21 (1865).

About twenty species. Chiefly in Central America and the Antilles, with outlying species in Florida, northern South America, the Andes, and southern Brazil.

KEY TO SPECIES.

1. Leaves acute or acuminate: inflorescence definitely paniculate.
  2. Scape-bracts and primary bracts ovate-acute: sepals 12 mm. long .... 1. *C. Berteroniana.*
  2. Scape-bracts and primary bracts narrowly lanceolate, acuminate: sepals 6 mm. long .... 2. *C. nutans.*
1. Leaves rounded-mucronate: inflorescence simple or of only a few spikes .... 3. *C. sessiliflora.*

1. **C. Berteroniana** (Schult. f.) Mez. About 8 dm. high: leaves 3 dm. long, blade 5 cm. broad, subtriangular, acuminate: scape erect, its bracts lance-ovate below, broadly ovate above, equaling or exceeding the internodes: inflorescence laxly bipinnate, 30 cm. long, 12 cm. wide; spikes long-stipitate, 6–30-flowered; primary bracts ovate-apiculate, much shorter than the sterile base of the spikes; floral bracts suborbicular, 8 mm. long: flowers perfect, suberect; sepals 13 mm. long, exceeding the petals, strongly asymmetric, even, yellowish; petals white, ovate-elliptic; style 1/3 as long as the ovary, ovary broadly ellipsoid: coma pale brown.—Mez in DC. Mon. Phan. ix. 621 (1896). *Tillandsia Berteroniana* Schult. f. in R. & S. Syst. vii. 1221 (1830). *Catopsis Mosenii* Mez in DC. Mon. Phan. ix. 622 (1896).

TRINIDAD: Aripo Savanna, 1920, *N. L. Britton, Broadway & Hazen 271* (NY); indefinite, *Fendler 819, Broadway 2773, 3238* (Mez!).
[Florida, British Honduras, the Antilles, British Guiana, Brazil.]

2. **C. nutans** (Sw.) Griseb. Stemless, 5 dm. high or less: leaves rosulate; sheaths broadly ovate; blades subtriangular, acuminate: scape slender, erect; scape-bracts narrowly lanceolate, acuminate: inflorescence laxly bi- or tripinnate; primary bracts like the scape-bracts, much shorter than the axillary spikes: spikes suberect, laxly flowered, long-stipitate; floral bracts broadly ovate, much shorter

than the sepals: flowers perfect, suberect; sepals broadly elliptic, 5–6 mm. long, strongly asymmetric, glabrous, nerved; petals elliptic, obtuse, white, slightly exceeding the sepals; ovary broadly ovoid.—Griseb. Fl. W. Ind. 599 (1864), not as used by Bak. *Tillandsia nutans* Sw. Prodr. 56 (1788).

TRINIDAD: woods of Morne L'Enfer, La Brea, *Crueger 2003* (Trin); Arima, 1901, *Dannouse* (Trin); Caparo, 1908, *Broadway 2675* (FM); Tabaquite, near El Paraiso, 1917, *Broadway 8949* (Trin); near Tabaquite, 1918, *Broadway* (Trin, G, NY); 1919, *Broadway* (NY); Maracas road to waterfall, 1924, *Broadway 5364* (FM, Mo); woods of Cumuto, 1–2 miles from railway south, 1925, *Broadway* (FM); indefinite, *Broadway 2806*, *Fendler 817* (Mez!).
[Antilles, Florida, Central America.]

Epiphytic.

3. **C. sessiliflora** (R. & P.) Mez. From 2 to 4 dm. high: leaves tubular-rosulate, 15 cm. long, blade sublinear, 22 mm. wide, rounded-apiculate: scape erect, its bracts remote: inflorescence simple or of only a few spikes, rhachis straight; floral bracts shorter than the sepals: flowers perfect, suberect, 10 mm. long; sepals suborbicular, 7.5 mm. long; petals ovate-lanceolate, slightly exceeding the sepals, white; style short but distinct; ovary large, ovoid.—Mez in DC. Mon. Phan. ix. 625 (1896). *Tillandsia sessiliflora* R. & P. Fl. Peruv. iii. 42, t. 271, fig. b (1802). *T. nitida* Hook. Exot. Fl. iii. t. 218 (1827). *Catopsis nitida* Griseb. Fl. W. Ind. 599 (1864). *C. nutans* as used by Bak. in Journ. Bot. xxv. 176 (1887), not Griseb.

TRINIDAD: Moruga, 1921, *N. L. Britton & Broadway 2443* (NY); Heights of Aripo, 1922, *Broadway 9914* (Trin, NY). TOBAGO: indefinite, *Eggers 5709*, *Broadway 4013*, *4354*, *4444*, *4831* (Mez!).
[Mexico and the Antilles to Peru and eastern Brazil.]

Epiphytic.

7. **BROMELIA** [Plum.] L.

Coarse herbs, spreading by subterranean stolons. Leaves usually rosulate, with large curved spines along the margin. Inflorescence sessile or stipitate, always paniculate, densely tomentose-lepidote in most species. Sepals free or somewhat united, obtuse or acute, rarely mucronulate. Petals dorsally united by the filaments, but their margins free, eligulate, fleshy in most species. Stamens included, often much conjoined by their filaments. Anthers narrow, acute. Ovary passing gradually into the thick pedicel, epigynous tube conspicuous to nearly lacking. Berry succulent, relatively large. Seeds few to many, flattened, naked.—Sp. Pl. 285 (1753).

Thirty-four species. Mexico and the Antilles to Argentina.

KEY TO SPECIES.

1. Inflorescence a densely capitate panicle, sessile, sunk in the leaf-rosette . . . . Subgenus *Karatas*.
  2. Floral bracts shorter than the sepals: scales on the inflorescence ferruginous, coarsely lacerate . . . . 1. *B. Karatas*.
  2. Floral bracts longer than the sepals: scales on the inflorescence white, divided so finely as to appear hair-like . . . . . 2. *B. humilis*.
1. Inflorescence an elongate rather open panicle, definitely stipitate. Subgenus *Eubromelia*.
3. *B. chrysantha*.

1. **B. Karatas** L. PINGUIN. Leaves densely rosulate, very narrowly triangular: inflorescence densely capitate, subconical, deeply immersed in the center of the leaf-rosette, densely ferruginous-tomentose throughout: floral bracts distinctly shorter than the sepals, linear-oblong, acute, spinulose-serrate: flowers 55 mm. long, passing indistinguishably into a short thick pedicel; sepals coriaceous, free, acute, entire, 22–25 mm. long; petals narrowly lanceolate, fleshy, glabrous, 30 mm. long, definitely exceeding the sepals, united for more than half their length; ovary few-seeded; style thick, trigonous: berry 80 mm. long, 20 mm. thick, greenish yellow, edible.—Sp. Pl. 285 (1753).

TRINIDAD: St. Ann's, 1923, *Broadway 5113*, 1922, *Broadway 5336* (FM).
[Mexico and the Antilles to Colombia and British Guiana.]

In hard lands under trees. Fruits edible.

2. **B. humilis** Jacq. Plants relatively small, less than 1 m. in diameter, propagating by leafy stolons: leaves densely rosulate, the outer recurving, the inner suberect and usually tinged with red or purple; sheath triangular-ovate, pale-tomentose-lepidote or glabrous; blade narrowly triangular, spinose-serrate: inflorescence densely capitate, sessile, sunk in the center of the leaf-rosette, densely pale-tomentose-lepidote: floral bracts lanceolate to linear, equaling the sepals: flowers 5–7 cm. long; sepals free, erect, entire, narrow, 20 mm. long; petals fleshy, purple at apex becoming white below, 40 mm. long, linear, obtuse, glabrous, connate for 1/3 their length; ovary cylindric.—Enum. Vindob. App. 306 (1762).

TRINIDAD: Chacachacare, 1890, *Hart 3751* (Trin. NY).
[Antilles, Venezuela.]

Terrestrial.

3. **B. chrysantha** Jacq. Terrestrial, 1–1.5 m. high: leaves densely rosulate, narrowly triangular: inflorescence scapose, paniculate, cylindric, ferruginous-tomentose; floral bracts broadly ovate, apiculate: flowers 45 mm. long, obscurely pedicellate; sepals broadly ovate,

mucronate, 10–12 mm. long; petals bright yellow, blade broad and conspicuous.—Hort. Schoenbr. i. 28, t. 55 (1797). *B. aurea* Britton, Bull. Torr. Bot. Club, xlviii. 328 (1922).

TRINIDAD: coastal hillside, western end of Monos Island, 1921, *Britton, N. L. & E. G., & Brown, M. S. 2736* (G, NY, TYPE of *B. aurea*).
[Venezuela, Colombia.]

8. **ARAEOCOCCUS** Brongn.

Low herbs. Leaves appressed-lepidote. Inflorescence laxly paniculate, scapose. Flowers small, inconspicuous. Sepals connate for half their length, rounded, glabrous. Petals free, reflexed at anthesis, eligulate. Stamens shorter than the petals, filaments free, anthers longer than the filaments, pollen grains ellipsoid with a pore at either end. Ovary terete, glabrous, no epigynous tube, style longer than the petals, stigma-lobes scarcely twisted. Ovules few (up to 10), borne at the top of the placenta, long-caudate. Seeds narrowly ovoid-cylindric, acute at either end.—Ann. Sci. Nat. ser. 2, xv. 370 (1841).

Four species. Trinidad, Tobago, Guiana, Brazil, Costa Rica.

1. **A. micranthus** Brongn. Leaves few, fasciculate, subpetiolate, spinulose-serrate, blades recurving, acute, 30–40 cm. long, 1–2 cm. wide: scape very slender, terete, erect; scape-bracts narrowly lanceolate, membranaceous, entire: inflorescence laxly subpyramidal, 15 cm. long, glabrous, its axes strongly geniculate, reddish: spikes laxly 3–8-flowered; floral bracts ovate, apiculate, 2–2.5 mm. long, spreading: flowers 5 mm. long, sessile; sepals 0.5–0.75 mm. long, suborbicular; petals 3–3.5 mm. long, with a short broad claw, abruptly acute; ovary subglobose, 2 mm. thick; ovules usually 2 in each cell: berry purple.—Ann. Sci. Nat. ser. 2, xv. 370 (1841).

TRINIDAD: Aripo Road via Arima, 4–5 mile posts, 1925, *Broadway 5708* (FM, S); 3–4 mile posts, 1927, *Broadway* (Mo); indefinite, *Prestoe 1672* (Trin). TOBAGO: the Widow, 1913, *Broadway* (G, NY); indefinite, *Hart 4143* (Trin).
[Guiana, the Amazon Basin.]

Epiphytic.

9. **HOHENBERGIA** Schult. f.

Mostly large showy herbs. Leaves densely rosulate, usually spinulose-serrate. Inflorescence scapose, compound. Spikes densely strobiliform, sessile or stipitate. Flowers sessile, compressed, mostly hidden by the floral bracts. Sepals strongly asymmetric, usually free. Petals nearly or quite free, naked. Pollen with 2 or 4 pores. Epigynous tube small or none. Ovules several to many in each cell.

Fruit dry, scarcely enlarged from the ovary.—Schult. f. in R. & S. Syst. vii. p. lxxi, 1251 (1830).

Twenty-seven species. Terrestrial and epiphytic. The Antilles, Venezuela, Guiana and Brazil.

1. **H. oligosphaera** (Bak.) Mez. Leaves about 1 m. long, 75 mm. wide, abruptly acute or rounded-apiculate, spines 1–3 mm. long, independent sclerenchyma bundles present: inflorescence tripinnate; primary bracts reflexed, broadly triangular, much exceeding the lower branches; spikes subglobose, stipitate; floral bracts with suborbicular base, contracted into a long-acuminate apex, 25–30 mm. long, purple: sepals acute, glabrous, carinate, 15–18 mm. long, the posterior ones distinctly longer than the anterior; petals acute, violet-purple; anthers acute; ovary glabrous.—Mez in DC. Mon. Phan. ix. 124 (1896). *Aechmea oligosphaera* Bak. Brom. 48 (1889). Pl. I, fig. 8.

TRINIDAD: Hort. Trin., 1890, *Broadway 3961* (Trin); Moruga, Edward Trace, 1916, *Broadway 8203* (Trin); forest, vicinity of Tabaquite, 1921, *N. L. Britton, Freeman & Nowell 2581* (NY, Trin, G). TOBAGO: Mason Hall, near the river, 1913, *Broadway 4517* (FM); 1914, *Broadway 4834* (US).
[Venezuela.]

Epiphytic.

10. WITTMACKIA Mez.

Large showy herbs. Leaves rosulate, abruptly acute or rounded, closely serrate. Scape central, often decurved, its bracts membranaceous. Inflorescence laxly bipinnate, composed of several elongate laxly polystichous spikes. Floral bracts minute, narrowly acuminate. Sepals nearly or quite free, mucronate, the right side extended into a broad membranaceous wing. Petals free, narrowly acute, recurved at anthesis, eligulate. Ovary glabrous, sessile, epigynous tube short. Style longer than the stamens. Ovules crowded, obtuse. Fruit dry, scarcely changed from the ovary. Seeds many, curved, obtuse, naked. —Mez in Mart. Fl. Bras. iii. pt. 3, 274 (1891).

Three species. The Antilles and Costa Rica to Guiana and Brazil.

1. **W. lingulata** (L.) Mez. Leaves about 1 m. long, 5–10 cm. wide, rounded-apiculate, finely spinulose-serrate: scape slender, at first light-floccose, later glabrous; scape-bracts membranaceous, lanceolate, entire or sparingly fine-dentate: inflorescence ample, its branches suberect to spreading and often arching; primary bracts spreading; floral bracts narrowed from a broadly ovate base into a bristle-like point, about equaling the ovary: flowers 12–18 mm. long, sessile; sepals glabrous, 3 mm. long without the terminal spine; petals 7–10

mm. long, acute; stamens shorter than the petals, anthers mucronate; ovary ellipsoid-cylindric to clavate, 7 mm. long.—Mez in Mart. Fl. Bras. iii. pt. 3, 275 (1891). *Bromelia lingulata* L., Sp. Pl. 285 (1753). *Billbergia odora* Miq. in Linnaea, xviii. 377 (1844). *Chevalliera lingulata* Griseb. Fl. Br. W. Ind. 591 (1864). *Aechmea lingulata* Bak. in Journ. Bot. xvii. 164 (1879). *Wittmackia odora* Mez in Mart. Fl. Bras. iii. pt. 3, 277 (1891).

TRINIDAD: Marabella, 1865, *Crueger 2000* (Trin); Santa Cruz Valley, 1874, *O. Kuntze 662* (NY); Moruga, Edward Trace, 1916, *Broadway 8204* (Trin); base of Mt. Tamana, 1920, *Britton, N. L. & E. G., & Hazen 1962* (NY, G); *Freeman 9070* (Trin); indefinite, *Fendler 833; Prestoe; Broadway 3236* (Mez!). TOBAGO: Mason Hall, near the river, 1914, *Broadway 4829* (NY, Bk); indefinite, *Eggers 5741; Meyer* (Mez!).

[Lesser Antilles, Costa Rica, Guiana, northern Brazil.]

Epiphytic.

11. **GRAVISIA** Mez.

Stemless plants with densely rosulate, linear, spinulose leaves. Scape central, large, its bracts entire. Inflorescence twice or thrice pinnate, many-flowered. Sepals free, erect, asymmetric, short-mucronate. Petals free, erect, bearing two coarsely crenate or deeply incised scales near the base. Stamens included, the first series alternate with the petals and free, the second more or less fused to the petals. Pollen grains with more than 5 pores. Ovary sessile, epigynous tube conspicuous. Ovules many, caudate, borne on the upper half of the placenta.—Mez in Mart. Fl. Bras. iii. pt. 3, 299 (1892).

Eight species. Antilles, Costa Rica, and northern South America.

KEY TO SPECIES.

Inflorescence dense, subglobose: primary bracts broad, suborbicular. 1. *G. aripensis.*
Inflorescence elongate with its lower branches remote: primary bracts lanceolate. . . . . 2. *G. aquilega.*

1. **G. aripensis** N. E. Brown. Leaves 6–10 dm. long; blade abruptly acute, mucronate, 5 cm. wide, teeth 1 mm. long: scape erect, 7–10 mm. thick, densely brown-lanate; scape-bracts slightly imbricate, broadly ovate, bright pink, soon glabrous: inflorescence dense, subglobose, 7–8 cm. long; primary bracts resembling the scape-bracts, suborbicular, almost entirely concealing the axillary branches; branches obscurely compound, densely fasciculate, many-flowered: floral bracts ovate-lanceolate, acute with a terminal mucro 3–4 mm. long, brown-lanate: sepals 15 mm. long, brown-lanate; petals exceed-

ing the sepals, deep blue, mucronate.—N. E. Brown in Bull. Torr. Bot. Club, liii. 466 (1926). Pl. II, figs. 13–14.

TRINIDAD: slopes of Aripo, 1921, *Freeman & Broadway* (Trin); Heights of Aripo, 1922, *Broadway 9917* (Trin, TYPE; phot. G); 1928, *Broadway 6894* (Mo); indefinite, 1862, *Crueger 1748* (Trin).
[Endemic.]

Epiphytic. On the basis of *Crueger 1748*, Mez records *Gravisia exsudans* (Lodd.) Mez from Trinidad, but a comparison of this specimen with the original plate by Loddiges shows the same differences as between it and *G. aquilega* as indicated in the foregoing key.

2. **G. aquilega** (Salisb.) Mez. Plant 1 m. or more high: leaves 1–1.5 m. long, 5–10 cm. wide: scape stout, densely farinose: inflorescence of 10–20 stipitate heads, interrupted-thyrsoid, 4 dm. long; primary bracts lanceolate, broad, reflexed, bright pink, the lower ones much exceeding the axillary heads; floral bracts ovate, pungent, shorter than the sepals, glabrous: flowers 36 mm. long, glabrous; sepals 14 mm. long, pungent; petals mucronate, flavous; ovary 10 mm. long, trigonous.—Mez in Mart. Fl. Bras. iii. pt. 3, 300 (1892). *Bromelia aquilega* Salisb. Parad. Lond. t. 40 (1805). *Aechmea aquilega* Griseb. Fl. Br. W. Ind. 592 (1864).

TRINIDAD: Marabella, 1868, *Finlay 1999* (Trin); Monos, 1889, *Broadway 3509* (Trin); Hort. Trin., 1889, *Broadway 3631* (Trin); 1890, *Broadway 3962* (Trin); Chaguanas, 1890, *Herb. Trin. 4213* (Trin); Aripo Savannah, 1893, *Broadway 5341* (Trin); Hort. Trin., *Herb. Trin. 5571* (Trin); Patos, 1917, *Williams* (Trin); 1924, *Williams, R. O. & A. A. 10716* (Trin); Teteron Bay, 1920, *N. L. Britton 481* (G, NY); Manzanilla, 1921, *Britton, N. L. & E. G. 2192* (G, NY); Oropuche, via Valencia, 1925, *Hombersley* (Mo); indefinite, *Fendler 841 in part, Broadway 2187* (Mez!). TOBAGO: Caledonia, *Herb. Trin. 5572* (Trin); Government House grounds, 1910, *Broadway 3639* (NY); indefinite, *Eggers 5677* (NY); *Meyer* (Mez!).
[Venezuela, Guiana, northern Brazil.]

Epiphytic.

12. AECHMEA R. & P.

Large or medium-sized stemless herbs. Leaves rosulate, usually linear. Scape usually conspicuous. Inflorescence of various types. Flowers usually sessile. Sepals often asymmetric, usually mucronate. Petals free, bearing two scales near the base or high up. Second series of stamens more or less joined to the petals. Pollen grains with 2 or 4 pores. Style shorter than the stamens, stigma lobes linear, often twisted. Ovules caudate or obtuse. Fruit usually dry. Seeds small, rugose, dark-colored, naked.—Fl. Peruv. Prodr. 47, t. 8 (1794).

Apparently 120 to 130 species. Mexico and the Antilles to northern Argentina.

KEY TO SPECIES.

1. Inflorescence compound.
  2. Sepals wholly unarmed: branches of the inflorescence polystichous-flowered........................Subgenus *Lamprococcus*.
    1. *Ae. fulgens*.
  2. Sepals mucronate at apex: branches of the inflorescence distichous-flowered.
    3. Margins of the floral bracts free from the rhachis. Subgenus *Euaechmea*.
      4. Floral bracts large, forming a tubular sheath about the ovary: branches of the inflorescence short and densely flowered........................2. *Ae. Mertensii*.
      4. Floral bracts minute, much shorter than the ovary, nearly flat: branches of the inflorescence long and laxly flowered.............................3. *Ae. porteoides*.
    3. Margins of the floral bracts joined to the rhachis: rhachis excavated........................Subgenus *Platyaechmea*.
      5. Spikes long-stipitate...........................4. *Ae. dichlamydea*.
      5. Spikes sessile or very short-stipitate..............Var. *trinitensis*.
1. Inflorescence simple, spicate.
  6. Sepals mucronate at apex, usually free............Subgenus *Pothuava*.
    5. *Ae. nudicaulis*.
  6. Sepals unarmed..........................Subgenus *Macrochordium*.
    6. *Ae. bromeliaefolia*.

1. **Ae. fulgens** Brongn. Leaves densely rosulate, 4 dm. long, 65 mm. wide, subensiform, minutely serrulate, rounded-apiculate: scape erect, glabrous, its bracts lanceolate, entire, roseate: inflorescence compound at base, simple for the upper half, about equalling the leaves, glabrous, red: spikes loosely 3–8-flowered, polystichous; floral bracts none: sepals 5 mm. long, rounded-emarginate, asymmetric, violet; petals 12 mm. long, erect, rounded, ligulate at base, violet changing to red; ovary ellipsoid, 7 mm. thick, epigynous tube conspicuous; ovules long-caudate, borne high in the ovary: berry scarcely enlarged from the ovary.—Brongn. in Ann. Sc. Nat. ser. 2, xv. 371 (1841), as *Aechmaea*.

TRINIDAD: Hort. Trin., 1888, *Broadway 2990* (Trin); 1890, *Broadway 4354* (Trin).
[Northern Brazil.]

Epiphytic. Introduced in cultivation.

2. **Ae. Mertensii** (Meyer) Schult. f. Leaves bulbous-rosulate, 2–6 dm. long; sheath large, ovate; blade 2–5 cm. wide, coarsely spinose-serrate, abruptly acute, mucronate: scape erect, bearing at its apex a tuft of reflexed lanceolate red remotely serrulate bracts: inflorescence at first dense, often becoming lax at maturity, 5–20 cm. long, 4–5 cm. wide, branches densely 4–12-flowered; floral bracts broadly reniform, long-spinose, enfolding the ovary: flowers 12–17 mm. long; sepals free, mucronate, 4–6.5 mm. long; petals yellow,

erect, mucronate, 11 mm. long, their scales fimbriate; ovary subclavate, glabrous, epigynous tube short but distinct; ovules few, long-caudate, borne high in the ovary.—Schult. f. in R. & S. Syst. vii. 1272 (1830). *Bromelia Mertensii* Meyer, Fl. Essequeb. 144 (1818). *Aechmea spicata* Mart. in R. & S. Syst. vii. 1273 (1830). *Ae. mucroniflora* Hook. in Bot. Mag. lxxxi. t. 4832 (1855).

TRINIDAD: Chaguanas, 1868, *Finlay 1998* (Trin); 1889, *Broadway 3714* (Trin); Piarco Savanna, 1893, *Broadway 5287* (Trin); Arima, 1905, *Dannouse* (NY); Princes Town, 1918, *Broadway* (Trin); Tabaquite, 1918, *Broadway* (NY); Dabadie, 1920, *Broadway* (NY); Valencia, 1920, *Britton, N. L. &. E. G., & Hazen 1010* (NY); St. Ann's, 1924, *Broadway* (FM, Mo); Laventille upper lands, 1930, *Broadway 7745* (G); indefinite, *Fendler 842, 843* (K); 1930, *Broadway 7431* (G); *2607* (Mez!).
[Guiana, Amazon Basin.]

Epiphytic.

3. **Ae. porteoides** Britton. About 1 m. high: leaves firm, 8 dm. long; sheath large, broadly elliptic; blade linear, 4–5 cm. wide, acute, spinulose-serrate, the spines 2–4 mm. long, black: scape arching (Broadway), terete, densely white-floccose; its bracts broadly lanceolate, membranaceous, pink, deciduous: inflorescence paniculate, equaling or exceeding the leaves, cylindric, very delicately white-stellate-tomentose throughout; lower primary bracts like the scape-bracts, often deciduous, upper primary bracts small and acicular, little larger than the floral bracts; branches fasciculate, spreading, with long sterile base, slender, very laxly few-flowered, strongly geniculate, 6–8 cm. long; floral bracts long-aciculose from a minute ovate base, much shorter than the ovary: flowers 2–3 cm. long, erect or nearly so, sessile; sepals 8–10 mm. long, strongly asymmetric, striate, mucronate with a spine 2–3 mm. long; petals deep violet-purple, 18 mm. long, with two nearly entire scales at base; ovary about 1 cm. long, much enlarged in fruit.—Bull. Torr. Bot. Club, xlviii. 328 (1922). Pl. II, figs. 7–10.

TRINIDAD: Mt. Tucuche, 1893, *Broadway 1997* (Trin); 1920, *N. L. Britton, Hazen & Mendelson 1342* (NY, TYPE, Trin); Heights of Aripo, 1922, *Broadway 9915* (Trin, G, NY); St. Clair, 1913, *P. McLean* (Trin); Morne Bleu, 1921, *N. L. Britton, Freeman & Bailey 2274* (NY); Blanchisseuse Road, 1925, *Broadway 5590* (G, FM); 1926, *Broadway 6008* (Mo); Ortinola Estate, Maracas, 1931, *Broadway 7460* (G); indefinite, *Herb. Trin. 1751, 6620* (Trin).
[Endemic.]

Epiphytic and terrestrial.

4. **Ae. dichlamydea** Bak. Half a meter or more high: leaves 6–8 dm. long, 4–7 cm. wide above the broad subovate base, lingulate, broadly acute, serrulate: scape stout, erect, its bracts lanceolate,

entire, bright red, showy: inflorescence ample, laxly bipinnate, tomentose-lepidote; primary bracts like the scape-bracts, mostly shorter than the sterile bases of the spikes: spikes long-stipitate, distichous, up to 16 cm. long and 3 cm. broad, 5–20-flowered, the rhachis excavated and attached to the sides of the floral bracts; floral bracts boat-shaped but not keeled, strongly nerved, 12–15 mm. long, short-mucronate: flowers sessile; sepals nearly free, mucronate; petals yellow: berry globose, 7 mm. in diameter.—Bak. in Journ. Bot. xvii. 133 (1879). *Ae. bracteata* as used by Bak. Brom. 55 (1889), not Griseb.

TOBAGO: Lot 42, 1913, *Broadway* (G, Mo); road from Charlotteville to Speyside, 1925, *Williams 11142* (Trin); Menna, 1914, *Broadway* (Mo); indefinite, *Greg* (BM, TYPE, phot. G); *Meyer* (Baker!); *Clemens 29; Eggers 5872* (Mez!).
[Jamaica, the Lesser Antilles.]

Epiphytic.

Var. **trinitensis** L. B. Smith (see p. 145). Sterile bases of the spikes short and stout. Pl. I, fig. 3.

TRINIDAD: Hort. Trin., 1890, *Hart 4089* (Trin, TYPE, phot. G); 1889, *Broadway 3627* (Trin); indefinite, 1867, *Prestoe* (K); *Herb. Trin. 916* (Trin).
[Endemic.]

Epiphytic.

5. **Ae. nudicaulis** (L.) Griseb. Leaves 5 dm. long, 6–10 cm. broad above the narrow sheath, linear, apiculate, strongly spinose-serrate: scape slender, erect or arching, white-floccose, its bracts erect, elliptic-acute, entire, red, membranaceous, congested at its apex: inflorescence simple, polystichous, cylindric, fertile throughout; floral bracts minute or sometimes wanting, entire: flowers 22 mm. long; sepals free, mucronate, strongly asymmetric; petals 12 mm. long, yellow, acute, bearing the fimbriate scales 4 mm. from the base; ovary subglobose, epigynous tube conspicuous; ovules short-caudate, borne at the middle of the placenta.—Fl. Br. West Ind. 593 (1864). *Bromelia nudicaulis* L. Sp. Pl. 286 (1753).

TRINIDAD: St. Ann's, 1923, *Broadway* (FM); Chaguanas, 1889, *Broadway 3730* (Trin); Hort. Trin., 1889, *Broadway 3630, 3513* (Trin); Marabella Pasture, 1868, *Herb. Trin. 2001* (Trin); indefinite, *Herb. Trin. 2422* (Trin); 1874, *O. Kuntze* (NY); *De Schach, Fendler 834, 835* (Baker!). TOBAGO: indefinite, *Broadway 4830* (Mez!).
[Mexico and the Antilles to southern Brazil.]

Epiphytic.

6. **Ae. bromeliaefolia** (Rudge) Bak. Leaves 6–10 dm. long, 5 cm. broad above the large ovate sheath, linear, usually rounded by decay of the point, coarsely spinose-serrate: scape stout, densely

white-lanate, its bracts ovate-lanceolate, entire, densely imbricate: inflorescence simple, polystichous, densely cylindric, 7–15 cm. long, 35 mm. wide, fertile throughout, densely white-lanate; floral bracts broader than long, truncate, thickened, 2-keeled: flowers 17 mm. long; sepals 7 mm. long, joined at base for 2 mm., emarginate; petals 15 mm. long, erect, emarginate, greenish-yellow soon turning black, scales borne 5 mm. from the base, fimbriate; ovary tomentose; ovules borne high in the ovary, long-caudate.—Bak. in Benth. & Hook. f. Gen. iii. 664 (1883). *Tillandsia bromeliaefolia* Rudge, Pl. Guian. 32, t. 50 (1805). *Macrochordium melananthum* Beer, Brom. 146 (1857).

TRINIDAD: Cabacera de Aripo, 1869, *Herb. Trin. 2002* (Trin); 1873, *Herb. Trin. 698* (Trin); Arima, 1874, *O. Kuntze 988* (NY); Carapichaima, 1892, *Alexander 5723* (Trin); Balandra, 1922, *Freeman 10505* (Trin); indefinite, *Fendler 837; Prestoe* (Baker!).
[Venezuela, Guiana, Brazil, Paraguay, Bolivia, Argentina.]

Terrestrial or epiphytic in forest.

13. **ANANAS** Mill. PINEAPPLE.

Leaves densely rosulate, scarcely enlarged at base, acuminate-pungent, spinose-serrate. Scape about equaling the leaves, spiny-bracted. Inflorescence densely strobiliform, crowned with a tuft of sterile bracts. Flowers sessile, violet or red. Sepals free, obtuse, slightly asymmetric. Petals free, erect, ligulate. Stamens included, pollen grains ellipsoid, with 2 pores. Ovary coalescing with the bracts and axis to form a fleshy compound fruit, epigynous tube short. Ovules borne near top of cell. Berry sterile in cultivated forms.—Gard. Dict. Abr. ed. iv. (1754).

A single rather variable species, native of Brazil and probably also the Guianas but widely cultivated.

1. **A. sativus** Schult. f. Typical specimens small, with fruit only slightly fleshy; cultivated specimens quite large, with a succulent fruit and aborted ovules.—R. & S. Syst. vii. 1283 (1830). *Bromelia Ananas* L. Sp. Pl. 285 (1753). *Ananassa sativa* Lindl. in Bot. Reg. xiii. sub t. 1068 (1827).

TRINIDAD: vicinity of Valencia, 1921, *Britton, N. L. & E. G. 2107* (NY).

Terrestrial; introduced and sparingly cultivated in Trinidad. Naturalized in the Mora Forest district.

14. **BILLBERGIA** Thunb.

Stemless plants. Leaves rosulate or rarely fasciculate, sheath large, blade linear, spinulose-serrate, often banded. Scape erect or arching, its bracts red, membranaceous. Inflorescence simple or compound. Flowers showy. Sepals free, erect, usually obtuse. Petals free, actinomorphic or zygomorphic, bearing two scales, claw long, blade elliptic. Stamens exserted at anthesis, both series free, or the second joined to the petals only as high as the scales. Pollen grains with longitudinal folds, but no pores. Style exceeding the stamens, epigynous tube large. Ovules many, obtuse.—Decad. Pl. Bras. iii. 30 (1823).

About 50 species. Chiefly Brazilian with outlying species in the rest of South America and in Central America and Trinidad.

KEY TO SPECIES.

Inflorescence cylindric: petals becoming spirally recurved at anthesis: ovary verrucose..1. *B. rosea.*
Inflorescence densely pyramidal: petals twisted, not spiral at anthesis: ovary not verrucose..........................2. *B. pyramidalis.*

1. **B. rosea** Beer. Leaves 1 m. long, 50 mm. broad above the narrowly ovate brown sheath, densely and finely spinulose-serrate, conspicuously spotted: scape slender, nutant, densely white-farinose, its bracts ovate-lanceolate: inflorescence simple, spicate, many-flowered, white-farinose throughout; bracts 4 mm. long, semiorbicular, obtuse: flowers sessile, 85 mm. long, erect; sepals narrowly triangular, 12, 10, and 10 mm. long respectively, verrucose; petals 65 mm. long, acute, with crenate scales, yellowish green; ovary broadly ovoid, densely granulose: berry yellow, globose, 10 mm. in diameter.—Brom. 128 (1857).

TRINIDAD: indefinite, Hort. Kew (Mez!).
[Dutch Guiana.]

Introduced in cultivation.

2. **B. pyramidalis** (Sims) Lindl. Leaves tubular-rosulate, 1 m. long, densely and finely spinulose-serrate, 6 cm. broad above the ovate base, often spotted: scape erect, white-farinose, its bracts lanceolate: inflorescence simple, pyramidal, 15 cm. long, white-farinose, rhachis stout, straight; floral bracts minute, acute: flowers 65 mm. long, obscurely short-stipitate; sepals 17 mm. long, pale reddish, densely farinose; petals 52 mm. long, obtuse, red, twisted at anthesis, scales fimbriate; anthers 7 mm. long, linear; ovary subcylindric, densely farinose, epigynous tube short.—Bot. Reg. xiii. sub t. 1068 (1827). *Bromelia pyramidalis* Sims in Bot. Mag. xlii. t. 1732 (1815).

TRINIDAD: Hort. Trin., 1890, *Broadway 4405* (Trin); St. Joseph, cultivated, 1917, *H. Fuller 8951* (Trin).
[Eastern Brazil.]

Introduced in cultivation.

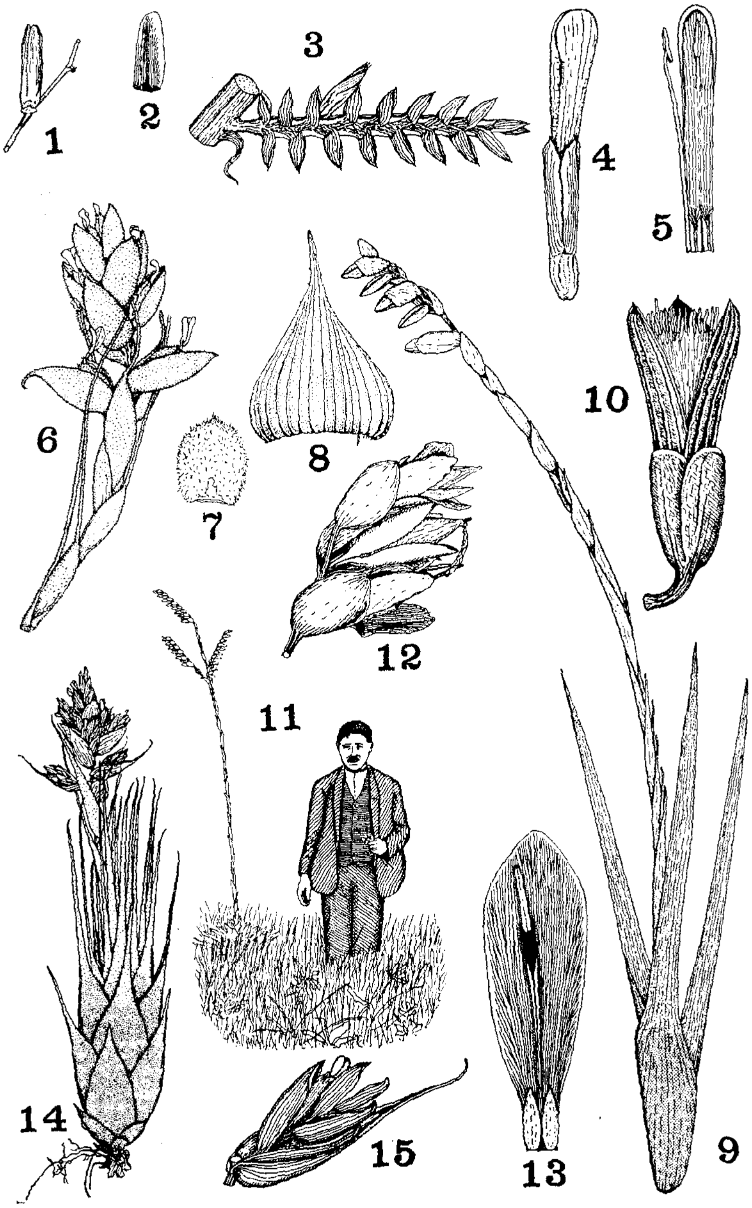
EXPLANATION OF PLATES.

PLATE I.

Fig. 1. DEUTEROCOHNIA MEZIANA O. Ktze. (*O. Kuntze*), flower and section of branchlet × 1.
2. Same, sepal × 1.
3. AECHMEA DICHLAMYDEA Bak. var. TRINITENSIS L. B. Smith (*Hart in hb. Trin. 4089*), spike × ½.
4. CANISTRUM MINUTUM (Mez) L. B. Smith (*L. B. Smith, Hoehne & Kuhlmann 1829*), flower × 1.
5. Same, petal and stamens × 1.
6. TILLANDSIA LATISEPALA L. B. Smith (*F. L. Herrera 3313*), inflorescence × ½.
7. Same, sepal × 1.
8. HOHENBERGIA OLIGOSPHAERA (Bak.) Mez (*N. L. Britton, Freeman & Nowell 2581*), floral bract × 1.
9. VRIESIA BROADWAYI L. B. Smith (*Broadway in hb. Trin. 9916*), habit × ¼.
10. Same (*N. L. Britton, Hazen & Mendelson 1258*), flower and capsule × 1.
11. VRIESIA HOEHNEANA L. B. Smith (designed from a photograph), habit much reduced.
12. Same (*L. B. Smith 1945*), terminal section of spike × ½.
13. Same (*Hoehne in hb. S. P. 4724*), petal and stamen × 1.
14. TILLANDSIA ARCHERI L. B. Smith (*W. A. Archer 1204*), habit × ¼.
15. Same, spike × 1.

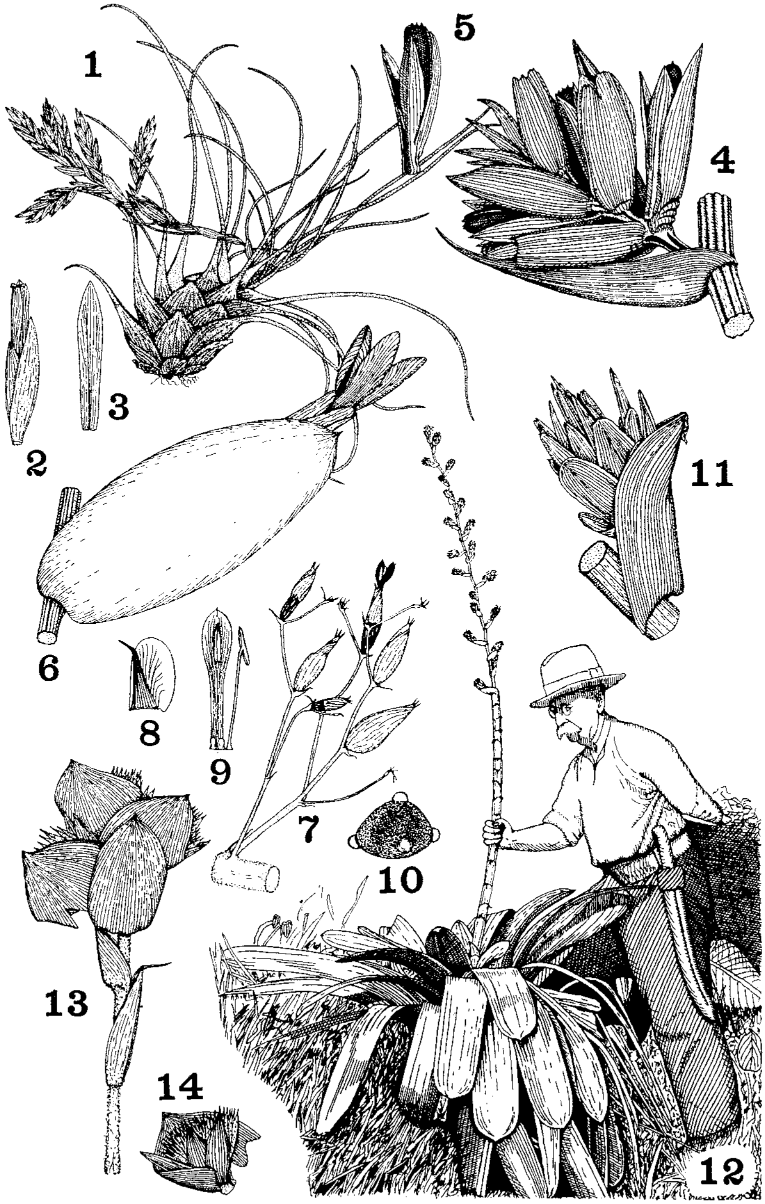
PLATE II.

Fig. 1. TILLANDSIA DIDISTICHOIDES Mez (*Fendler 827*), habit × ¼.
2. Same (*Broadway 7433*), floral bract and flower × 1.
3. Same, petal and stamen × 1.
4. GUZMANIA PLICATIFOLIA L. B. Smith (*Tonduz in inst. phys.-geogr. nat. Costaric. 12, 533*), spike × 1.
5. Same, flower and capsule × 1.
6. GUZMANIA ROSEA L. B. Smith (*Lehmann K 355*), floral bract and flower × 1.
7. AECHMEA PORTEOIDES Britton (*Broadway 7460*), fascicle of branches × ½.
8. Same, sepal × 1.
9. Same, petal and stamens × 1.
10. Same, pollen grain × 250.
11. GLOMEROPITCAIRNIA ERECTIFLORA Mez (*J. R. Johnston 303*), spike × ½.
12. Same (designed from a photograph), habit much reduced.
13. GRAVISIA ARIPENSIS N. E. Brown (*Broadway 9917*), habit × ¼.
14. Same, rear view of branch × ¼.



Figs. 1–2, Deuterocohnia Meziana O. Ktze.; 3, Aechmea dichlamydea Bak. var. trinitensis L. B. Sm.; 4–5, Canistrum minutum (Mez) L. B. Sm.; 6–7, Tillandsia latisepala L. B. Sm.; 8, Hohenbergia oligosphaera (Bak.) Mez; 9–10, Vriesia Broadwayi L. B. Sm.; 11–13, V. Hoehneana L. B. Sm.; 14–15, Tillandsia Archeri L. B. Sm.

Plate II.



Figs. 1–3, Tillandsia didistichoides Mez; 4–5, Guzmania plicatifolia L. B. Sm.; 6, G. rosea L. B. Sm.; 7–10, Aechmea porteoides Britton; 11–12, Glomeropitcairnia erectiflora Mez; 13–14, Gravisia aripensis N. E. Brown.

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A SUGGESTION CONCERNING PLATO'S ATLANTIS.

BY W. A. HEIDEL.

A SUGGESTION CONCERNING PLATO'S ATLANTIS.

By W. A. Heidel.

So much has been written about Atlantis in ancient and modern times that one hesitates to take up again the hackneyed theme. In doing so I plead the only possible justification, that I have a new suggestion to offer which, though incapable of proof, is at least reasonable and in accord with the context of Plato's thought.

The relation to Plato's thought is of prime importance; for, as Henri Martin[1] has shown, there are in ancient times no references or allusions to Atlantis except such as are based on the statements of Plato in the *Timaeus* and the *Critias*. Therefore the large number of interesting hypotheses which either ignore this fact or find their support in modern geological discoveries, which Plato could not possibly either know or surmise, may be set aside for the present. Many of them have unquestionable value, but not for Plato.

Plato[2] represents the tale of Atlantis as told to Solon in Egypt by the priests of Sais, who professed to have documentary evidence for it in ancient records going back to the foundation of their city. These records did not confine themselves to the history of their own city and land, but included the notable deeds of all lands,[3]—in other words, they were a sort of chrestomathy, reminding one of the conception of the Father of History:[4] "What Herodotus the Halicarnassian has learnt by inquiry is here set forth: in order that so the memory of the past may not be blotted out from among men by time, and that great and marvellous deeds done by Greeks and foreigners and especially why they warred against each other may not lack renown." Is it not strange that Egyptologists have found no evidence of universal history among the records of Egypt? Solon is said to have brought the tale to Athens where Dropides recorded it and passed it on to the elder Critias who in turn told it as a 'tale of a grandfather' to the Critias of Plato's dialogues. Is it not strange that there is no other allusion to this precious story? There is less trace of it than of the epic which Homer wrote, relating the battle on the Isles of the Blest between the Heroes gathered there and the Impious Damned, of which we read in Lucian's *True Story*. Of the latter, though the remainder was lost, Lucian has at least preserved the first line. Again, Plato tells us that the primitive Athens was founded, and the glorious war between that state and Atlantis was fought, not less than nine thousand years before Solon's time, while Sais was founded a full thousand years later. Supposing that the alleged record was contemporary with the founding of the city, it was thus long subsequent to the events and to the utter destruction of the states concerned. Needless to say that existing Egyptian records give no hint of a tradition regarding either Atlantis or primitive Athens.

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[1] T. Henri Martin, *Études sur le Timée de Platon*, pp. 319 ff.

[2] *Tim.* 21 e ff.

[3] One suspects that Tacitus, like Plato on occasion, enjoyed puncturing the inflated reputation of the Egyptian priests for superior wisdom and knowledge of affairs beyond their own borders, when he said (*Hist.* IV, 83) they were *Ponti et externorum parum gnari*.

[4] Herod. I, 1. I generally quote the translation of Godley, occasionally adapted.

Could Plato have more clearly indicated that what was to be related was fiction? What has imposed on some scholars and led them to accept, or at least not to reject outright, the historical character of Plato's account, is the fact that he appealed to the authority of the Egyptian priests. Before the Egyptian records were deciphered and their character made known the awed respect generally accorded to everything purporting to derive from that fountain of wisdom might in a measure be excused. It is more difficult to make allowances for the attitude of some of the most reputable scholars of to-day, who, while compelled to acknowledge a certain Hellenic color in the supposed priest-tales, obstinately close their minds to the obvious indications of the tradition. This is not the place to discuss this problem in detail; but after a comprehensive study for more than a dozen years of the entire Greek tradition regarding Egypt, and especially regarding the Egyptian priests, I do not hesitate to say that it was from beginning to end the vehicle of Greek speculations.[5] It is still possible to discover some of the motives of the early writers who resorted to this fiction. Suffice it for the present to say that one motive was to give authority to hypotheses and ideals that the thoughtful Greeks had formed; and there is good evidence that Plato and other men of intelligence in the fourth century were well aware that the appeal to the Egyptian priests was a convenient fiction. This does not, of course, imply either that the point at issue is false or that the views expressed are without foundation. In the present case the appeal to the Egyptian priests has special significance; for it gives us a hint as to the quarter in which we are to look for the data that suggested the fiction. We shall do well to examine more closely Plato's statements regarding Atlantis.

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[5] I hope shortly to publish a study of the priest-tales in the Second Book of Herodotus.

We must recall, in the first place, that in the beginning of the *Timaeus*[6] Socrates remarks that he desires to see the state therein described set in motion and functioning in a crisis such as the wars which all states are called upon to wage. It is the state of which Plato had confessed[7] that it mattered not at all whether it ever existed or would exist otherwise than as a pattern laid up in heaven. Critias is to satisfy the desire of Socrates in the dialogue which bears his name, and he gives his credentials, so to speak, by telling how he came by the story of Atlantis; but before he takes up his proper theme the philosopher Timaeus is introduced. The latter proceeds to set forth an elaborate philosophy in the dialogue named after him; but by way of preface Critias says, "Let me proceed to explain to you, Socrates, the order in which we have arranged our entertainment. Our intention is, that Timaeus, who is the most of an astronomer amongst us, and has made the nature of the universe his special study, should speak first, beginning with the generation of the world and going down to the creation of man; next, I am to receive the men whom he has created, and of whom some will have profited by the excellent education which you have given them; and then, in accordance with the tale of Solon, and equally with his law, we will bring them into court and make them citizens, as if they were those very Athenians whom the sacred Egyptian record has recovered from oblivion, and henceforward we will speak of them as Athenians and fellow-citizens." The obvious implication of this statement is made even plainer in words just before[8] uttered by Critias: "And now, Socrates, to make an end of my preface, I am ready to tell you the whole tale. I will give you not only the general heads, but the particulars as they were told to me. The city and citizens, which you yesterday described as in fiction, we will now transfer to the world of reality. It shall be the ancient city of Athens, and we will suppose that the citizens whom you imagined were our veritable ancestors, of whom the priest spoke; they will perfectly harmonize, and there will be no inconsistency in saying that the citizens of your republic are these ancient Athenians."

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[6] 19 b ff.

[7] *Repub.* 592 a f.

[8] *Tim.* 26 e d, tr. Jowett.

Plato here again as clearly as possible indicates the fictitious character of the narrative[9] which Critias is to offer for the entertainment of Socrates. There is, however, something even more significant in the statement just quoted. Though there is an evident playfulness in the form, the program sketched is striking enough to arrest attention. We naturally ask in what relation the promise stands to the fulfilment? So far as the *Timaeus* is concerned the extremely brief statement of the program may be said to suffice, although we have in this dialogue a complete, though sketchy and poetically conceived treatise Περὶ Φύσεως. As for the task reserved for Critias, the incomplete nature of his contribution makes it uncertain just what it would have included. But certain things are clear in regard to both works. To speak first of the *Timaeus*, the world is said to have been created by the uncreated God, who brought into existence four species of 'animals' (animate beings) corresponding to the four elements; first of all the created gods (the fixed stars and planets), answering to fire; then birds, sea and land animals (including man). The traditional gods of mythology are mentioned, but their nature is not discussed. From men spring women and other animals.[10] So far the scheme includes, though in a different spirit, the subject matter not only of the treatises Περὶ Φύσεως, but also what the poets offered in theogonies and catalogues, bridging the interval between the gods and mortal man. One can also discern the ancient division which marked two periods in the existence of, say, man,—the first of supernatural or spontaneous origin, the second by natural generation.[11] In the *Critias*, as we shall see, the interest centres in history and geography. This division of the field of science is especially noteworthy because it is obviously not an afterthought, but part of a considered program. How fully it would have been realized, if the plan had been brought to completion, we have no means of knowing.

But here arises another question, Did this scheme originate with Plato? The answer must be emphatically in the negative; for it is plainly involved not only in the Hesiodic poetry, to which allusion has been already made, and in the genealogists who wrote in prose, but most clearly in such works as the *Genealogies* and the *Tour of the Earth* by Hecataeus of Miletus. Regarding this matter I have elsewhere[12] written at some length and do not wish here to repeat what I then said; but it is important to draw attention here to the clear parallel in the program of Plato, because the *Timaeus* and the *Critias,* taken together, are clearly intended to cover, in Plato's own way, the field of early scientific thought. That being the case, the place of history and geography in the scheme is made obvious; and we are the better prepared to see where Plato got the inspiration for the tale of Atlantis.

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[9] One naturally thinks of the *Phaedrus*, 274 c ff, where Plato in his story of Thoth turns the tables on the supposed wisdom of the Egyptians derived from their ancient written records. Phaedrus then says, 275 b ὦ Σώκρατες, ῥᾳδίως σὺ Αἰγυπτίους καὶ ὁποδαποὺς ἂν ἐθέλῃς λόγους ποιεῖς.

[10] This is unmistakably a playful suggestion.

[11] ἐξ ἀλλήλων.

The tale told by Critias is too long to repeat here, but a summary of the introduction is desirable. Solon, we are told, went to Sais, a city situated at the apex of the Delta of the Nile, whose tutelary deity, the goddess Neith, was identified with Athena, the patroness of Athens. There he was received with great honor by the priests, both because of long-standing friendship between their respective cities and because of the goddess whom they served in common. Solon, who, as tradition said, had gone abroad for the sake of speculation,[13] sought out the priests who were most versed in ancient lore and in his conversation with them discovered that the Greeks had no knowledge worth mentioning about antiquity. Wishing to draw out his hosts on that subject he told the story of the deluge of Deucalion, and by tracing the genealogy of his descendants tried to reckon up the number of years that had elapsed since that event. Thereupon an aged priest, very much amused at his apparent simplicity, told him that the Greeks were all children and their genealogies childish. He referred Solon to other events which, he said, were known also to the Greeks, such as the escapade of Phaëthon, which had resulted in a general conflagration of the world and an alteration[14] in the course of the sun. Such catastrophes recur periodically, alternating with cataclysms that purge the earth and destroy civilizations. In other lands there escape at such times only illiterate mountaineers who, having no written records, in a few generations lose all recollection of what had been before. The deluge of Deucalion was not the first,[15] but there were at least two earlier ones, in one of which primitive Athens and Atlantis were simultaneously overwhelmed. From all these catastrophes Egypt had escaped, thanks to her savior, the Nile.[16] The goddess Neith-Athena first founded Athens, receiving the seed of the race from Gê and Hephaestus, and afterwards founded Sais. The ideal institutions which she gave to the former city may still be seen existing in Egypt. Proclus, in his *Commentary on the Timaeus*,[17] reports that, according to Crantor (a member of the Academy who lived about the turn of the fourth and third centuries, and wrote the first commentary on the *Timaeus*), Plato was taunted by his contemporaries with not being the inventor of his ideal state (the *Republic*) but with copying the institutions of Egypt, in consequence of which Plato had referred to the Egyptian priests the story of the antediluvian Athenians and Atlantians and had represented them as once living under that form of government. Such a perversion was possible only if one took the history as true (Crantor thought Atlantis once existed); but that such a view of the *Republic* was possible shows, as do Plato's own words, that many of his own speculations on the ideal state were at least foreshadowed in the Greek tradition about Egypt.

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[12] *Anaximander's Book, the Earliest Known Geographical Treatise (Proceedings of the American Academy of Arts and Sciences,* Vol. 56, No. 7 (April, 1921), pp. 263 ff., 268 ff.) In writing then I unaccountably overlooked the scheme of the *Timaeus* and the *Critias,* which fully confirms what I there said.

[13] Herod. I, 29 κατὰ θεωρίης πρόφασιν, surely not merely 'to see the world.' Cf. Plato, *Phaedo* 77 e f.

[14] Plato says παράλλαξις, *Tim.* 22d, which Jowett translates "declination." A comparison with Herod. II, 142 and other passages where the phenomenon is referred to, shows what he meant.

One cannot contemplate this tale without thinking of Herodotus at every turn. The last statement at once arrests our attention. In Herodotus one finds everywhere expressed the conviction that Greek institutions were derived from Egypt: it is neatly to pay back the speculations which were based on the superior antiquity of Egypt by making primitive Athens a thousand years older than Sais and the institutions of Egypt a derivative from an antediluvian Athens. One sees at once that Plato is amusing himself, because he recognizes the speculative character of the Greek tradition regarding the land of the Nile. But that is not all. One cannot fail to note that there is a very close relation between the tale told by Critias and a passage in Herodotus, which likewise appeals to the records of the Egyptian priests. There[18] we read that they showed him that from the first human king, Menes, down to Sethos there were 341 generations, or

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[15] *Crit.* 112 a.

[16] One does not readily see how the Nile could save Egypt from either flood or a world-conflagration. Can one suppose that Plato took this seriously?

[17] 23 e.

[18] Herod. II, 142 f.

11340 years, "in all which time they had had no king who was a god in human form, nor had there been any such thing either before or after those years among the rest of the kings of Egypt.[19] Four times[20] in this period (they said) the sun rose contrary to his wont; twice he rose where he now sets and twice he set where he now rises; yet Egypt at these times underwent no change, neither in the produce of the river and the land, nor in the matter of sickness and death.[21] Hecataeus the historian was once at Thebes, where he made for himself a genealogy which connected him with a god in the sixteenth generation. But the priests did for him what they did for me (who had not traced my lineage). They brought me into the great inner court of the temple and showed me there wooden figures which they counted up to the number they had mentioned, for every high priest sets there in his life-time a statue of himself; counting and pointing to these the priests showed me that each inherited from his father,—they went through the whole tale of figures back to the earliest from that of him who had lateliest died. Thus when Hecataeus had traced his descent and claimed that his sixteenth forefather was a god, the priests too traced a line of descent according to the method of their counting; for they would not be persuaded by him that a man could be descended from a god: they traced descent through the whole line of 345 figures, not connecting it with any god or hero."

I hope before long to discuss at length the relation between these two stories. For the present purpose it will suffice to place them side by side and draw attention to the obvious fact that there is a relation between them without attempting to define it. There can be no doubt that Plato had in mind either the statement of Herodotus or that of Hecataeus to which reference is made. Though the account of Herodotus is vague, showing that he did not really understand the points involved, it has much in common with Plato's: there are the same cosmic periods and the same reversal of the course of nature. But while in Plato Egypt is expressly excepted from the general catastrophes, which Greek myths associated with Phaëthon, Deucalion, and Atreus, there is no mention of these myths in Herodotus; and while Egypt is said to have suffered no evil effects, the calamity which befel the rest of the world, though implied, is ignored. Again, Plato represents Solon as held up to ridicule by the priest for recounting Greek genealogies, but "saves his face" by saying that the wise Athenian had done so only for the purpose of drawing out the priests on the subject of ancient history, while Herodotus does his best to make Hecataeus really ridiculous in recounting his genealogy. Yet it was this same Hecataeus who began his *Genealogies* with the words, fortunately preserved,[22] "Thus saith Hecataeus the Milesian: I write this as it appears to me to be true; for the tales of the Greeks are, as it seems to me, many and ludicrous." He was presumably thinking above all of their genealogies: he found them many, that is to say divergent, and ridiculous. Is it likely that he took his own very seriously? It is to be observed that the space of time, not reduced to a definite number of generations, reckoned by Plato approximates the historical period which, according to Herodotus, the Egyptian priests verified for Hecataeus.

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[19] If an Egyptian priest said that, he was a heretic; for the orthodox view, from early times, was that the king was quite literally the "son of the Sun-god." Is it not obvious that we are dealing with the fiction of a Greek rationalist?

[20] Really, only *two* changes are implied. Herodotus here, as always where a question of science is involved, displays his want of comprehension.

[21] Note how vague this statement is! Did Herodotus understand what was meant? Modern editors give no evidence that they do.

Herodotus himself,[23] we know not on whose authority, reports that Solon had visited Egypt. He may have found a statement to that effect in Solon's own poems, though, if he did, one is surprised to find no reference to it in either Herodotus, Plato, or Aristotle. Plutarch, to be sure, quotes a line[24] from Solon in which his biographer says that Solon himself mentioned his voyage, but the quoted words only mention the Canobic mouth of the Nile. Whatever the fact,[25] Solon's visit to Egypt was part of the tradition about Egypt. The complex of motifs in the stories of Solon and Hecataeus in Egypt points unmistakably to the historico-geographical line of tradition, which concerned itself to a surprising extent with Egypt. We may be sure that Plato found in it the inspiration for his fiction of Atlantis.

This tradition, as has already been said, is thoroughly imbued with speculation, especially with the spirit of inquiry which aimed at the establishment of a rational chronology and the determination of the limits of the known world. In the sixth and fifth centuries B. C.

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[22] *Fr.* 332 (*H. G. F.*, I, p. 25).

[23] I, 30.

[24] Solon, *fr.* 6, Diehl: Νείλου ἐπὶ προχοῇσι Κανωβίδος ἐγγύθεν ἀκτῆς.

[25] The questions concerning the reported travels of Solon are critically discussed by Ivan M. Linforth in *Solon the Athenian* [*U. of Cal. Publications in Classical Philology*, Vol. 6 (1919)], Appendix 6, pp. 297 ff.

the dependable data for so ambitious an attempt were unavoidably few; the result, therefore, was far from satisfactory, but the effort reveals the stirring of the scientific spirit. If we observe the manner in which the then desperate problem was approached, we see that the methods then followed are precisely the same as those which in modern times have led scholars with data ampler and more critically verified to erect a frame more nearly satisfying the demands of reason. Some of these methods will presently be made clear as we proceed.

One of the greatest obstacles to a due appreciation of the work of the pioneers in this field is the sad wreck of the literature of the sixth and fifth centuries. There is clear evidence of the existence in ancient times of a number of maps[26] and a considerable body of writing on the subjects of history and geography; but what remains to us gives only the names of a few authors (presumably an incomplete list) and scattered fragments, until we reach the last quarter of the fifth century. Herodotus, for our purposes the sole writer whose work is intact, is extremely unsatisfactory, because he rarely cites his real authorities and, when he uses their works, excerpts their data negligently and incompletely. We should often be completely baffled, were it not for later writers with the help of whose data we can still reconstruct an intelligible picture. For the Second Book of Herodotus, with which we shall be chiefly concerned, we have the accounts of Diodorus and Strabo, whose sources are generally independent of Herodotus, but belong to the same line of tradition. We cannot here enter into a detailed discussion of this complex question; but it will be seen in the sequel that certain outlines and interests appear throughout, and for the most part the design is not the one sketched by Herodotus, but rather what his account presupposes.

That Plato, for the purpose of the *Critias*, made a study of this tradition is so plain that it was recognized by Henri Martin, and Wilamowitz also expressed the opinion that this work was the result of the philosopher's interest in geography. This is perhaps not quite the best way of stating the fact: it seems fairer to say that Plato was still primarily concerned with the problems of the ideal state. In the *Republic* such a state had been sketched in outline; but, as Socrates intimates in the *Timaeus*, a static picture was less likely to appeal to the beholder than one which portrayed action. But where, seeing that the state was an utopia, was it to be located? We know what Homer did. Where did he locate the blessed Phaeacians? In a place whence Odysseus returns in a trance. Later writers[27] use the term ἐξωκεανισμός for this practice, in which imaginary scenes are depicted while giving the semblance of truth by setting the whole in relation to known things and places. How prevalent this practice was, not only among older poets but also among historians and geographers, is shown by Herodotus where he is speaking of the limits of the known world.[28] "It would seem that the fairest blessings have been granted to the most distant parts[29] of the world." . . . "It is but reasonable that the most distant parts of the world, as they enclose and wholly surround all other lands, should have those things which we deem best and rarest." The known world is perhaps a rather drab affair, but, like the death's-head in the *Merchant of Venice*, it is enclosed in a golden casket. The Isles of the Blessed have ever been to seek. One might fancy he was reading Hakluyt. It was primarily for the setting that Plato had recourse to suggestions of the skirts of the world; but there can be no doubt that many of the ideal arrangements embodied in his ideal state likewise derive from the historico-geographic tradition, which, as we have said, was deeply imbued with the spirit of utopian speculation. This obvious fact has been strangely overlooked with the result that the most absurd theories have been proposed, such as those relating to the supposed reference of the *Ecclesiazusae* of Aristophanes to the *Republic*.

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[26] An interesting study of them will be found in J. L. Myres, "An Attempt to Reconstruct the Maps Used by Herodotus," *Geographical Journal*, Dec., 1896. I hope soon to add some important details overlooked by Professor Myres.

Those dialogues of Plato which on any ground whatever must be assigned to the earlier period of his literary activity reveal an exclusive preoccupation with problems which may be described as logical, ethical (in the wider sense), and educational. The subjects falling under the head of 'science,' in which he shows an interest, are almost exclusively mathematical. But from a certain point onward, say from the beginning of the *Republic*, Plato manifests an increasing familiarity with the facts and speculations relating to astronomy, geography, and history. Astronomy, on the mathematical side, interested him greatly, and clearly was a subject of study in the

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[27] Strabo I, 3, 36, 39; VII, 3, 6.

[28] III, 106, 116.

[29] He uses the phrase ἐσχατιαὶ τῆς οἰκουμένης, which was the technical term for the limits of the world.

Academy. Geography and history were not regarded as sciences in the Platonic and Aristotelian sense, and were relatively neglected. So far as history occupied the attention of either the Academy or the Lyceum, it was in relation to special subjects or to speculation on the good state. How little even Aristotle concerned himself with descriptive geography is patent to every reader of his works. Heraclides of Pontus and Dicaearchus seem to have shown an interest in speculation and in mathematical and physical cosmology and geography rather than in the type of history and geography cultivated by the Ionians. It was among the pupils of the 'philosopher' Isocrates, notably by Ephorus, that the older tradition was carried on. From this review one may infer the relation of Plato to the subjects which here concern us.

But we must now return to the story of Atlantis. This is described[30] as an island,— one of a number, by which one might pass, as it were on stepping stones, to the opposite continent which surrounded the true ocean,—but really as a continent, because it was larger than Europe and Africa put together. It lay in the Atlantic, which was then navigable, before the Pillars of Heracles. Here Plato playfully alludes to ancient theories of the world as an island washed on all sides by the sea; but he represents the earth as containing a number of such islands, each in itself an οἰκουμένη. In the island of Atlantis dwelt a race of men who had conquered Africa as far as Egypt, and so came into collision with the primitive Athens. The *Critias* was to have told the history of that epic contest,[31] in which Athens, deserted by her allies, was compelled to stand alone; but, as in the Persian War, she defeated the invaders and liberated all the inhabitants of Europe and Africa. "But afterwards there occurred violent earthquakes and floods; and in a single day and night of misfortune all your warlike men in a body sank into the earth, and the island of Atlantis in like manner disappeared in the depths of the sea. For which reason the sea in those parts is impassable and impenetrable, because there is a shoal of mud in the way; and this was caused by the subsidence of the island." In the shoals off the Straits of Gibraltar we have, as it were, the token[32] by which the revenant proves the reality of the other world. The way in which Plato refers to this supposed obstacle to navigation makes it clear that he depends on it to confirm his fiction. That this was a datum of the old geography we shall presently see; but that is not all. Like accomplished creators of fiction in all ages, he multiplies the circumstances which give verisimilitude to his story. Aristotle says,[33] "Homer more than any other has taught the rest of us the art of framing lies in the right way. I mean the use of paralogism. Whenever, if A is or happens, a consequent, B, is or happens, men's notion is that, if the B is, the A also is—but that is a false conclusion.[34] Accordingly, if A is untrue, but there is something else, B, that on the assumption of its truth follows as its consequent, the right thing then is to add on the B. Just because we know the truth of the consequent, we are in our own minds led on to the erroneous inference of the truth of the antecedent." Plato had no need of being instructed on this head by his pupil, for he knew his Homer very well. It seems reasonably certain that in speaking of the "true ocean" and the "true continent" beyond it in the far west he had in mind the Homeric Oceanus and the lands beyond it;[35] but there is no doubt that this conception passed to some of the Ionian philosophers and geographers. For the details of his picture of Atlantis he was, however, indebted chiefly to suggestions derived from Ionian geographers. Thus, when he speaks[36] of the series of canals constructed around the capital of Atlantis as circular, "formed as with compasses," he was no doubt thinking of the ancient maps of the earth which Herodotus and Aristotle tauntingly described as drawn as if with compasses and surrounded by the River Oceanus.[37] In the description of the island also there is more than one hint that it was modelled after the Ionian οἰκουμένη. The range of mountains surrounding the island and falling abruptly into the sea recalls conceptions of the Milesian philosophers which were at least in part accepted by geographers; the proportions of the central plain suggest those of the οἰκουμένη approved by Democritus;[38] and the location[39] of the plain itself "near the sea in the middle of the whole island" rather definitely hints that Plato had in mind the *orbis terrarum* surrounding the Mediterranean. Besides, in mentioning the continents, Plato divides the world either into Europe and Asia, which was the most ancient scheme, or he adds Libya as a third, a subdivision of Asia.[40] Who popularized the tripartite scheme is uncertain; but it is at least as old as Hecataeus.[41]

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[30] *Tim.* 24 e f.

[31] *Crit.* 108 e.

[32] Cp. the golden napkin which Rhampsinitus brought back from Hades after playing at dice with Demeter (Herod. II, 122). The motive is frequently met in folklore.

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[33] *Poet.* 1460 a 19 ff., tr. Bywater. Cf. Strabo I, 2, 9.

[34] Bywater, *ad. loc.*, well refers to *Soph. Elench.* 167 b 1 ff. for a fuller discussion of the fallacy. It is really the "fallacy of multiple causes," which Bacon and John Stuart Mill unaccountably thought the Greeks did not recognize.

[35] Cf. Theopompus, *ap.* Aelian., *V. H.* III, 18.

[36] *Crit.* 113 d.

[37] Herod. IV, 36; Arist., *Meteor.* 362 b 13. The early conception of the world is reflected in the pattern of the shields of Achilles and Heracles, on which see Daremberg-Saglio, *Dict. des Antiq.*, *s. v. Astronomia.* It is possible also that Plato had in mind proposals for city-planning such as those of Hippodamus and the one attributed to Meton in Aristophanes' *Aves*, 1000 ff.

Plato was, of course, intentionally harking back to ancient views, because they were most in keeping with his invention. He was living in a period of transition as regards geography; he was well aware of the new conception of the earth as a sphere, but why should he dash his dream-world by introducing the latest novelty? Besides, the theory was as yet cosmological rather than geographical: the zones were charted in the heavens, but no one knew where to locate them on the earth's surface.[42] Tentative beginnings were being made on the basis of which Aristotle could venture a guess as to the circumference of the globe, probably on data furnished by Eudoxus and Dicaearchus; but an actual map embodying the new conception was probably not attempted before Eratosthenes. It is noteworthy that Ephorus, Aristotle, and even Polybius (at least at times) used old Ionian maps; and even Hipparchus preferred to follow them wherever satisfactory determinations based upon the newer conception were wanting.

In basing his invention on ancient maps and geographical conceptions Plato was only doing what could hardly be avoided; but he could count on the acceptance of at least enough to answer his purpose of creating an atmosphere of reality. We have, then, to examine the tradition of ancient geography for the necessary points of support for the fiction of Atlantis. Since the maps of the Ionians have long since ceased to exist, we are compelled to reconstruct them, or so much of them as we require, from the literature of history and geography. In so doing we shall have to touch on a variety of questions which call for discussion; but in the end we hope to find a plausible explanation of the way by which Plato came to suggest the former existence of the island of Atlantis.

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[38] 118 c; cf. Diels, *Vorsokr.*^{3} II p. 65 *fr.* 15 B. The statement of Agathemerus that Democritus was the first to regard the οἰκουμένη as oblong (προμήκης) is certainly wrong, even the proportions may have been suggested before, though perhaps not in a formal description of the earth.

[39] *Crit.* 113 c.

[40] *Tim.* 24 b e; Herod. II, 16, IV, 42.

[41] Pindar, *Pyth.* IX, 8.

[42] Even Polybius could refer localities to the parallels in use in old Ionian maps. How Posidonius *ap.* Strabon. II, 2, 2, could attribute the discovery of the five geographical zones to Parmenides is still a mystery. There is no good reason for supposing that he knew even the celestial zones.

The text of Plato offers two principal complexes on which he relies. The first is connected with the supposed shoals lying before the Straits of Gibraltar, which serve as evidence of the former existence of the island;[43] the second is that of successive catastrophes[44] which divide the history of the race into periods. The latter serves primarily the purpose of disposing of the dream-world which the fancy of the poet-philosopher has conjured up. We have already referred to the notion of periodic catastrophes, which was a favorite of Plato and Aristotle. That it did not originate with them is evident from the text of Herodotus,[45] which makes it certain that Hecataeus had used it, presumably as Plato did, to account for the disparity between the length of the historical period in Greece and Egypt. But of course the materials out of which the theory was constructed lay ready to hand in Greek mythology and in the four ages of Hesiod. Plato recognizes it as Greek rather than Egyptian when he makes the priest say[46] to Solon, "There is a story, which even you have preserved, that once upon a time Phaëthon, the son of Helios, having yoked the steeds of his father's chariot, because he was not able to drive them in the path of his father, burnt up all that was upon the earth, and was himself destroyed by a thunderbolt." The myth is certainly not Egyptian, but the notion that in all such catastrophes Egypt was spared is part of the old tradition about that favored land;[47] for it was generally believed that Egypt was not subject to earthquakes,[48] and we have seen that Egypt escaped when primitive Athens and Atlantis were suddenly overwhelmed by that disaster.

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[43] *Tim.* 25 d.

[44] *Tim.* 22 c ff.

[45] II, 142 f.

[46] *Tim.* 22 c.

[47] [Arist.] *Probl.* XIV, 15 obviously is thinking of Egypt: διὰ τί οἱ ἐν τοῖς θερμοῖς τόποις σοφώτεροί εἰσι ἢ ἐν τοῖς ψυχροῖς; . . . . ἢ διὰ τὸ πολυχρονιώτερον τὸ γένος εἶναι τοῦτο, τοὺς δὲ ὑπὸ τοῦ κατακλυσμοῦ ἀπολέσθαι, ὥστε εἶναι νέους πρὸς γέροντας τοὺς ἐν τοῖς ψυχροῖς τόποις πρὸς τοὺς ἐν τοῖς θερμοῖς οἰκοῦντας. Olympiodorus *In Arist. Meteor.* fol. 27 a: ἀρχαιότατοί εἰσιν Αἰγύπτιοι ἐπειδὴ οὔπω ὑπὸ κατακλυσμῶν ἀπόλλυνται. Cf. Plato, *Tim.* 22 c ff.

For our present purpose, however, the group of notions connected with the Atlantic is the more important. For Homer, as we have remarked, the earth was a land-mass divided by the Mediterranean and encircled by the River Oceanus. In the west Odysseus crosses and re-crosses this stream. As geographical knowledge increased, the river became an ocean, which was called Atlantic because Atlas on its confines supported the vault of heaven. Plato, as we have seen, says that in former times this ocean was navigable, and one could pass from one of the continental islands to another and to the true continent beyond: that this is meant to do justice to Homeric geography is obvious. But Plato says that when Atlantis was engulfed there remained, to testify to its former existence, muddy shoals that proved to be insuperable obstacles to navigation. That no such shoals exist need not be said; what concerns us is the tradition.

For reasons that we do not know the notion got about and was widely accepted that the Pillars of Heracles were the limits of the world: it even became proverbial, in so much that Pindar repeatedly uses the proverb to point the moral that bounds are set to all human aspirations and achievements.[49] But why the Pillars of Heracles? To whom this name is due we do not know; but there are interesting possibilities suggested by a certain passage in which Pindar says,[50] "But, if the son of Aristophanes, being fair to look upon, and doing deeds that befit the fairness of his form, embarked on the highest achievements of manly prowess, no further is it easy for him to sail across the trackless sea beyond the pillars of Heracles, which that hero and god set up as far-famed witnesses of the furthest limit of voyaging. He quelled the monstrous beasts amid the seas,[51] and tracked to the very end the streams of the shallows, there where he reached the bourne that sped him home again; and he made known the limits[52] of the land" (say rather, of the earth). One sees at once that the notions here associated with the Pillars of Heracles are not those of a mere proverbial saying. They call for more careful scrutiny.

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[48] Seneca, *Q. N.* VI, 26, 1: Poteram ad hoc probandum abuti auctoritate magnorum virorum, qui Aegyptum numquam tremuisse tradunt. . . 3: Sed movetur et Aegyptus et Delos. For Delos, see Thucyd., II, 8, 2. Cf. *Orac. Sibyl.* XIV, 297.

[49] Pindar, *Ol.* III, 43 f.; *Pyth.* X, 29; *Nem.* IV, 69; *Isth.* IV, 29. The *Hippolytus* of Euripides (acted 428 B. C.) also twice (vv. 746, 1053) mentions the Atlantic as impossible of navigation.

[50] *Nem.* II, 19 ff., tr. Sandys.

[51] The dread sea-monsters hindering navigation in the Atlantic are repeatedly mentioned by Avienus, 78, 117, 580.

We note first of all that here, at earth's remotest bound, we find mentioned the shallows which Plato treats as survivals of a lost continent. Of this we shall presently have to speak at greater length. These shallows are evidently thought of as the cause of Heracles' return, because he had reached the furthest limit of voyaging. But the pillars are not only memorials of the hero's expedition, but also a sort of *limes*[53] set up by a world conqueror. Such pillars set by conquerors are of fairly common occurrence in Greek literature. Besides those of Heracles in the west, we read of similar pillars of Dionysus in India and of Sesostris in Thrace, Ethiopia, and elsewhere,[54] obviously as indications of the extent to which their conquests extended. The tradition of the victorious progress of gods like Dionysus and Osiris, or of heroes like Heracles and Sesostris, is not easy, if indeed it is possible, to trace; but at a certain point it assumes a quasi-scientific character. Here it is made evident in regard to Heracles, who "made known the limits of the earth." The interest is distinctly geographical. This is equally evident, as we shall presently see, in the case of Sesostris, and one cannot fail to discern the same character in the "history" of Osiris and of Dionysus as we read it, *e. g.*, in Diodorus. We are all familiar with the geographical interest that attaches to the conquests of Alexander and the Romans: it is so plainly revealed in the historical literature that he who runs may read it.[55] But it will doubtless be looked upon as hazardous if one suggests that it is equally plain in the reference of Pindar to the Pillars of Heracles. Yet, why should one doubt it? Pindar is not presenting a new thought that has just struck him: he is using for his purpose what some one else has said, and though it is impossible to identify his source, we may be sure that it belongs to the historico-geographical tradition. The more in detail one studies any tradition, the more elements in it does one discover as belonging to it from the beginning. This applies more especially to the methods: the facts or data multiply in time, but the way in which one approaches and utilizes them, though it may become more refined, generally continues to be followed. In the case of Heracles we are of course dealing with myth or legend; but did not Greek 'history' begin with the effort to extract by rationalization the essential truth from the myths and legends of the past?

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[52] Here Pindar used the poetical phrase γᾶν φράδασσε, but *Ol.* II, 19 he spoke almost as a geographer, νῦν δὲ πρὸς ἐσχατιὰν Θήρων ἀρεταῖσιν ἱκάνων ἅπτεται οἴκοθεν Ἡρακλέος σταλᾶν. Cf. Suidas, *s. v.* Γάδειρα: μέχρι τούτων ἐλθὼν ὁ Ἡρακλῆς, καὶ μὴ δυνάμενος διελθεῖν, στήλας παρὰ τῶν ἐγχωρίων λαβὼν ἱδρύσατο, δηλῶν δι' αὐτῶν ἄχρι τῶν τῇδε βατὴν εἶναι τὴν γῆν καὶ τὴν θάλασσαν.

[53] Cf. Pliny, *N. H.* III, 4 (of the shallows at the straits of Gibraltar): Frequentes quippe taeniae candicantis vadi carinas territant. qua de causa limen interni maris multi eum locum appellavere. Proximis autem faucibus utrimque impositi montes coercent claustra, Abila Africae, Europae Calpe, laborum Herculis metae, quam ob causam indigenae columnas ejus dei vocant creduntque perfossas exclusa antea admisisse maria et rerum naturae mutasse faciem. This is probably an echo of the view of Strato which is discussed after Eratosthenes by Strabo I, 3, 4 ff. Polybius IV, 39 ff. also refers to the same topic, obviously after Strato; but the basis of the latter's theory was laid by Xanthus and Hecataeus.

[54] Strabo III, 5, 5; Herod. II, 102, 106 (Sesostris in Thrace); IV, 87 (Darius at the Bosporus, cf. Strabo XVI, 4, 4, where Sesostris similarly marks his passage from Arabia to Ethiopia at the Straits of Dira); Dio Prus. 623 (Dionysus in India); Diodorus I, 20, 1 tells of the pillars which Osiris set up everywhere in his march through the world; Demosth., *De Cor.* 154. Charax of Pergamum, fr. 16 (*H. G. F.*, III, 640) says that the Pillars of Heracles were formerly called the Pillars of Cronus διὰ τὸ μέχρι τῶν τῇδε ὁρίζεσθαι δῆθεν τὴν ἀρχὴν αὐτοῦ. One suspects that the realm of Cronus here mentioned, was Elysium, and therefore the Pillars of Cronus marked its eastern boundary. Lucian, *V. H.* II, 28 sets up a pillar in the Isles of the Blest, inscribed with verses furnished by Homer, to mark the end of his voyage.

In Pindar we have therefore a witness to the fact that the historico-geographical tradition about the turn of the sixth and fifth centuries concerned itself with the question of the limits of the earth and in that connection mentioned shallows near the Pillars of Heracles as explaining why the hero had gone no further.[56] The rôle played by Heracles in the story of the lands of the west[57] is known in a general way to all students,—Dionysius of Halicarnassus makes much of it,—but I am not aware that any one has seriously studied it. Here we cannot consider it further, but must pursue our own inquiry. The outer limits of the earth were at first simply the favorite site of utopias, as the lands of the Ethiopians[58] or the Phaeacians;[59] as a place of banishment, as for the Titans,[60] or the abode of the blessed heroes, translated to Elysium;[61] but the former interest, while of course it continued throughout ancient times, and still continues, had come, with the extension of knowledge, to engage the serious attention of geographers, as one clearly sees from the discussions of Herodotus to which we shall presently refer.

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[55] Cf. Pliny, *N. H.* V, 51; Strabo XV, 1, 2.

[56] Cf. Soph., *Trach.* 1100.

[57] It is obvious that in this story the "Egyptian Heracles" was largely concerned. This fact adds to the interest of the problem and raises questions of great importance for the history of Greek literature.

The question concerning the limits of the land inevitably involved the question whether the ocean was or was not continuous.[62] The serious occupation with these questions must go back at least to the beginnings of cartography, associated by tradition with Anaximander of Miletus, who is reported to have died shortly after 546 B.C. Of his map we know directly nothing at all in detail; but we may be sure that in their main outlines the early maps of which we get reports in later writers generally reproduced his, though there were of course minor changes and, in particular, refinements and additional details resulting from different theories and the growth of knowledge, more especially regarding the "inner sea," the Mediterranean, and the adjacent lands. In the latter respect the map of Hecataeus undoubtedly marked a notable advance, for he is said to have marvellously improved that of his predecessor. His preoccupation with the questions regarding the limits of the world is, however, no less evident; for he is known to have shared the view that the land was surrounded by the ocean and that the continents were separated by channels reaching from it to the Mediterranean.[63] In the state of knowledge then existing he could not possibly have indisputable information on all points: he was obliged, as geographers have always been, to supplement his own observations by reports. How much credence he actually gave to these reports we have no means of knowing; but one may surmise that, being human, he accepted such as confirmed his own opinions which were, of course, not purely a priori guesses. Between Hecataeus and Herodotus there obviously came into existence other maps, though we cannot attach a name to one of them.

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[58] Hom., *Od.* I, 23.

[59] Hom., *Od.* VI, 205.

[60] Hes., *Theog.* 731. Lucian, *V. H.* II, 27 places the Isles of the Impious beyond the Isles of the Blest.

[61] Hom., *Od.* IV, 561 ff.

[62] Plutarch, *De Facie in Orbe Lunae*, 9, p. 645. Cf. Strabo I, 1, 8; I, 2, 26.

[63] Cf. F. Jacoby, in *Pauly-Wissowa* VII, 2702 ff.

The fact, however, that Eratosthenes, when he sketched the history of cartography, does not seem to have deemed any other geographer before Hellanicus (who supplied no map) worthy of mention creates the presumption that these maps were of minor importance. The tone adopted by Herodotus in referring to the maps then in existence is consistent with the view that they were essentially variants of that of Hecataeus, whom he chose to ridicule while being absolutely dependent on him.

The question whether the ocean was one or divided by land-bridges was evidently much discussed. For reasons easily intelligible the dispute centered chiefly on Africa. This fact is of some importance because, as we have before observed, the older view regarded the world as divided into two continents, Europe on the north, Asia on the south. Apparently not long before the turn of the sixth and fifth centuries Asia was divided into Asia and Libya, but neither the division nor the line of demarkation was at once accepted. This suggests that the division of Asia and Libya and the question whether Libya (Africa) could be circumnavigated are closely connected. The Isthmus of Suez was of course known, but as there was at that time, or had recently been, in use a canal from the Nile to the Red Sea, this point of contact could be ignored. For one who limited Libya to that part of Africa west of Egypt the question would assume another form if, as seems to have been the case with Hecataeus, he derived the Nile from the southern ocean—probably in southwest Africa. The point that then needed to be determined would be, whether there was an isthmus in that part of the outer ocean.

Strabo says,[64] "All who have coasted along the shores of Libya, whether they started from the Red Sea or from the Pillars of Heracles, after advancing some distance always turned back, because they encountered some obstacles, and therefore left in the minds of most people the conviction that the interval was blocked by an isthmus." Whatever one may think of this sweeping statement, it clearly indicates a popular belief that Africa could not possible be circumnavigated. Herodotus himself tells a story which might be taken as confirming that view. We are told[65] that Sataspes, son of Teaspis, an Achaemenid, was sent out by Xerxes to circumnavigate Africa, beginning at the Pillars of Heracles, but returned without accomplishing his mission. Really he was afraid of the length and loneliness of the voyage, but he excused his failure by alleging that after sailing many months his ship was unable to proceed farther, but was held fast. The alleged cause of the stoppage has been differently understood, but there can be no reasonable doubt that it was shoal water.[66] That Herodotus was acquainted with other accounts which asserted the possibility of sailing round Africa we shall presently see; but there is ample evidence that from early times it was popularly believed that Africa was joined to Asia. In the earlier period the connection was supposed to be on the eastern side of Africa; after the acceptance of the globe earth a connection between Africa on the west with Asia was imagined.[67]

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[64] I, 2, 26.

[65] Herod. IV, 43. From the closing words of the paragraph one may conclude that Herodotus himself received either the story, or what he regarded as confirmation of it, from an unnamed man of Samos.

Strabo himself, agreeing with Eratosthenes and Posidonius, rejected these views, believing that the ocean was one and not divided into several by land-bridges; but it is noteworthy that Hipparchus, who in the absence of satisfactory proof to the contrary always preferred to follow the charts and opinions of the early Ionians, was at least doubtful about the continuity of the ocean.[68] Polybius, in a passage[69] which is based on old maps and has rather marked affinity to the notions of Ephorus, says that no one up to his time had been able to determine with regard to those parts of Asia and Libya, where they approach each other in the neighborhood of Ethiopia, whether the continent is continuous to the south, or is surrounded by the sea. Even Ptolemy, following Marinus, accepted a land connection between eastern Asia and Africa, on the east.[70] Procopius[71] in a passage evidently based on an old geographical source,—whether map or treatise, we do not know,—says that the source of the Nile is in India; and Alexander the Great, obviously influenced by the old tradition, when he found 'crocodiles' and certain plants common in Egypt in the Indus and its tributaries, was convinced that these were the head waters of the Nile.[72] He imagined that the river of India was lost in desert lands and reappeared in Egypt: he even organized an expedition to explore the course of the Indus, but abandoned the project when he learned that it emptied into the Arabian Sea. That this tradition was ancient is made especially plain, not only by the course of Io's wanderings foretold by Prometheus,[73] but by the entire Greek tradition regarding India, where the comparison with Egypt and Ethiopia is constantly expressed or implied,[74] and by the discussion about the Ethiopians. This latter subject is too complex to be treated in this connection, but one point may be brought out. Homer had spoken of the Ethiopians as the remotest race of mankind, divided in twain, some living at the rising, others at the setting of the sun.[75] This gave rise to all sorts of disquisitions among geographers, some maintaining that the northern Indians were like the Egyptians, the southern like the Nubians;[76] but Posidonius, who did not accept the connection between India and Africa, was at pains to point out differences between them.[77]

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[66] Herodotus uses the word ἐνίσχεσθαι; Xenoph., *Anab.* VII, 4, 17 and Apollon. Rhod., *Argon.* I, 10 f. suffice to define the meaning. Strabo I, 2, 26 obviously understood it to refer to shallows. Pseudo-Scylax (Müller, *Geogr. Graeci Minores* I, 93) says διὰ βραχύτητα θαλάττης καὶ πηλὸν καὶ φῦκος, and Pliny, *N. H.* III, 4, quoted above, n. 50, refers to reefs. Suidas *s. v.* ἄπλωτα πελάγη· τὰ Ἀτλάντικα καὶ ἑσπέριος ὠκεανὸς καὶ ἑῷος· ἄπλωτον γὰρ τὸ ἄπλευστον is vague to as the cause; but *s. v.* Γάδειρα· ἣν (*sc.* θάλατταν) διεκπλεῦσαι οὐκ ἐνῆν βραχώδη καὶ σκοτοειδῆ τυγχάνουσαν is clear. See also Arist., *Meteor.* II, 1, 354 a 22–27 and Ideler's note *ad loc.*, vol. I, pp. 504 ff.

[67] See p. 215, n. 92.

[68] Strabo I, 1; Philoponus, *De Creat.* IV, 5.

[69] III, 38. I hope to discuss this soon in connection with the problem of the parallels of the Ionian maps.

[70] Ptolem., *Geogr.* IV, 9; VII, 5, 2; VI, 16. Cf. Berger, *Gesch. d. wiss. Erdkunde,*^{2} p. 24 and *idem, Ber. u. Verh. d. Leipz. Ges. d. Wiss., Phil.-Hist. Cl.*, vol. 50 (1898), p. 89.

[71] *De Aedific.* VI, 1, 6.

The prevailing view in the older times, then, was that Asia and Africa were united at other points than at the Isthmus of Suez, and the point of connection in Asia was generally supposed to be India. So long as this conception dominated the picture of the world there was no special reason for increasing the number of continents from two to three. To us this division seems so natural that we find it difficult to conceive that it was not always so. Glancing at a map of the world we note how much Africa exceeds Europe in area: it falls little short of Asia. But we must recall that Libya was originally a coastal strip along the Mediterranean with its undefined hinterland. That the question of the third continent was connected with that of the limits of the landmass is made sufficiently clear by the context of Herodotus. In a general geographical account where he is notoriously indebted to Hecataeus,[78] after discussing the boundaries of Europe and traversing Asia to India, the remotest land toward the east, he says,[79] "Such is Asia, and such its extent. But Libya is on the second promontory; for Libya comes next after Egypt. The Egyptian part of this promontory is narrow . . . but after this narrow part the promontory which is called Libya is very broad. I wonder, then, at those who have mapped out and divided the world into Libya, Asia, and Europe; for the difference between them is great, seeing that in length Europe stretches along both the others together, and it appears to me to be beyond all comparison broader.[80] For Libya shows clearly that it is encompassed by the sea, save only where it borders on Asia." This passage is noteworthy, among other reasons, for the manifest confusion in the use of the name Libya, which means now the entire continent, including Egypt, since it is reckoned as adjoining Asia at the Isthmus of Suez, and again is said to come next after Egypt, and so refers to the land of the Libyans only. Herodotus is thus himself guilty of the kind of inconsistency which he criticizes in Hecataeus.

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[72] Strabo XV, 1, 25; Arrian VI, 1. The presence of crocodiles, wherever found, always to the Greeks suggested the Nile (Herod. II, 32). The 'crocodiles' found in India (Herod. IV, 44) were neither many nor dangerous to man, said Nearchus (Strabo XV, 1, 45).

[73] Aesch., *Prom.V.* 807 f. The "dark race in a far-away land" are the 'Ethiopians' of India.

[74] The evidence is to be found almost everywhere, and there is no need of giving it.

[75] *Od.* I, 22 ff.

[76] Strabo XV, 1, 2; cf. Herod. VII, 70, Pliny, *N. H.* VI, 70.

[77] Strabo II, 3, 7. On the other hand, Syncellus, *Chron.* p. 151 (Müller, *H.G.F.* II, 609), supposedly after Manetho, says Αἰθίοπες ἀπὸ Ἰνδοῦ ποταμοῦ ἀποστάντες πρὸς τῇ Αἰγύπτῳ ᾤκησαν.

The statement in this connection that Libya is clearly surrounded by the sea is especially to be noted. Rather unexpectedly this fact is given as proof of the relatively small size of the third continent; but when Herodotus confesses[81] that the eastward extension of Asia is unknown, one wonders how he can be so confident that Libya and Asia taken together are not equal in extent to Europe, whose eastern and northern limits he asserts have not been discovered, since no one can show that it is surrounded by the sea. He is obviously criticizing a geographer who held the opposite view, as did Hecataeus. We have, therefore, no reason to suppose that the geographer whom he is generally following in this excursus cited the circumnavigation of

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[78] Cf. F. Jacoby, in *Pauly-Wissowa* VII, 2684 f.

[79] Herod. IV, 41 f.

[80] Cf. Herod. IV, 45.

[81] IV, 44.

Africa as proof of its dimensions: that is rather a new turn given to the matter by Herodotus. His predecessor quite certainly was concerned primarily with the boundaries of the continents and with the question whether the ocean was continuous. This will become clear as we proceed. It is worth noting that this geographical dissertation on the charts is connected with the conquests of Darius, as the western limit of the world at the Pillars of Heracles was fixed by the campaign of that hero-god.

After saying that Libya must be relatively small, because it is clearly encompassed by the sea, Herodotus adds,[82] "This was proved first (as far as we know) by Necos, King of Egypt. He, when he had made an end of digging the canal which leads from the Nile to the Arabian Gulf, sent Phoenicians in ships, charging them to sail on their return voyage past the Pillars of Heracles till they should come into the northern sea and so to Egypt. So the Phoenicians set out from the Red Sea and sailed the southern sea; whenever autumn came they would put in and sow the land, to whatever part of Libya they might come, and there await the harvest; then, having gathered in the crop, they sailed on, so that after two years had passed, it was in the third that they rounded the Pillars of Heracles and came to Egypt. There they said (what some may believe, though I do not) that in sailing round Libya they had the sun on their right hand. Thus the first knowledge of Libya was gained; next comes the assertion of the Carthaginians." There follows the story of Sataspes summarized above.

The story of the voyage of the Phoenicians sent out by King Necho has been much discussed, chiefly however with regard to its credibility as an historical fact, a matter which does not primarily concern us.[83]

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[82] IV, 42.

[83] The *prima facie* evidence for the voyage is of the best; for the datum about the position of the sun is hardly to be explained except upon the assumption that it was actually reported. It is true that Egyptians who went into Nubia might see the sun to the north at the time of the summer solstice, but it is quite certain that Greeks (and here we are undoubtedly dealing with Greeks) had had no such experience, excepting soldiers who under Psammetichus got as far south as Abu Simbel, just below the tropic. Was it likely that *they* would note and report the slight inclination to the north? Greeks in the fifth century had no notion of climatic differences resulting from differences of latitude except at the extreme north and south, and, as was remarked above, the zones, as geographical belts, were then unknown. The only serious argument against the acceptance of the report is that it was ignored by

We are rather interested in incidental points. To be sure, the reference to the position of the sun as the expedition rounded the southern extremity of Libya is of capital importance, even apart from the question whether it establishes the truth of the narrative or not. What is certain is that the story was not invented by a Greek, though it is equally certain that Herodotus got it from a Greek geographer; who that geographer was we shall presently see reason to suspect. A second point of some moment is the connection in which this voyage is brought with the completion of the canal from the Nile to the Red Sea. Though the text of Herodotus does not assert that the expedition started from the Nile, and elsewhere he expressly says that it set out from the Red Sea, one can hardly doubt that his source intended it otherwise. The question with regard to the digging of this canal will presently recur in a form that suggests a departure on the part of Herodotus from the text of his source. A third point of interest is the time required by the circumnavigation of Libya. Usually it is assumed that this was nearly three years, but a knowledge of Greek usage makes it certain that the phrase "after two years had passed, in the third" suggested to Herodotus a period just over two years.[84] This, no doubt, accounts for his view that the circumnavigation proved the relatively small extent of Libya. Finally, there is a bare reference to a story of the Carthaginians, which presumably confirmed the possibility of sailing round Africa; but we know nothing more about it. Whether his source referred to the voyage of Hanno we cannot say.

The whole situation is one of extraordinary complexity, not only because of the points already noted, but also because of the confusion everywhere apparent in the attitude of Herodotus to the connection between Libya and Asia. That he regarded as hardly justified the distinction between them as separate continents we have already seen. Furthermore he could find no entirely satisfactory boundary between them, if one must be drawn. The difficulty was in part due

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the serious geographers of the fourth and third centuries. That fact raises a most difficult problem, which is presumably connected with the neglect of the geographical treatise of Hecataeus about the same time. We do not know the reason for this, and in the absence of satisfactory clews even conjecture is useless.

[84] In reference to years it is parallel to "a year and a day." See my book, *The Day of Yahweh*, p. 415, n. 1, and Eisler, *Weltenmantel und Himmelszelt*, p. 138.

to the fact that the name Libya was ambiguous, having been at first applied to North Africa west of Egypt and only later made to include all Africa; but there were apparently other things that troubled him, among which one must probably reckon the fact that he used more than one map. He seems, however, to have followed in the main one which divided Asia into "promontories" (ἀκταί), and Jacoby has made it all but certain that it was the map of Hecataeus. We need not reproduce this entire scheme, which divided Asia roughly into strips from north to south, again divided into northern and southern halves. That the eastern strip, which included India, was similarly divided we do not positively know, but may probably infer from the fact that Herodotus says,[85] "Other Indians adjoin the town of Caspatyrus and the Pactyic country, northward of the rest of India; these live like the Bactrians and are of all Indians the most warlike." The town of Caspatyrus, as we know, was mentioned by Hecataeus,[86] and its location, vague as it is, was doubtless due to him. After the mention of India Herodotus proceeds to say,[87] "Such is Asia, and such its extent. Libya is on one of these promontories; for Libya adjoins Egypt." This obviously implies that Egypt and Libya belong to Asia. He is speaking, no doubt, of the second promontory of western Asia, which, he says,[88] "beginning with Persia, stretches to the Red (Arabian) Sea, being the Persian land, and next the neighboring country of Assyria, and after Assyria, Arabia; this promontory ends,—but conventionally only,—at the Arabian Gulf, whereunto Darius brought a canal from the Nile." The fact that this tract is said to end, conventionally only, at the Red Sea, is significant; for it again implies the ancient conception of the continuity of Asia and Africa. By noting that the canal from the Nile to the Red Sea, which elsewhere he attributes to Necho, was built by Darius, this geographical excursus is again connected with the campaigns of that king. That this reference is significant will appear as we proceed.

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[85] III, 102.

[86] *Fr.* 179 (Müller, *H.G.F.* I, 12) Κασπάπυρος, πόλις Γανδαρική, Σκυθῶν ἀκτή. That the town is placed in the Σκυθῶν ἀκτή shows that it was well to the north; for Hecataeus assigned the entire north to the Scythians, as the Ethiopians held the south. The modern attempts to locate it and the Pactyic country are pure guesses, made on the assumption that the expedition of Scylax proceeded from there down the Indus.

[87] IV, 41.

[88] IV, 39.

This same geographical excursus, which is concerned with the delimitation of the continents, leads us further. After telling of the voyages of discovery which disclosed the fact that Africa could be circumnavigated, Herodotus continues,[89] "But as to Asia, most of it was discovered by Darius. There is a river Indus, in which so many crocodiles are found that only one river in the world has more. Darius, desiring to know where this river issues into the sea, sent ships manned by Scylax, a man of Caryanda, and others in whom he had confidence; these set out from the city of Caspatyrus and the Pactyic country, and sailed down the river towards the east and the sunrise till they came to the sea; and voyaging over the sea westwards, they came in the thirtieth month to that place whence the Egyptian king sent the Phoenicians afore-mentioned to sail round Libya. After this circumnavigation Darius subdued the Indians and made use of this sea. Thus it was discovered that Asia, saving the parts towards the rising sun, was in other respects like Libya." This account, like that of the voyage of the Phoenicians round Africa, has been the subject of much discussion, chiefly concerning the question whether it is fact or fiction. As has already been said, for our present purpose it matters little whether the report is false or true, though it is difficult to accept the view that it is pure invention. Scylax, coming from Caryanda in Caria, was presumably a Greek, and it is not unlikely that, if not a physician like Democedes,[90] whom Darius placed at the head of an expedition of reconnaissance into Greek lands, he was more or less in touch with the nearby medical schools of Cnidus and Cos; for he was evidently chosen as a man of science, and in the earlier period all such were concerned more or less with medicine. The data regarding the flora of India and the region of the Caspian which date from the time of Darius suggest an interest in such things; and if the king was concerned for revenue, as well as glory, in his conquests, it is well to remind oneself that trade with outlying parts of the world must then have been confined to articles, like simples and precious metals, whose value is great in proportion to their bulk. Whether Scylax wrote an account of his voyage is open to question; but it is fairly certain that Hecataeus mentioned him and related his story, which he may have received from his own lips.

As for the content of the story, it is obvious that the reported expedition was at least suggested by the discovery of 'crocodiles' in the rivers of India. Crocodiles were known to the Greeks from the Nile, which is of course the other river, mentioned by Herodotus, where the animal abounds. The finding of crocodiles always suggested the Nile, as one sees from another passage in Herodotus,[91] who relates the story of some young Nasamonians and their venturesome exploit in the deserts of western Libya, where they found a river with crocodiles. The historian concludes that it must have been the Nile, whose source is therefore assumed to be in the western part of Africa. No doubt the ancient belief that eastern Asia and Africa were united by a land-bridge was thought to be confirmed by the discovery of crocodiles in India.[92] But obviously the expedition of Scylax, by following the river to its mouth and then winning its way by sea to the Arabian Gulf, had for our geographer refuted the ancient belief; for we are told "Thus it was discovered that Asia, saving the parts towards the sunrise, was in other respects (or "for the rest") like Libya," which, in the context, can only mean that along its southern side it was everywhere washed by the ocean. This would confirm the results of the voyage of the Phoenicians.

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[89] IV, 44.

[90] On him cf. Herod. III, 125, 129–137. Many modern explorers likewise have had political as well as commercial and scientific objects in view.

The question then arises which river Scylax was supposed to have descended. I do not doubt that Herodotus really meant the Indus River, and for the most part the same view must have been entertained by later geographers; but, as one looks at the map, one sees that a voyage down the Indus and westward from its mouth to the Red Sea would not by itself have justified the conclusion that there was no land-connection between India and Africa, and that is distinctly implied. This objection will be met by saying that our maps cannot be taken as a guide in such matters, since the Greeks in the sixth and fifth centuries B.C. had a very imperfect conception of the world; and, of course, this is quite true. But we have a definite text to deal with, and that contains several statements with which we must reckon. First of all we note that the course of the river which Scylax descended is said to be eastward, a matter about which an intelligent man could not well be mistaken. Again, the length of time required by the voyage, given as thirty months, is extravagant both in comparison with that of the slightly more than twenty-four credited to the Phoenicians who circumnavigated Africa and also in relation to the actual distance to be traversed. We also hear that after Darius sent out the expedition he conquered the Indians. The implication of this statement is obviously dubious, depending, first, upon the way in which we conceive the expedition to have been initiated, and, secondly, upon the point from which Darius started after the expedition was made. We do not know whether Darius had already reached the Indus[93] before he conceived the plan of the voyage or whether he sent it out, while he was in his homeland, to reconnoitre the Orient, as he despatched Democedes to the west. The comparison of the two expeditions, which must at once suggest itself, rather favors the latter alternative. If that be assumed, and there is really nothing to disprove it, the fact, if it be a fact, that Darius did not push his conquests much beyond the Indus, has no significance in relation to the story of Scylax. Nor does the statement that the expedition set out from Caspatyrus and the Pactyic country help us in the least, because we cannot identify these localities. All that we have to go upon is the statement that these places lay northward of the rest of the Indians and that their inhabitants were the most warlike of the Indians and in their mode of life resembled the Bactrians.

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[91] II, 32 f.

[92] The presence of elephants in India and Africa was another indication that they might be connected. Aristotle, *De Caelo* 298 a 9 ff. διὸ τοὺς ὑπολαμβάνοντας συνάπτειν τὸν περὶ τὰς Ἡρακλείους στήλας τόπον τῷ περὶ τὴν Ἰνδικήν, καὶ τοῦτον τὸν τρόπον εἶναι τὴν θάλατταν μίαν, μὴ λίαν ὑπολαμβάνειν ἄπιστα δοκεῖν· λέγουσι δὲ τεκμαιρόμενοι καὶ τοῖς ἐλέφασιν, ὅτι περὶ ἀμφοτέρους τοὺς τόπους τοὺς ἐσχατεύοντας τὸ γένος αὐτῶν ἐστίν, ὡς τῶν ἐσχάτων διὰ τὸ συνάπτειν ἀλλήλοις τοῦτο πεπονθότων. For that reason also, no doubt, Plato, *Crit.* 114 e, mentions elephants in Atlantis.

The identification of the river in question with the Indus rests, then, solely on the name, unless we stress the agreement of later writers, who may merely be following the interpretation of Herodotus. One can readily comprehend that the Ἰνδὸς ποταμός, which to the original narrator might have meant only "the Indian River," would be understood to be the river of India best known to the Greeks, the Indus; but there is no certainty that this represents the thought of the authority to whom Herodotus was indebted for the story of the expedition of Scylax. One will readily concede that the indications of the text of Herodotus, aside from the identification of the river as the

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[93] We do not even know that Darius himself conducted the expedition to India: he may have heard by report of the crocodiles. Who were the "Indians" whom Darius subdued? Did he not go beyond the Indus? The story certainly implies that he did, but we cannot say how far.

Indus, are better satisfied by the assumption that it was the Ganges[94] that Scylax descended; for it might lay claim to the title "the Indian River" as well as the Indus, if not better.

Without asserting positively that it was in fact the Ganges to which Hecataeus referred, one may say that another group of data, some of which likewise come from Herodotus, make that conclusion seem not improbable. These data relate to the Egyptian king whom Herodotus calls Sesostris and Diodorus names Sesoosis. They present a most interesting complex, which has hardly received the attention it deserves. Egyptologists, while holding that much of the Greek tradition regarding this king is apocryphal, being an exaggerated popular legend reflecting the exploits of various Egyptian conquerors,[95] have reached no agreement about their identity. Much less have they discerned the essential character of the tradition. That it is apocryphal, and based on Egyptian legends is certain; but it is not this fact that renders it of special interest to us. Our earliest document of this tradition is the story of Sesostris given by Herodotus in his baffling list of the kings of Egypt; but a close scrutiny of his statements and of the later reports about this remarkable character renders it quite certain that the Father of History was in this imperfectly reproducing an earlier writer whose account he altered at some important points.

To an inattentive reader the story of Sesostris as told by Herodotus will probably appear straightforward and simple; but its specious simplicity is deceptive. This king is dated[96] in the twelfth generation before Amasis, which by Herodotus' reckoning of three generations to a century would be equal to four hundred years, hence we should place him *ca.* 935 B. C. But Herodotus likewise tells us that the Colchians were Egyptians, either deserters or a detachment purposely left behind from the army of Sesostris.[97] This, he assures us, he himself suspected on the ground of certain points in which Egyptians and Colchians agreed, before he heard this from another. Who this other was we do not learn from him, but there are interesting clews. A Greek would be interested in the Colchians chiefly for one of two reasons, either because they dwelt about the Phasis River, which was the usually accepted boundary between Asia and Europe,[98] or because the Argonauts went there in quest of the Golden Fleece. In this case both these interests obviously played a part, the former for geography, the latter for chronology. The chronological reference, though in no way indicated by Herodotus, becomes at once apparent when one notes that the second successor of Sesostris on the throne of Egypt was Proteus,[99] who was reigning when Menelaus and Helen arrived in the land of the Nile on their return from Troy. Now this latter datum is unmistakably connected with the Trojan epoch, a fact made doubly certain by the fact that in Greek chronology the Argonautic Expedition commonly preceded the Trojan War by a generation.[100] These connections were of course made by someone who was concerned to establish synchronisms between Greek and Egyptian chronology and who reckoned by generations. The fact that Herodotus cites the Egyptian priests as his authorities need not detain us even for a moment, for that claim, whether made by Herodotus or by his source, is a fiction, like others of the sort. But Herodotus seems not merely to ignore the relation of these data but to be actually unaware of it; for he elsewhere[101] dates Homer about the same time here given for the Argonautic expedition and again[102] says that the Trojan War was about eight hundred years before his own time,—say, *ca.* 1235 B. C. That the entire chronology of the Egyptian kings is vitiated by a fundamental confusion is true,[103] but it cannot account for so egregious a blunder as this. I can explain it only on the assumption that Herodotus had made notes which he combined without fully realizing what he was doing. Strabo,[104] from whatever source and on whatever grounds, dates Sesostris "before the Trojan times." The geographical interest in the Colchians is brought out by the fact that they are mentioned in connection with the arrival of Sesostris at the Phasis on his victorious march through the world. The presumptive source of Herodotus thus combined two interests, that of geography, which utilized the data of military and other expeditions for the determination of the boundaries of the continents, and that of chronology, which was built upon a scheme of generations and international synchronisms.

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[94] J. L. Myres, "An Attempt to Reconstruct the Maps Used by Herodotus," *Geographical Journal*, Dec. 1896, p. 623, long ago made the suggestion that the Ganges was intended.

[95] It seems to be the view most commonly held that the Sesostris of the Greeks combines (and greatly exaggerates) exploits of several kings of the Twelfth Dynasty with those of Rameses II. and III., who reigned some 700 years later. According to Tac., *Ann.* II, 60 Theban priests must have identified Sesostris with Rameses; *ibid.*, VI, 28 Sesostris is said to have lived a phoenix-period (1461 yrs.) before Amasis.

[96] Herod. II, 102.

[97] Herod. II, 103 ff.

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[98] See Herod. II, 103; IV, 45. That this was the view of Hecataeus of Miletus is certain (Jacoby, Pauly-Wissowa VII, 2704 f.). Jacoby; *l. c.* and 2722, makes the questionable suggestion that Hecataeus identified the Phasis with the Hypanis. It is quite probable that Hecataeus did not have an accurate knowledge of the geography of the lands lying between the Euxine and the Caspian; but Herod. II, 103 leaves no doubt about the Phasis being that of Colchis. Hecataeus thought that the Phasis communicated with the Caspian, which is turn communicated with the outer ocean (*H.G.F.*, fr. 339). Presumably he had heard of the ancient trade-route which followed for the most part the valleys of the Cyrus and the Phasis rivers. A vague report might readily lead to the assumption that it was one river, and since the Phasis was known to empty into the Euxine, it would naturally be thought to issue from the Caspian.

[99] Herod. II, 112.

[100] Aeetes, son of Helios, is obviously supposed to have been the first king of the Colchians, since he had no human father from whom he might inherit. Jason took to wife Medea, daughter to Aeetes; consequently the origin of Colchis falls two generations before the Trojan epoch.

Though Sesostris is recognized as a great conqueror, "subduing every nation to which he came," Herodotus gives no connected account of his conquests. He is apparently diverted by the mention of certain "pillars" which the great king set up.[105] One is at first inclined, from his account, to suppose that Sesostris was setting up pillars everywhere he went, merely from a love of ribald jests at the expense of his victims; but one questions this conclusion when one compares the fuller account given by Diodorus of the campaigns of Sesoosis. Even Herodotus says that Sesostris passed through the continent erecting these monuments until he passed from Asia to Europe and overthrew the Scythians and Thracians. "So far and no farther, I think, did the Egyptian army proceed; for in their country are the pillars to be seen erected, but none beyond it." Thus the pillars here also become a sort of *limes* of the Egyptian empire, though they serve other purposes as well. The same situation reappears in Diodorus, but with a difference. After speaking of the

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[101] Herod. II, 53, Homer and Hesiod are dated not more than 400 years before his own time, say, *ca.* 835 B. C.

[102] II, 145.

[103] Cf. Ed. Meyer, *Forschungen zur alt. Gesch.* I, 164.

[104] XVII, 1, 25.

[105] Herod. II, 103.

Colchians, he says,[106] "In the same way he brought all the rest of Asia into subjection and the greater part of the Cyclades Islands. Crossing over into Europe and traversing the whole of Thrace he was in danger of losing his army by reason of the scarcity of food and the difficulties of the country. Therefore, *establishing his boundaries in Thrace, he erected pillars in many places he had won.*" It is true that he goes on to speak of these pillars very much in the same vein as Herodotus, but he brings out more clearly the relation of these monuments to the *limes* of the empire. Taken by itself this fact is not decisive; but when considered in connection with others, to be presently considered, it at least raises the question whether the account of Diodorus, though certainly not uninfluenced by Herodotus, does not more faithfully reproduce the old tradition regarding Sesostris.

The statement of Herodotus about the campaigns of Sesostris is meager, though he is represented as a great conqueror. Aside from an expedition to the east, to which we shall have to give our attention presently, the only countries specified as conquered by him are Colchis, Scythia, Thrace, Syria, Asia Minor, and Ethiopia, and some of these only by implication as lands in which his taunting monuments were erected. The account of Diodorus, however, is not only fuller but more clearly reveals the geographical interest which is merely implied in the story of Herodotus. A close juxtaposition of the two texts would be instructive, but this may be left to the reader, if he desires. We will take up a number of special points that seem to throw light on the character and history of the tradition.

They relate, says Diodorus,[107] that Sesoosis achieved the most illustrious and greatest exploits up to his time; but since regarding him not only the Greek historians but the Egyptian priests[108] and encomiastic poets differ among themselves in their statements, I shall endeavor to give the account that is both most probable and most in accord with the indications in the country.[109] At his birth his father brought together from all Egypt all the boys born at the same time and had them reared in common, on terms of the utmost frankness. Their training was for war, and they formed an intimate band, which subsequently furnished the officers for the army of Sesoosis. The first expedition undertaken, partly for hunting and partly for conquest, led into Arabia, which was then for the first time reduced to subjection. Next he turned to Libya, the greater part of which he subjugated while still quite young. After the death of his father, when he succeeded to the throne, he conceived the idea of conquering the world, in which he was encouraged by oracles and dreams his father had had. Once his object was determined, he set about creating the conditions necessary to success: he must secure the unlimited loyalty of his army and prevent revolution at home during his absence. Consequently he bound all his subjects to him by well considered benefactions and noble conduct on his own part. Among other measures he instituted the system of nomes, dividing the country for purposes of administration and taxation; and levying a vast army, commanded by his companions from childhood, he parcelled out the best lands in Egypt among them, so that they might be men of independent means and devote themselves entirely to war. Then he first proceeded against the Ethiopians in the south and conquered them; next he turned eastward, sending a fleet of four hundred ships into the Arabian Sea, while he went by land, conquering the islands and the entire mainland of Asia, as far as India,—not only what Alexander later won, but he even went farther, crossing the Ganges and reaching the ocean. The Scythians also he conquered as far as the river Tanais (Don) which divides Europe from Asia;[110] on this occasion leaving a detachment of his army, which constituted the Colchian nation. Next, as we have seen, he passed into Europe and after traversing Thrace, turned back, marking the limits of his empire by erecting pillars. Imposing vast annual tribute and bringing incomparable numbers of prisoners of war and objects taken as booty, he returned home, enriching his army and his country, having surpassed all who went before him in the magnitude of his exploits.

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[106] Diod. I, 55, 6 f. Of course Diodorus could not forego the pleasure of repeating with variations the stories about the pillars which we find in Herod. II, 106. What was first conceived as a method of verifying legend has become a means of diversion for the reader.

[107] Diod. I, 53, 1–10.

[108] This is significant of the character of the tradition; cp. Diod. I, 15, 2.

[109] Cf. the spirit shown by Isocrates, *Busiris* 30, 34–35, where it is evident that things variously assigned but belonging to the Greek tradition about Egypt are attributed to Busiris. That so much is given to Sesostris shows the importance of the first account of that king.

So runs the story of the conquests of Sesoosis. The impression one gets from Diodorus is that in the main the foregoing account was the uniform tradition,[111] though certain things, such as the comparison with Alexander, must of course be later additions. Diodorus then proceeds to speak of the works accomplished by Sesoosis after his return by means of his captives. First of all[112] he busied himself with building temples in all Egyptian cities; then he had the level or the dikes of all cities raised to a height sufficient for their protection during the period of the Nile floods. Next, canals were constructed throughout all the land below Memphis, to serve as means of communication and as barriers against possible invaders. The latter purpose also inspired the construction of a wall across the Isthmus. The remainder of Diodorus' story is for the most part a rehearsal of things told by Herodotus, but embellished with additions.

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[110] Cp. Herod. IV, 45.

[111] Where, as in I, 56, 5–6, Diodorus is aware of widely divergent reports, he is at pains to note them.

What chiefly concerns us, therefore, is the comparison of the two accounts of the campaigns of Sesostris. Here, as we have already remarked, Diodorus is both fuller and more orderly, and the purpose of giving an account of the world in connection with the progress of conquest, which obviously underlies the story told by Herodotus, is much more clearly indicated. One might suppose that this was due to Diodorus or to his probable source, Hecataeus of Abdera; but that is not at all probable. With the exception of the expeditions to Arabia and Libya, and the education of the boyhood companions of Sesostris, all the elements of the story are present in the older tradition. Even the statesmanlike preparations before the departure abroad are probably indicated in earlier statements regarding the division of Egypt into nomes. I am well aware that we are dealing with fiction in the guise of history, and that writers of fiction may exercise their imagination on very meager data. It was Flaubert, I believe, who, when it was suggested that he must have delved deeply in archaeology before writing *Salammbô,* quietly remarked that one could obtain a good deal of information from a dictionary. Quite true; but we shall see that in the case of Sesostris we are dealing with a clearly marked tradition, to which more than one contributed, but which was in essentials known to Herodotus.

Let us, then, return to Herodotus. He says[113] that Sesostris apportioned the land among all the Egyptians, assigning an equal square parcel to each, and based taxation thereon. If the lot of any was diminished by the river, it was surveyed and the tax reduced proportionally. This gave rise to geometry. The equal division of lands was a utopian measure, proposed *e. g.* by Phaleas.[114] Nothing is said here about the division of Egypt into nomes or the setting aside of lands for the warrior class by Sesostris, as one finds these measures assigned to him by Diodorus. These, together with the division of the people into castes, are brought forward by Herodotus[115] in a description of the institutions of Egypt, in such a way as to make it clear that they all were thought to belong together. They were in fact measures which were constantly recommended by political theorists in the fifth and fourth centuries, as by Hippodamus,[116] Plato, and Aristotle. One is tempted indeed by the manner in which Diodorus speaks of the nomes to suspect that the motive imputed to Sesostris was that of dividing the people to prevent revolution,[117] as the institution of the ten tribes by Cleisthenes was designed to prevent the collusion of neighboring demes. It was an age of political theorizing fostered by social unrest and the founding of colonies. Herodotus says[118] he cannot positively decide whether the Greeks learned from the Egyptians to set apart the warrior class, because practically all foreign peoples held soldiers in higher esteem than artisans; but Isocrates in the *Busiris*,[119] Plato,[120] Aristotle,[121] and Dicaearchus[122] had no hesitation on that score, and Aristotle expressly attributes the institution to Sesostris. It is difficult to escape the conclusion that the source of Herodotus said the same.

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[112] I, 57, 1 ff.

[113] Herod. II, 109.

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[114] Arist., *Polit.* 1266 a 36 ff.

[115] II, 164 f.

[116] Arist., *Polit.* 1267 b 30 ff.

[117] Suidas *s. v.* Σέσωστρις says that the king changed the customs of the Egyptians, wishing to humiliate them, and passed laws requiring the men to perform the offices of women and *vice versa*. Nymphodorus, in the 3d century B. C. (?), fr. 21 (*H.G.F.* II, 380) enumerates all the instances of paradoxical customs mentioned by Herod. II, 35 f, and attributes them to the desire of Sesostris to render the men effeminate. This notion certainly antedates Diodorus and is hardly due to Hecataeus of Abdera; for it is quite in keeping with the older tradition. Jebb on Sophocles, *O. C.* 337 makes the improbable suggestion that Nymphodorus originated the notion. At least as far back as Plato the tradition regarding Egypt is colored by the conception that the Egyptian institutions and customs were due to wise legislators. Plato even ridicules the notion! Wiedemann, *Herodots Zweites Buch*, 148, curiously pronounces this view one of late date.

[118] II, 167.

[119] 15. Cf. 17 and Herod. VI, 60.

[120] *Tim.* 24 a f.; *Legg.* 745.

[121] Polit. 1329a 40, 1329 b 23.

[122] Schol., Apollon. Rhod., *Argon.* IV, 272.

Again Herodotus says[123] that Sesostris employed the prisoners of war in digging all the canals existing in Egypt. We should expect to find included in the number the canal connecting the Nile with the Red Sea, which certainly existed in his own time. It is said to have existed as early as the twelfth Dynasty[124] and to have been renewed by Necho and Darius,[125] while Aristotle[126] and Strabo[127] say it was begun by Sesostris. Does it not seem clear that for his own purposes Herodotus departed from the old tradition?

Herodotus tells a curious tale[128] about this king: "Sesostris was the only Egyptian king who also ruled Ethiopia. To commemorate his name, he set before the temple of Hephaestus two stone statues of himself and his wife. . . . Long afterwards Darius the Persian would have set up his statue before these; but the priest of Hephaestus forbade him, saying that he had achieved nothing equal to the deeds of Sesostris the Egyptian; for Sesostris (he said) had subdued the Scythians besides as many other nations as Darius had conquered, and Darius had not been able to overcome the Scythians; therefore it was not just that Darius should set his statue before the statues of Sesostris, whose achievements he had not equalled. Darius, they say, in view of that pardoned the priest." This story is for several reasons remarkable and deserving of special attention. That it is an invention, and, like the rest of the tale of Sesostris, a Greek invention, may be safely assumed; but this only adds to its interest for us. It displays, to be sure, a spirit of banter toward the Persian king, which is characteristically Greek, but there is no trace of hostility to him such as might have been expected after Darius attempted the conquest of Greece. Moreover, in the comparison of the respective achievements of the two kings there is no hint of those campaigns of

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[123] II, 108.

[124] Breasted, *A History of the Ancient Egyptians*, 1914, p. 159. What evidence there may be for this statement I do not know, unless it be that of Aristotle and Strabo. Professor Steindorff in Baedeker's *Egypt*, 1929, p. 193 f., says the earliest authenticated attempt was made by Necho, and the canal first completed by Darius. Of the latter fact there is unquestionable evidence in monuments erected by Darius.

[125] Herod. II, 158.

[126] *Meteor.* 352 b 19 ff.; cf. Pliny, *N. H.* VI, 165.

[127] Strabo I, 2, 31.

[128] Herod. II, 110.

Darius, which might have afforded "the priest of Hephaestus" with ample grounds for an even more derogatory estimate of the prowess of the Great King. Does not this go far toward dating the invention about 500 B. C.?[129] One can see the comparison of the two kings throughout the account of the campaigns of Sesostris, which, with whatever later embellishments and additions, is apparently in its essentials a unit dating from before the time of Herodotus, who gave excerpts from it and modified it is some details.

In view of this a statement of Herodotus[130] regarding Sesostris assumes for us especial significance: "This king, said the priests, set out with a fleet of long ships from the Arabian Gulf and subdued all the dwellers by the Red (Arabian) Sea, till as he sailed on he came to a sea which was too shallow for his vessels." If now we compare with this story the account of Diodorus[131] we see how this was understood, at least in later times: "After fitting out his army he made an expedition first against those Ethiopians[132] who dwell to the southward, and by force of arms compelled the nation to pay tribute of gold, ebony, and ivory. Then he despatched into the Arabian Sea a fleet of four hundred ships (being the first of his nation to build long ships), and he conquered both the islands in those regions and the parts of the mainland along the coast as far as India, while he proceeding with his army by land subjugated the whole of Asia,—not only that which was later won by Alexander of Macedon, but also certain peoples whose lands he did not penetrate; for he crossed the Ganges River and overran the whole of India up to the ocean." The comparison with the conquests of Alexander is, as has been already said, of course a later variant, almost certainly substituted for an expressed or definitely implied comparison with those of Darius. But, if that be so, the statement that Sesostris reached the (eastern) ocean is of capital importance. It fits in perfectly with the statement of Herodotus regarding the expedition sent by Darius under Scylax to descend "the Indian River," which ran eastward, into the sea, and with the obvious implication of that account that up to the eastern ocean the boundaries of Asia were made by the sea. Whether the Ganges was actually known in the sixth century, I do not venture to say; but that Diodorus (or Hecataeus of Abdera?) was not far wrong in his interpretation of the tradition about Sesostris is reasonably clear. That the historians of Alexander could not allow the claims made for Sesostris in regard to India, is easily intelligible: they would rob their hero of his proudest laurels.[133]

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[129] Jacoby, Pauly-Wissowa VII, 2717, speaking of the account of Scythia given by Hecataeus of Miletus, says: "Sie ist offenbar als Ganzes durch Herodot verdrängt. Dieser benutzt Aristeas und eigene Erkundigung; aber es erscheint unzweifelhaft, dass ein grosser Teil seiner Angaben auf Hekataios zurückgeht der . . . sich der durch Dareios' Expedition gewonnenen Kenntnisse als erster bedienen konnte."

[130] II, 102.

[131] I, 55, 1.

[132] Attention was called above (p. 204, n. 54) to the parallel between the monuments of Darius at the Bosporus and those of Sesostris at Dira.

Surveying the story of Sesostris as a whole one readily sees that it is a fiction. Based, no doubt, on popular legends current in Egypt, it was elaborated by a Greek who made it the vehicle of a historical and geographical reconstruction, partly perhaps to take the wind out of the sails of Darius. But if that be so, where and what were the shoals that Sesostris is reported to have encountered? At first sight it would seem that we could say no more than that they were located somewhere to the eastward in the Arabian Sea and that they proved an obstacle to the king's further progress. But are we not entitled to draw further inferences? The definite comparison with Darius; the statement that "the Indian River" flows eastward; the conclusion of Herodotus that the expedition of Scylax revealed that all Asia "saving the parts toward the rising sun" was washed by the sea; and the statement of Diodorus that Sesoosis overran the whole of India up to the ocean;—all this creates the presumption that the fleet of Sesostris was supposed to have gone as far as the eastern limit of the southern shore of Asia.

Why, then, the shallows there,[134] barring further progress? Was it merely the counterpart of the shallows off the Pillars of Heracles, in the sense of the proverbial exhortation to recognize the limits of human achievement—a variant of the Delphic γνῶθι σεαυτόν? There is something to be said for such a view, because, as Bury[135] says, "it was a principle of the early Greek geographers who arose about the end of the sixth century in Ionia that the features of the earth were symmetrically arranged." Thus Herodotus, despite his frequent sarcasms at the expense of the Ionian cartographers, himself falls into their way of thinking when, in deriding the notion of the Hyperboreans, he says,[136] that if there are Hyperboreans we must likewise assume Hypernotians. But was there nothing more to suggest the notion of shallows in those parts? We have seen that the geographer whom Herodotus follows in his sketch was greatly interested in the limits of the continents and the possibility of circumnavigating them. The problem of Africa in particular had engaged his attention, and the expeditions of the Phoenicians under Necho and of Scylax under Darius had given him conclusive evidence that there was no land connection between them except at the Isthmus of Suez. Our study has shown, however, that not all geographers accepted that view, but some at least held to the ancient belief that Asia, specifically India, was united to the African Ethiopia by a land-bridge. It is obvious that the geographer who told the story of the expedition of Scylax and that of the eastern campaign of Sesostris had abandoned that belief. That this was Hecataeus of Miletus I do not doubt. But if it was, then we may venture a further guess. That Hecataeus was deeply interested in physical geography is as certain as anything can be; for it cannot be denied that Herodotus owed to him his description of the φύσις of Egypt. Furthermore, the reference in Herodotus to the cataclysms which had had disastrous results except in Egypt must be derived from Hecataeus, who is mentioned immediately afterwards. And it is to this very context, as we have seen, that Plato appeals in support of his fiction of Atlantis. What, then, shall we say regarding the shoals in the eastern ocean? Were they not the remains and the reminders of a cataclysm which had engulfed the ancient land-bridge between Asia and Africa?

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[133] Megasthenes *ap.* Strabon. XV, 1, 6, denied the expedition of Sesostris to India, but admitted those of Heracles and Dionysus, no doubt because they were gods and worthy exemplars of Alexander.

[134] Cf. Suidas, *s. v.* ἄπλωτα πελάγη, quoted above, p. 208, n. 66.

[135] *History of Greece*, p. 236. Bury based his statement chiefly on the article by J. L. Myres, "An Attempt to Reconstruct the Maps Used by Herodotus," *Geographical Journal*, Dec. 1896, p. 608 f.

It is not claimed that this demonstrates the origin of the notion of Atlantis,—such matters are hardly susceptible of demonstration,—and this essay was conceived as a suggestion for the consideration of scholars, or if one will, as a *divinatio*. One may, however, affirm with confidence that the direction in which the solution of this old problem is sought is not far wrong. Certainly it is more reasonable to look for the answer in Greek history and geography than in the latest findings of modern geology. Granted that in Mesozoic times there was land-connection successively between Africa and India and between Africa and South America, and that in a later geological age there was a considerable body of land in the Atlantic that was engulfed in consequence of volcanic action,—what reason is there for supposing that Plato could by any means have got either knowledge or report of them?

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[136] IV, 36; cf. Strabo I, 3, 22. See also Herod. II, 26, Arist., *Meteor.* 362 a 32 ff., 362 b 30 ff.

If one is disturbed by the evidence that speculation played so large a rôle in the beginnings of history and geography, which, like most of the sciences, were founded by the Greeks, one may be comforted by the reflection that in all things the factor of greatest importance is the idea; once that is put forward, positive and negative evidence is bound to be discovered, and upon the data thus brought to light a structure can gradually arise that may rightly command the assent of a schooled intelligence. In history and geography the sixth century B. C. was prolific of ideas; within restricted areas even the application of many of them has been approved by the modern world, and in more general ways it is obvious that the methods then suggested, though requiring refinement and criticism, are those which experience has shown to be the best that man can devise.

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GYROMAGNETIC EXPERIMENTS ON THE PROCESS OF MAGNETIZATION IN WEAK FIELDS.

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California Institute of Technology.

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§1. Qualitative theory and early experiments. When a rod of iron (or other magnetic substance) is rotated about its axis it becomes longitudinally magnetized, and in the direction opposite to that in which it would be magnetized by an electric current flowing around it in the direction of rotation.[1] The magnetization thus produced is not due to the rotation of the rod as a whole, however, but to the rotation of the magnetic elements which are entrained in the motion of the rod. Hence the idea arose that if the rod, at rest, were magnetized transversely, and the magnetization rotated by a two phase electrical system in a plane normal to its axis, longitudinal magnetization should result.

In or about 1921 Dr. Lewi Tonks began an investigation based on this idea, but abandoned it because his apparatus was inadequate for the purpose.[2] The subject was again, a little later, in 1922, and independently, attacked by J. W. Fisher,[3] in London; but the extraneous disturbances were so great that no useful conclusions could be reached. Fisher attacked the problem again, and with much greater success, in 1925,[4] when he obtained null effects with magnetite and iron. These later experiments were made by magnetometer methods, and at frequencies from 2×10^4 to 5×10^4 per second. In the first group, on magnetite in an applied field of about 4 gausses, deflections of 1 mm or less (and in the wrong direction when they occurred) were obtained; while rotation of the rod at the same frequency would have given about 35 cm. In the second group, on magnetite and iron, with applied field strengths of about 100 gausses, deflections of a few mm (and in the wrong direction) were obtained; while rotation at the same frequency, in the case of magnetite, would have given about 13 cm.

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[1] S. J. Barnett, Phys. Rev. 6, 1915, p. 239, *et al.*

[2] From information given to the author by Professor A. P. Wills and Dr. Tonks.

[3] J. W. Fisher, *Proc. Phys. Soc. Lond.*, 34, 1922, p. 177.

[4] J. W. Fisher, *Proc. Roy. Soc.* (A), 109, 1925, p. 7.

As Fisher points out in his paper, the gyrostatic magnetization would not be expected to appear unless the magnetic elements suffer at least a mean rotation when the field and the magnetization vector rotate. He at first expected them to do so, and he planned by his experiments to determine the gyromagnetic ratio much more readily than by experiments on the Barnett effect, in experiments on which high frequencies are not practicable.

§2. ROUGH QUANTITATIVE THEORY ON MOST FAVORABLE ROTATION HYPOTHESIS. Even if rotation of the magnetic elements with the field occurs, however, it can be shown[5] that on the most favorable hypothesis Fisher's experiments could be expected to give only a small fraction of the magnetization produced by rotation of the rod at the same frequency. The most favorable hypothesis assumes that when an element rotates with respect to the rod the torque due to adjacent elements and acting to prevent its alignment with the axis is identical with that which acts when all the elements rotate together, *i. e.* when the whole rod rotates. It is in all probability very much greater.

The order of magnitude of the effect to be expected on the above mentioned most favorable hypothesis can be obtained by the following rough process. Let μ denote the moment of the magnetic element, θ the angle it makes with the axis of the rod, N the number of elements per unit volume of the rod, and I the axial intensity of magnetization (always made initially as nearly zero as practicable). Then

$$I = \Sigma\mu \cos \theta$$

the summation extending over the unit volume.

If now an axial magnetic intensity H_0 is applied to the rod a change δI in I will result such that

$$\delta I = \Sigma\delta\mu \cos \theta = \Sigma\mu \sin \theta(-\delta\theta).$$

Now $-\delta\theta$ is proportional to the torque exserted on the element by H_0. That is

$$-\delta\theta = A\mu H_0 \sin \theta$$

where A is a constant for a particular element and will be assumed identical for all elements. Thus

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[5] S. J. Barnett, *Phys. Rev.* 27, 1926, p. 115 (Abstract).

$$\delta I = AH_0\mu^2 \sum_0^\pi \sin^2\theta = AH_0\mu^2 N \cdot 2/3$$

The rotating cross-field produces an intensity of magnetization $I_\perp$ normal to the axis of the rod. For simplicity it will be assumed that this intensity of magnetization is due to N' elements all oriented in its own direction. Then

$$\frac{N'}{N} = \frac{I_\perp}{I_\infty}, \text{ or } N' = N\frac{I_\perp}{I_\infty}$$

where I_∞ is the saturation intensity of magnetization of the material. If now the rotating field entrains with its own frequency (ν) each of the N' elements in each unit of volume, the effect on each will be identical with that of an axial magnetic intensity $H' = 2\pi\rho\nu$ (Barnett effect). An axial intensity of magnetization $\delta I'$ will result such that

$$\delta I' = A'H'\mu^2 \Sigma \sin^2\theta$$

where A' is constant like A. Since $\sin\theta = 1$ for every element involved, this equation may be written

$$\delta I' = A'H'\mu^2 N'.$$

From what precedes it follows immediately that

$$\frac{D}{D_0} = \frac{\delta I'}{\delta I} = \left(\frac{A'}{A}\,\frac{3}{2}\,\frac{I_\perp}{I_\infty}\right)\frac{2\pi\rho\nu}{H_0} = C\frac{2\pi\rho\nu}{H_0}$$

where D and D_0 are the deflections of the measuring instrument produced by δI and $\delta I'$.
Thus

$$D/D_0 = C\ 2\pi\rho\nu/H_0, \text{ or } D = CD_0\ 2\ \pi\rho\nu/H_0 = C\alpha \qquad (1)$$

where α is the deflection which would have resulted had the whole rod been rotated with the same frequency ν.

If the length of the rod is great in comparison with the diameter, it is easy to show that the cross-magnetization $I_\perp$ is given to a close approximation by the equation

$$I_\perp = \left(\frac{\mu - 1}{\mu + 1}\right)\frac{H_e}{2\pi}$$

where μ is the permeability of the rod and H_e is the field intensity when the rod is removed. When the permeability is far greater than unity, as in all the experiments with which we are concerned here, it is sufficient to replace the above equation with the following:

$$I_\perp = \frac{H_e}{2\pi}.$$

Thus we have for C, with all sufficient exactness,

$$C = \left(\frac{A'}{A}\right)\frac{3}{2}\frac{H_e}{2\pi I_\infty}. \tag{2}$$

Even when A'/A is put equal to 1, its maximum possible value, *and one far beyond its probable actual value*, C is always a small quantity, so that the value of D to be expected is much less than α.

In Fisher's first experiments on magnetite powder, in which the deflections D (always in the wrong direction and doubtless due to accidental and systematic errors) obtained were less than 1 mm. and calculation makes α about 35 cm., C was only about 1/400 with A'/A put equal to unity; so that on the most favorable hypothesis deflections as great as 1 mm should hardly be expected. In the last experiments with solid magnetite, in which D was a few mm. (again in the wrong direction) and calculation makes α about 13 cm., $I_\perp/I_\infty$ was estimated to be about 1/20, making $C = 1/13$. Hence the above hypothesis calls for deflections of the order of 1 cm. only. Experiments on iron powder also gave results of the same character.

§3. Early experiments by the author. When Fisher's second paper appeared I had under way an investigation of the same effect in permalloy wire (presented by the Bell Telephone Laboratories) by a method of electromagnetic induction.[6] The permalloy wire was

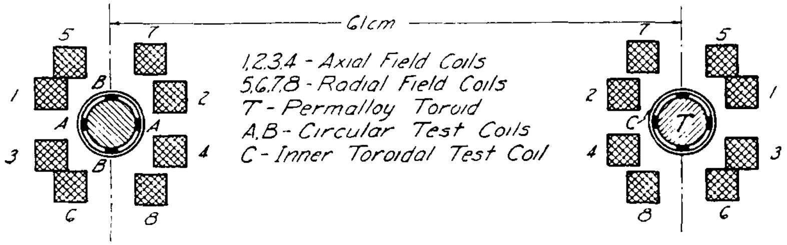
61cm
5
7
1
B
2
A
A
3
B
4
6
8
1,2,3,4 - Axial Field Coils
5,6,7,8 - Radial Field Coils
T - Permalloy Toroid
A,B - Circular Test Coils
C - Inner Toroidal Test Coil
7
5
2
1
C
T
4
3
8
6

Fig. 1.

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[6] For assistance in this work I am indebted to Mr. W. A. Arnold and Mr. J. S. Campbell.

formed into a toroid (Fig. 1) with a mean diameter of about 2 ft. and a circular cross-section with diameter about 2 in. After being annealed it was wound with tape and with narrow single layer circular test coils A, B, suitable for measuring the radial flux across its section, and also the flux across a section normal to the axis of symmetry. Another single layer coil *C* in the form of a close helix was wound over the whole length of the toroid; and over this, a continuous set of coils *D* of a great many turns, not shown in the figure, to make possible the measurement of minute changes of flux around the ring. The coil system *D* was connected with a ballistic galvanometer. The toroid with its coils was mounted in the annular space within two sets, 1–4 and 5–8, of parallel and coaxial circular coils, each set consisting of four coils. The mean diameter of each set was equal to that of the permalloy toroid. This toroid was coaxial with the coils, which were so placed that one set produced an approximately uniform radial field in the region occupied by the toroid, the other an approximately uniform field parallel to the axis of the toroid. The best arrangement had to be found by trial, and is illustrated in the figure. The whole system was placed in a toroidal vessel filled with molten beeswax and rosin, which, when it had cooled, held all parts solidly together. The solid system, looking somewhat like a huge doughnut, was then removed from the vessel and mounted on a wooden support with its axis in the direction of the earth's magnetic field.

The two sets of coils were connected to a two phase electrical system with a frequency of 50 cycles per second, which produced a rotary field throughout the permalloy ring, tending (according to the above theory) to magnetize it gyrostatically in one direction or the other around the ring according to the direction of rotation of the field. Calibration of the apparatus showed that it was sensitive enough to measure the effect calculated above (on the hypothesis $A = A'$); but the extraneous disturbances (largely due to heat) were so great as to mask completely any effect of this magnitude, and the investigation was temporarily abandoned.

The toroidal arrangement was adopted in order to eliminate the effect of fluctuations of the earth's magnetic field, and to avoid end effects and thus to take full advantage of the great permeability of permalloy.

§4. NEW EXPERIMENTS. GENERAL METHOD. It has always seemed to me practicable to go much beyond either of these investigations, and I have recently completed a piece of work which does so. The method used in the work just described is closely related to that of the first work on magnetization by rotation;[7] while the method used in Fisher's later work is closely related to that of the later work on mag-

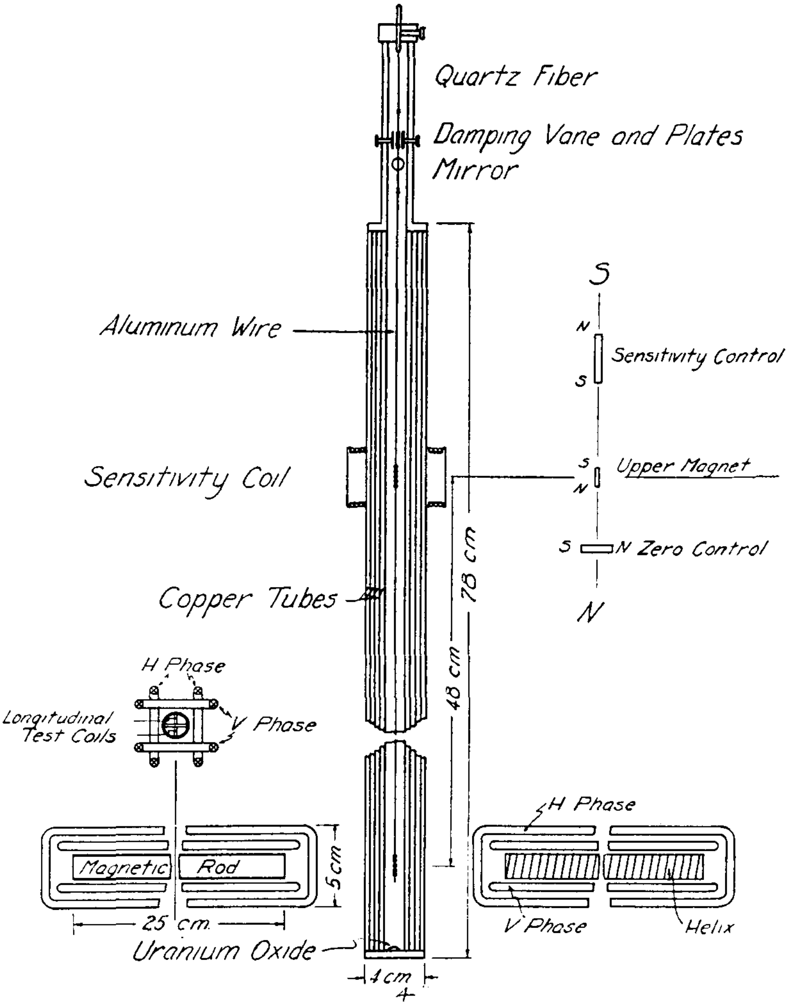
Quartz Fiber
Damping Vane and Plates
Mirror
Aluminum Wire
Sensitivity Coil
Copper Tubes
78 cm
48 cm
S
N
S
Sensitivity Control
S
N
Upper Magnet
S
N Zero Control
N
H Phase
Longitudinal Test Coils
V Phase
Magnetic
Rod
5 cm
25 cm
H Phase
V Phase
Helix
Uranium Oxide
1 cm

Fig. 2.

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[7] S. J. Barnett, *l. c.* in § 1.

netization by rotation[8]—viz., a magnetometer method. A magnetometer method has been used in the new work in preference to an electromagnetic induction method because it is possible with it to use smaller samples of magnetic material, to use higher frequencies, and to attain greater sensitiveness.

In the new work,[9] two similar round rods of the material to be studied (Fig. 2) were mounted with their axes horizontal and normal to the magnetic meridian. Half way between the centers of the rods, and in the same horizontal plane, was the lower magnetic system of an astatic magnetometer. The rods were placed usually east and west of the magnetometer (axial arrangement, Fig. 2) with their axes in the same straight line; but in some of the work they were placed north and south, with the line between their centers in the magnetic meridian (equatorial arrangement). Suitable test coils were wound upon the rods, and suitable magnetizing coils were provided to produce equal horizontal and vertical field intensities when supplied with high frequency currents from a two phase vacuum tube converter. Magnetometer deflections due to the reversal of the direction of rotation were obtained, and all measurements were made which were needed for the calculation of the quantities α and I_1/I_∞.

§5. THE MAGNETIC RODS AND THE TEST COILS. In the early part of the work, in default of better material, rods were built up of the finest iron wire obtainable here (0.007 inch diameter); but they were discarded because the eddy currents were too great. They were replaced with rods built up from small discs of compressed permalloy dust and iron dust, presented by the Bell Telephone Laboratories. From the rings supplied, flat discs 9/16 inch in diameter were cut out and packed closely, with cement between them, in bakelite tubes 5/8 inch external diameter and 1/32 inch thick, and squeezed together in a vise. The rods thus formed were about 25 cm. long. Over each bakelite tube three test coils were closely wound, viz., two ten-turn longitudinal coils at right angles to each other, and, over them for nearly the whole length of the tube, a helix—all of silk enamel copper wire No. 32. The whole system, with suitable leads, was then mounted in bakelite end rings, and inserted in a much longer bakelite tube of greater diameter, by which the whole could be easily handled and

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[8] See especially S. J. Barnett and L. J. H. Barnett, *Proc. Amer. Acad. of Arts and Sciences* 60, No. 2, 1925, pp. 125–216.

[9] S. J. Barnett, *Phys. Rev.* 43, 1933, p. 384 (Abstract). The predicted numbers quoted should be divided by 2.

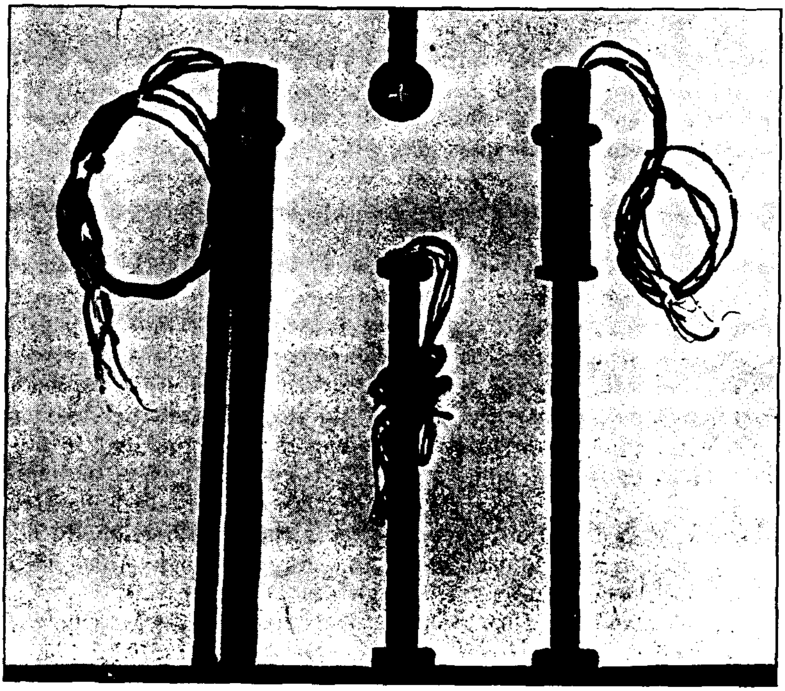

FIG. 3.

adjusted with reference to the magnetometer and magnetizing coils. Exactly similar coils were wound on bakelite rods provided with suitable holders for placing them exactly in the positions alternatively occupied by the coils on the magnetic rods. The arrangements are illustrated in Fig. 3 and Fig. 4.

The permeabilities of the iron and permalloy dust are about 35 and 75, respectively, in weak fields, and their saturation intensities of magnetization about 0.9 those of the solid materials.

§6. THE MAGNETIZING COILS. The magnetizing coils (Fig. 2) were wound in two groups, one to surround each rod or set of test coils. Each group consisted of two pairs, one pair (H coil) designed to produce an approximately uniform horizontal intensity in the region to be occupied by the rod; the other (V coil) a similar and equal vertical intensity. The approximate dimensions are indicated in Fig. 2. Each coil contained 2 x 100 turns of No. 32 silk enamel copper wire.

The mean width of each coil was about 4.0 cm., and the mean distance between the two halves of each coil about 2.2 cm. The coils of each group were wound between hardwood supports on a bakelite tube whose internal diameter was just larger than the external diameter of the rod-holders, and all were properly taped in place and provided with suitable leads. Each group was mounted horizontal in a wooden frame, which was attached by wax to a heavy marble slab mounted on a heavy wooden table under the magnetometer. See Fig. 4. The copper tubes forming the greater part of the magnetometer case (§7) extended downward through a hole in the marble slab. The two pairs of H coils were connected in series, and likewise the two pairs of V coils. To equalize the circuits, the longer coils in one group were made H coils, the longer coils in the other group, V coils. (One group should be turned through a right angle in the figure.)

§7. THE MAGNETOMETER AND ITS CONTROLS. The magnetometer is a modification of the one used in the later work on magnetization by rotation, and is illustrated in Fig. 2. Each system of the astatic pair was built of 11 magnets similar to those used earlier. An extra length of aluminum rod was inserted between the upper system and the mirror in order to make possible better protection of the moving parts from the high frequency magnetic field. For this purpose five coaxial copper tubes, each 1/16 inch thick, coaxial with the suspension and long enough to project well beyond the magnets, were made to replace the lower part of the former magnetometer case. A small piece of uranium nitrate at the bottom of the inclosed space prevented difficulties from electrification of the suspended system.

A small Helmholtz coil centered on the upper magnet made it possible at any time to determine the sensitivity of the instrument.

The sensitiveness and zero were adjusted with great facility by the control system used in earlier investigations.[10] Two small magnets, or groups of magnets, acting chiefly on the upper system, are used. One of them, with its axis parallel to that of this system, alters the sensitivity by motion toward or from the system without changing (much) the zero; the other, with its axis normal to that of the system, alters the zero by motion to or from the system without changing

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[10] See S. J. Barnett, *A sine galvanometer for determining in absolute measure the horizontal intensity of the earth's magnetic field. Researches of the Department of Terrestrial Magnetism,* Carnegie Institution of Washington, Vol. 4, 1921, pp. 373–394 (See p. 388). Also, S. J. Barnett and L. J. H. Barnett, *Proc. Amer. Acad. of Arts and Sciences,* Vol. 60, No. 2, 1925.

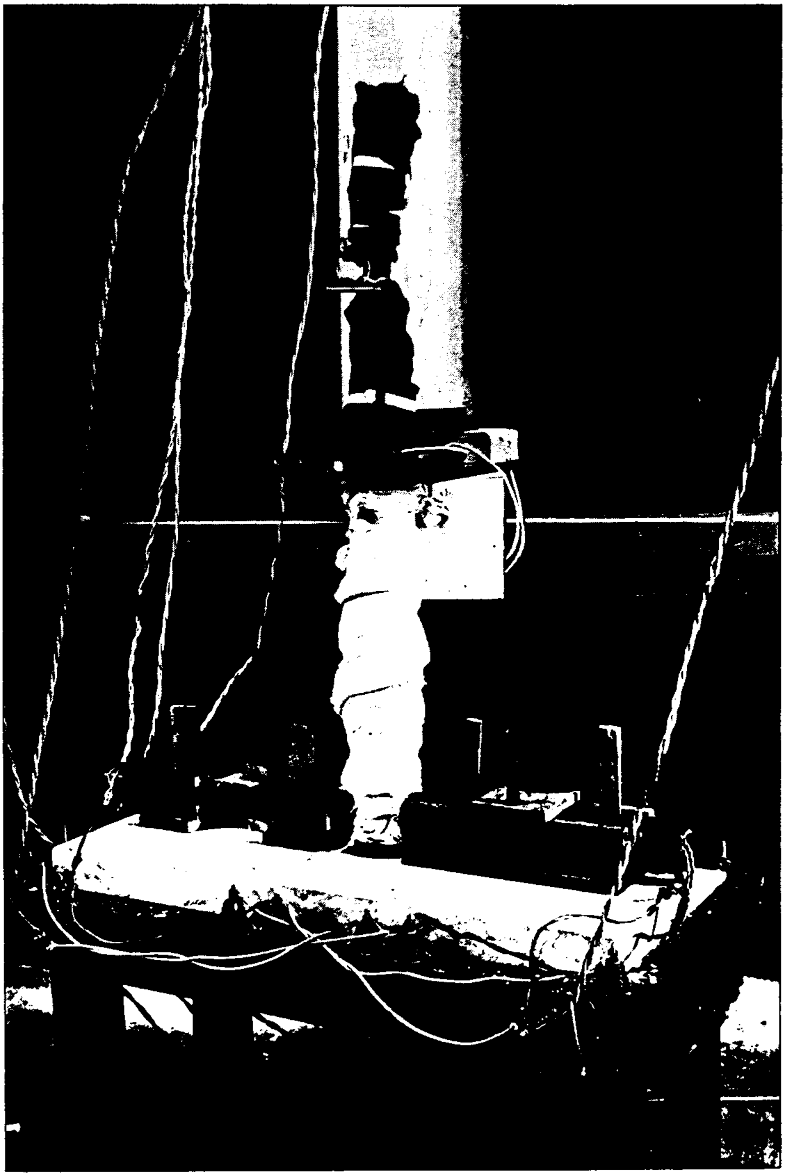

Fig. 4.

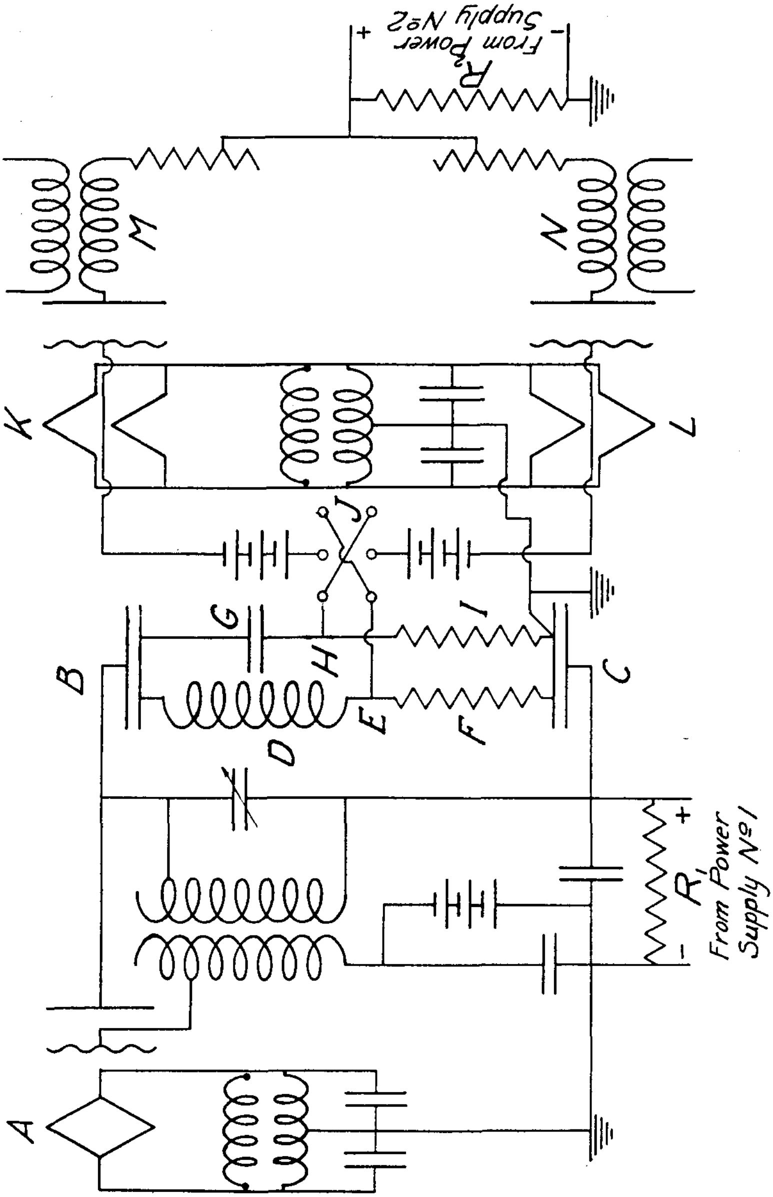
A
B
C
D
E
F
G
H
I
J
K
L
M
N
R_1
+
-
From Power Supply N°1
R_2
+
-
From Power Supply N°2

Fig. 5.

(much) the sensitivity. Gross adjustment by slides and fine adjustment by micrometers are provided. (Electric coils with adjustable currents can of course replace the magnets, and have sometimes been used, but not in this investigation.) The difficulty of this investigation, as well as that on magnetization by rotation, would have been greatly increased without this method of controlling the magnetometer.

The magnetometer, with its controls, was mounted in brass supports on the end of a heavy beam of marble bolted to and projecting out from a heavy concrete shelf supported by the floor and wall of the laboratory.[11]

In order to screen the magnetometer thermally and to prevent the formation of convection currents the magnetometer was covered almost completely with a thick coat of felt and other insulating material. See Fig. 4.

§8. THE POWER SUPPLY. Single phase power was obtained from the mains of the local transmission system at 50 cycles per second, and transformed by a vacuum tube system for use at the higher frequencies needed in two phases in quadrature. See Fig. 5.

A tuned-plate vacuum tube oscillator A[12] produces a voltage of constant frequency and of approximately sine form between the terminals B and C. Through two condensers this voltage is impressed on the two partial circuits DEF and GHI in parallel. D has inductance, but only negligible capacity and resistance; G, capacity, with negligible inductance and resistance; and F and I, resistance without appreciable capacity or inductance. The resistance of F is made equal to the reactance of D, and the resistance of I equal to the reactance of G. With these adjustments made, the voltages along F and I, whose lower ends are connected together, are in quadrature. Through a switch J, and biasing batteries, one of these voltages is impressed between the grids and filaments of a pair of 50 watt tubes K in parallel; the other, between grids and filaments of a similar pair L.[13] All these filaments are in parallel and supplied by the same filament transformer, the center of whose secondary is connected to the junction between F and I.

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[11] For assistance in mounting the magnetometer and the magnetizing coils I am indebted to Dr. Otto Beeck.

[12] For advice with regard to this part of the apparatus I am indebted to my colleague Professor S. S. Mackeown.

[13] This arrangement was used in the latter part of the work, a single battery having been connected between the junction of F and I and all the filaments in parallel in most of the work.

The power for the oscillator was furnished by an AC-DC full-wave vacuum tube converter with three-stage filter, the power for the 50 watt tubes by a similar converter of much larger capacity. The condensers and chokes were so large as to make very constant terminal voltages certain.

The positive terminal of the larger converter was connected through adjustable resistors and the primaries of two Western Electric high frequency transformers *M* and *L* to the plates of the two pairs of 50 watt tubes, the negative terminal to the center-tap of the filament transformer.

The secondaries of the transformers *M* and *L* were connected separately (Fig. 6) through reversing switches and similar adjustable condensers to the terminals of the magnetizing coils. By means of the condensers the two secondary circuits were tuned sharply to the oscillator frequency. In this way equal harmonic currents in quadrature were made to traverse the two sets of magnetizing coils, designated as H-phase and V-phase coils. By means of switches not shown in the figure a Weston thermo-ammeter could be thrown into either secondary at will, and a D. C. Weston ammeter into either of the plate circuits. Also, switches made it possible to exchange at will the transformer coils in the secondary circuits.

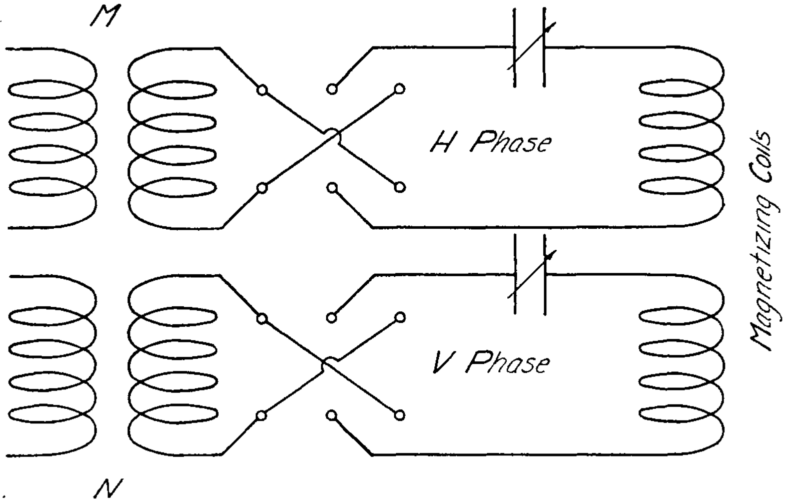

FIG. 6.

§9. THE FIELD PRODUCED BY THE MAGNETIZING COILS. To study this, one of the longitudinal test coils wound on its bakelite support was inserted in the position normally occupied by a magnetic rod in one of the groups of magnetizing coils, and turned about until it was parallel to the turns of one of the pairs of magnetizing coils. Then the flux through the test coil, and hence the mean magnetic intensity, were determined by the common ballistic method for a measured current in the magnetizing coil. This was repeated with the other longitudinal test coil and the other magnetizing coil of the group; then the whole process was repeated for the other group. The resulting values of the mean magnetic intensity per unit current were, as would be expected, nearly the same, the mean being about 30 gausses per ampère.

With the currents ordinarily used in most of the high frequency work, this makes the intensity of the rotating field about 15 gausses.

§10. TESTS FOR ABSENCE OF EDDY CURRENTS. The principal tests were made in two parts: (1) One of the longitudinal test coils was inserted in the proper position and turned about until it was parallel with the turns of one of the magnetizing coil pairs, as in §9. With a ballistic galvanometer in the test coil circuit, the mean throw D_B was determined when a known steady current was reversed in the magnetizing coil. The process, with the same current, was then repeated with a longitudinal test coil on one of the magnetic rods replacing the coil on bakelite, the mean throw D_M being determined; and the whole process was repeated for each of the materials to be studied. For each substance the ratio D_M/D_B was determined. (2) Instead of the steady direct current, a steady high-frequency current was passed through the magnetizing coils, and the electromotive forces E_M and E_B in the longitudinal test coil measured, both for the magnetic material and for the bakelite. For each of the magnetic materials the ratio E_M/E_B was calculated.

At the frequency 14650 cycles per second, E_M/E_B was found equal, or very nearly equal, to D_M/D_B, showing that no eddy currents of any consequence existed.

At the frequency 21000 cycles per second, E_M/E_B was somewhat less than D_M/D_B in the case of the permalloy, with which alone this frequency was used.

The practical absence of eddy currents at the lower frequency also follows from the fact that the currents in the magnetizing coils (the secondary circuits being tuned) were practically identical whether the magnetic rods were in their holders within the magnetizing coils, or were removed.

§11. TESTS FOR THE DIRECTION OF ROTATION OF THE FIELD. For every arrangement used the direction of rotation was directly determined by means of a small washer of brass or small aluminum ring carried by fine threads or bundles of silk fibres and inserted in the region within the field coils. There was thus never any uncertainty in this matter.

§12. QUADRATURE OF THE TWO PHASES. As stated above, the two phases must be in quadrature if the impedances D and F, Fig. 5, are equal, and the impedances G and I equal. Tests for these equalities were made by interchanging D and F, and by interchanging G and I. When the interchanges produce no alterations in the secondary currents of the appropriate phases, the impedances in each branch are equal and the secondary currents in quadrature.

Another test is to measure the electromotive force induced in one of the longitudinal test coils wound on the magnetic rod as the rod and coil are turned about the horizontal axis. If the electromotive force remains constant the currents are in quadrature.

Both these tests were made (but not usually at the same time) in this work and the currents were always found to be in, or nearly in, quadrature.

§13. ELIMINATION OF ERROR DUE TO RESIDUAL MAGNETIZATION. Inasmuch as it is never possible to obtain exactly identical currents in the magnetizing coils for the two directions of rotation, and as residual magnetization—a function of eddy currents and temperature—is always present to a greater or less degree, it was necessary to eliminate possible error from this source. This was done simply by interchanging the two rods and reversing their directions. The mean result is independent of the error. As a matter of fact, however, the error vanished in this work, even in the case of the iron dust, which could not be demagnetized by any means so well as the permalloy dust.

§14. ELIMINATION OF ERRORS DUE TO LEAKAGE BETWEEN PRIMARY AND SECONDARY CIRCUITS. If there is leakage between the circuits of one of the phases, let us say the H-phase for definiteness, and if the field of the H coils through asymmetry has a component normal to the magnetometer magnet or along the magnetic rods, then, if we close the V switch *on the right* (for definiteness) and *reverse the H switch*, the magnetometer deflection D' will be the sum $D_R + D_L$ of two parts, one (D_R) due to the effect of rotation being looked for (if any), and one (D_L) due to the leakage. If now we set the V switch *on the left,* and then reverse the H switch *in the opposite direction,* the deflection D'' will clearly be equal to $D_R - D_L$. Thus the mean of D' and D'' will be equal to D_R. In making the observations quoted here this procedure was always applied to both phases, although no appreciable effect of leakage was manifest with the permalloy rods, or with rods absent. In every set, equal numbers of observation were made for each of the cases: (1) H-switch left, V-switch reversed right to left and left to right; (2) H-switch right, V-switch reversed right to left and left to right; (3) V-switch left, H-switch reversed right to left and left to right; (4) V-switch right, H-switch reversed right to left and left to right. The mean double deflection for (1), (2), (3), and (4), proper attention being paid to directions of reversal, was thus independent of leakage in either secondary.

§15. THE AXIAL AND EQUATORIAL ARRANGEMENTS. In all the final work with the magnetometer in the axial position the ends of the rods were about 7 cm. from the magnetometer axis; while in that with the equatorial position the axes of the rods were about 9 cm. from this axis.

§16. THE ABSOLUTE AND ROD SENSITIVITIES AND COILS. The small coils in the Helmholtz pair centered on the upper magnetometer magnet (Fig. 2) each had 3 turns with mean diameter 2.9 cm. and were at the mean distance apart of 3.45 cm. Their constant was thus 0.69 gauss per ampère. Reversal of a current 1.55 volt/80,000 ohms through these coils ordinarily gave a deflection, A_0, of about 12 cm. or more. This 12 cm. deflection corresponds to a deflection of 1 mm. for 1.1×10^{-7} gauss. The deflection A_0 obtained in this way will be referred to as the *absolute sensitivity.*

The helices wound longitudinally over the rods had about 710 turns in 23.2 cm. The reversal of a current 1.55 volt/180,000 ohms through the helices on both rods in series produced a double deflection R_0 which will be referred to as the *rod sensitivity.*

The magnetic intensity at the center of a rod impressed by the current mentioned in the helices is approximately $H_0 = 3.3 \times 10^{-4}$ gauss. In the work described here R_0 ranged from about 18 cm. to about 37 cm.

If the two helices wound on bakelite are made to replace the helices wound on the magnetic material, the reversal of the same helix current produces a double deflection B_0, which, for the same value of A_0, is of course much smaller than R_0.

For the experiments described here the mean values of R_0 for iron dust at the lower frequency, permalloy dust at the lower frequency, and permalloy dust at the higher frequency were 20.9 cm., 31.4 cm., and 33.1 cm., respectively. For the values of A_0 corresponding to these values of R_0, the mean values of B_0 were 1.5 cm., 1.4 cm., and 1.4 cm., respectively. Thus the corresponding mean values of D_0 were 19.4 cm., 30.0 cm., and 31.7 cm., respectively.

§17. PROCEDURE IN MAKING A SET OF OBSERVATIONS. A complete set of observations, aside from the currents in the magnetizing coils, which were read at proper times, comprised the following determinations, which were made in the order indicated: (I) Absolute sensitivity A_0 (7 scale readings) with magnetic rods removed; (II) double deflections D due to reversal of H-switch from right to left and left to right (3 groups of 11 scale readings each) with V-switch on left, and deflections due to reversal of V-switch from right to left and left to right (3 groups of 11 scale readings each), with H-switch on the left, the two sets of groups being "sandwiched"; (III) repetition of (I), but with the magnetic rods in place; (IV) rod sensitivity R_0 (7 scale readings); (V) repetition of (II), but with the rods in place; (VI) repetition of (V), but with the roles of the V and H switches interchanged; (VII) repetition (IV); (VIII) repetition of (III); (IX) repetition of (II) (rods removed), but with the roles of the V and H switches interchanged; (X) repetition of (I). All observations were made on a time schedule, the mean times for right and left readings in each group being identical. Between (II) and (III), between (V) and (VI), and again between (VIII) and (IX), the magnetizing coils and rods (when present) were cooled by a fan motor. A complete set of observations required over five hours to complete.

§18. SUMMARY FOR ONE SET OF OBSERVATIONS. As an example, the observations on permalloy for November 6, 1932, are summarized here. The Roman numerals refer to the groups of §17, the Arabic numerals designate the order of the sub-groups. Only the mean result for each sub-group is given.

Rods removed (Air)

I. $A_{0A} = 21.76$ cm.
Mean value of current = 315 m. a.
X. $A_{0A} = 20.77$

Mean $A_{0A} = 21.27$

	V-Switch L		H-Switch L	
II. D_A =	− 0.16 cm.	(1)	+ 0.36 cm.	(2)
	+ 0.04	(3)	− 0.04	(4)
	+ 0.15	(5)	− 0.11	(6)
Mean	+ 0.01		+ 0.07	

Mean + 0.04

	V-Switch R		H-Switch R	
IX. D_A =	+ 0.18 cm.	(1)	− 0.33 cm.	(2)
	− 0.07	(3)	+ 0.07	(4)
	+ 0.05	(5)	− 0.13	(6)
Mean	+ 0.05		− 0.13	

Mean − 0.04

Mean D_A = 0.00 cm.

Rods in Place

III. A_{0M} = 13.98 cm.	IV. R_0 = 27.77 cm.
VIII. A_{0M} = 15.01	VII. R_0 = 28.98
Mean A_{0M} = 14.50	Mean R_0 = 28.38

	V-Switch L		H-Switch L	
V. D_M =	− 0.04 cm.	(1)	+ 0.02 cm.	(2)
	+ 0.10	(3)	− 0.03	(4)
	− 0.14	(5)	+ 0.26	(6)
Mean	− 0.03		+ 0.09	

Mean + 0.03

VI. D_M =	− 0.11 cm.	(1)	− 0.17 cm.	(2)
	− 0.14	(3)	− 0.05	(4)
	− 0.17	(5)	− 0.04	(6)
Mean	− 0.14		− 0.09	

Mean − 0.11

Mean D_M for Permalloy − 0.04 cm.

It happens in this case that the mean deflection D_A for air is 0.00. Had this not been the case, the procedure would have been to multiply D_A by the ratio A_{0M}/A_{0A} and to substract this from D_M to get the effect on the permalloy alone.

The mean deflection for permalloy is *negative;* whereas in this case a positive deflection, if any, should have occurred according to the theory.

The magnetometer was in the equatorial position.

§19. SUMMARY OF RESULTS FOR IRON DUST. Eight complete sets of observations were made upon iron. The mean deflections D, in cm., classified according to type of reversal, are as follows: (VL) − 0.075; (HL) + 0.264; (VR) + 0.051; (HR) − 0.245, mean of all 0.00. Here VL means that the V-phase switch was set on the left, and the H-phase switch reversed; HR, that the H-phase switch was set on the right, and the V-switch reversed, etc. The positive sign means a deflection in the direction predicted by theory; the negative sign, one in the opposite direction. The results show a decided leakage effect, eliminated from the mean.

For the eight sets the mean deflections and errors (leakage eliminated) are as follows: Air alone, + 0.01 ± 0.02; iron alone, 0.00 ± 0.03; iron—air, − 0.01 ± 0.03.

Five sets were made with the rods turned in one direction, three with the rods interchanged and reversed. The two groups give − 0.02 ± 0.04 and + 0.01 ± 0.01, respectively.

The mean value of the currents was 337 m. a., and the mean value of R_0, 20.9 cm.

§20. SUMMARY OF RESULTS FOR PERMALLOY DUST. At the lower frequency, about 14,650/sec., fourteen complete sets of observations were made on permalloy. With the conventions adopted in §19, the mean deflections classified according to type of reversal are (VL) + 0.07 ± 0.10; (HL) − 0.03 ± 0.12; (VR) − 0.05 ± 0.12; (HR) 0.00 ± 0.07. The corresponding observations on air give (VL) − 0.00 ± 0.04; (HL) + 0.01 ± 0.05; (VR) + 0.04 ± 0.06; (HR) − 0.02 ± 0.07. Here the effect of leakage, if existent, is very small.

For the fourteen sets the mean deflections and errors (leakage eliminated from each set) are as follows: Air alone + 0.03 ± 0.07; permalloy alone, − 0.01 ± 0.13; permalloy—air, − 0.04 ± 0.14.

The mean value of the current was 352 m. a. and the mean value of R_0, 31.4 cm.

Three sets were obtained at the frequency 21,000, giving the mean deflection + 0.03 ± 0.02. The mean value of R_0 was 33.1 cm., and the mean value of the current 171 m. a.

§21. CALCULATION OF MAXIMUM DEFLECTIONS TO BE EXPECTED ON ROTATION HYPOTHESIS. From (1) to (2), §2, we have, when A'/A is put equal to unity,

$$D = \frac{3}{2} \cdot \frac{\rho\nu}{I_\infty} \cdot \frac{H_e}{H_0} \cdot D_0. \qquad (1)$$

For compressed iron dust and permalloy dust, whose densities are about nine-tenths those of the homogeneous metals, we may take the saturation intensities of magnetization (I_∞) as about 1550 and 1550 x 11/21, respectively.

For permalloy at the lower frequency the mean value of H_e was about 30 x 1.41 x 0.35 gauss (§§9, 20), the rms. value of the current having been read by the ammeters. The values of D_0 and H_0 were 30.0 cm. and 3.3 x 10^{-4} gauss (§16). The lower frequency was about 14,650 per second, and ρ may be taken as m/e x 1.05 e.m. units, or 1.05 x 0.57 x 10^{-7} e.m.u. Hence, by substitution in (1), we obtain $D = 2.1$ cm. In the same way, for iron at the lower frequency, we obtain $D = 0.7$ cm.; and for permalloy at the higher frequency, 21,000 per second, we obtain $D = 1.5$ cm. The last value should be reduced somewhat because eddy currents were appreciable. All the values are slightly too high because the two phases were not, in general, in perfect quadrature.

As the results show, the deflections actually obtained were in all cases far smaller than the values of D calculated above on the most favorable hypothesis.

§22. FURTHER DISCUSSION OF RESULTS. When the magnetization vector rotates, as in these experiments, the conditions are far more complicated than when a rod itself rotates, and it is impossible to give any detailed theory of what happens if the elements themselves rotate. The results of the experiments, together with discussions of the subject with others, especially Professor Einstein and Professor Epstein, have convinced me that any rotation of the elements which could possibly occur would be accompanied by changes of magnetization much less than the minimum detectable in this work ($A' << A$ in §2) and that the most probable interpretation of the null effect is, as Professor Einstein suggested two years ago, that the elements do not rotate with the field at all, but have their moments periodically *reversed* by the rotating field.

Thus, for simplicity, we may imagine half the elements to have moments with vertical and axial components only, and half to have moments with horizontal and axial components only. The alternating vertical intensity will then change the vertical cross magnetization by reversing moments of the first group, and the alternating horizontal intensity will change the horizontal cross magnetization by reversing moments of the second group. Thus the magnetization vector will rotate without any rotation of the elements, and thus without the production of any longitudinal magnetization by a gyromagnetic process.

There have long been other indications favoring the hypothesis that in the early part of the magnetizing process (weak fields) the magnetization proceeds by such quantum jumps, or reversals, of the elements, and not by changes in their orientation; and the work described here, which has to do with weak fields only, strongly supports the other evidence. In strong fields, however, the indications are that changes of magnetization are due to gradual changes in the orientation of the elements. If this is the case, a repetition of the investigation described here, but with fields strong enough nearly to saturate the magnetic material, should yield positive results instead of a null effect. But there are obviously great difficulties in the way of such an investigation. Professor Einstein has suggested to me an equivalent investigation in which it is hoped that the difficulties will be less, and work on this has been started.

§23. FURTHER ACKNOWLEDGMENTS. I am under many obligations for help in addition to those mentioned in what precedes. The Board of Research of the University of California, and the California Institute of Technology, have given generous support to the work. Special mention must be made also of the Carnegie Institution of Washington and the National Research Council for indispensable equipment; of Dr. Arnold, Dr. Buckley, and the Bell Telephone Laboratories for materials, equipment, and information of great importance; of Dr. Vaino Hoover, who, as research assistant, has given me very valuable help in the recent part of the investigation; and of the instrument-makers of the University, and the Institute, whose work has been excellent. The experiments have been conducted in the Norman Bridge Laboratory of Physics of the Institute.

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THE SIGNIFICANCE OF CELLS AS REVEALED BY THEIR POLYHEDRAL SHAPES, WITH SPECIAL REFERENCE TO PRECARTILAGE, AND A SURMISE CONCERNING NERVE CELLS AND NEUROGLIA.

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Introduction.—The literature of this neglected phase of cytology is scanty, and has been reviewed in the previous papers of this series.[1] In summary, Brisseau-Mirbel (1802 and 1809) pictured in three dimensions what he considered to be typical cells of vegetable parenchyma; and Kieser (1815 and 1818), recognizing that Mirbel had drawn truncated rhombic dodecahedra, adopted and exploited the idea, believing it to rest on a firm physical basis. But cells are so irregular that biologists took only a waning interest. In D'Arcy W. Thompson's remarkable book, described by its author as "all preface—an easy introduction to the study of organic Form by the methods which are the common-places of physical science"—the question of the shape of cells is again introduced, for physicists had meanwhile provided *a priori* reasons for assuming cells to be tetrakaidecahedral. Yet balls of putty compress into dodecahedra, and Professor Thompson, influenced apparently by the 12-rayed stars of Juncus pith, was inclined toward the older view.

Knowing that the actual shape of certain cells could be determined by laborious application of the wax-plate reconstruction method, I began what is now more than a ten years' study of cell forms. The initial question has been definitely answered. Tissues are not composed of rhombic dodecahedral cells, truncated or otherwise, for those shapes have characteristic tetrahedral angles which cells avoid. In a mass of cells of approximately uniform size, the average cell has 14 faces of contact with its neighbors: it is a tetrakaidecahedron. Data in proof of this are gathered in Table I.

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[1] Lewis, F. T. The typical shape of polyhedral cells in vegetable parenchyma and the restoration of that shape following cell division. Proc. Amer. Acad., 1923, vol. 58, pp. 537–552.

—— A further study of the polyhedral shapes of cells. I. The stellate cells of *Juncus effusus;* II. Cells of human adipose tissue; III. Stratified cells of human oral epithelium. Proc. Amer. Acad., 1925, vol. 61, p. 1–34.

As previously published, 100 cells of elder pith, taken at random, have the number of contacts shown in the top line of the table, with an average of 13.97 facets. One hundred fat cells, also previously reported, are in the line below, with an average of 14.01 facets. Fifty precartilage cells of the tadpole, here presented for the first time, complete the tabulation; they average 13.9 facets. Altogether the 250 cells have an average of 13.97 facets. Not only from this counting of actual faces, but from geometrical considerations to be explained presently, we conclude that cells massed in tissues are primarily tetrakaidecahedra.

TABLE I.

No. of Facets	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Elder pith 100 cells	1	1	2	0	2	8	8	21	16	19	10	2	3	6	1	
Human fat 100 cells				2	4	5	15	13	20	15	17	4	2	2	1	
Precartilage 50 cells				1	1	4	9	8	10	7	3	3	2	1	0	1
Total	1	1	2	3	7	17	32	42	46	41	30	9	7	9	2	1

No longer should the cautious histologist evade this issue by using the term 'polyhedral.' Schaffer, quoting the observation that fat cells have from 9 to 20 facets, comments that it is idle to pursue the subject further—es wohl müssig erscheint, die Zahl der Kontaktflächen näher zu erörtern.[2] But when Nemiloff can write that "viele beschriebenen histologischen Strukturen sind sozusagen „Gottesmutterhaar", welches nur Gläubige sehen können,"[3] any feature of cells definite enough for mathematical expression seems worth investigating. What causes underlie the tetrakaidecahedral shape?

The essential requirement for the production of 14-hedra.—It is quite evident that tetrakaidecahedra are the three-dimensional expression of forces which produce the characteristic patterns of epithelial mosaics. Since it is easier to deal with plane surfaces than with solids, several preliminary studies have been made of the patterns of simple epithelium in surface view.[4] In such mosaics, as is well known, usually three cells meet at every corner. Exceptionally 4-rayed vertices occur, and sometimes frequently. The effect which they have upon the pattern is definite and can be mathematically expressed. But at the outset it may be assumed that only 3-rayed vertices occur. Then, if the mosaic entirely covers any solid body, whether a sphere, a cucumber-shape, or the complex form of an embryo, its polygons will have twelve less sides than if they were all hexagons.[5] This loss of sides is in connection with the rounded form of the body, and a deficiency of six would be expected at either end of the cucumber. If the polygons are small and there are thousands or millions of them, their average number of sides in any little flat area will be exactly six, except as chance may lead to such incidents as including a pentagon in the patch and excluding the balancing heptagon, or to including some of the polygons contributing to the deficiency of twelve sides as just mentioned.[6]

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[2] Schaffer, J. Die Stützgewebe. Hdb. d. mikr. Anat. d. Menschen, herausgegeben von W. v. Möllendorff, Bd. 2, Tl. 2, 1930, p. 75.

[3] Nemiloff, A. Über einige Widersprüche in der heutigen Anatomie und Histologie. Anat. Anz., 1932, Bd. 73, p. 362.

[4] Lewis, F. T. The effect of cell division on the shape and size of hexagonal cells. Anat. Rec., 1926, vol. 33, p. 331–355.

Thus it comes about that if many central points or nuclei are scattered at random over a flat surface, which is then subdivided into polygons so that each contains a nucleus, and the polygons invariably form 3-rayed vertices with their neighbors, the resulting mosaic will duplicate the cellular pattern in consisting of polygons with an average of close to six sides. Hence it may be inferred that hexagonal cells are not circles compressed into hexagons: they arise from the subdivision of a syncytial or plasmodial mass, by the formation of walls separating each nuclear territory from those that adjoin it. *Cell division is essentially a continuation of this process.* Before any walls have formed, nuclear division is tantamount to cell division; but after the walls have appeared, nuclear division merely produces bi-

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Lewis, F. T. The correlation between cell division and the shapes and sizes of prismatic cells in the epidermis of Cucumis. *Ibid.*. 1928, vol. 38, p. 341–376.

—— A volumetric study of growth and cell division in two types of epithelium,—the longitudinally prismatic epidermal cells of Tradescantia and the radially prismatic epidermal cells of Cucumis. *Ibid.*, 1930, vol. 47, p. 59–99.

—— A comparison between the mosaic of polygons in a film of artificial emulsion and the pattern of simple epithelium in surface view (cucumber epidermis and human amnion). *Ibid.*, 1931, vol. 50, p. 235–265.

—— Mathematically precise features of epithelial mosaics: observations on the endothelium of capillaries. *Ibid.*, 1933, vol. 55, p. 323–341.

[5] There would be a further deduction of two sides for every 4-rayed vertex, of 4 sides for every 5-rayed vertex, etc.

[6] The mathematical conditions requisite for finding an average of precisely six sides in a given patch of cells have been formulated by W. C. Graustein, Annals of Mathematics, 1931, vol. 32, p. 149–153. The deduction of twelve sides in the mosaic covering a solid, or lining a cyst, is explained by the writer in the Anatomical Record, 1933, vol. 55, p. 323–341.

nucleate cells, and cell division awaits the formation of the intervening partitions. The new cell walls, like the old, form 3-rayed intersections only, and therefore the cells continue to be, on the average, hexagonal. Intersections with more than three rays are unstable and consequently eliminated.

The instability of a quadruple intersection may be simply illustrated with a model, using four light elastic bands of equal length and thickness (the rubber 'elastics' for securing parcels). On a drawing-board make two lines crossing at right angles (though other angles serve as well), and along these lines insert four thumb tacks equidistant from the point of intersection. The distance chosen should be two or three times the length of the unstretched elastics. Join two of the elastics end to end making one of double length, and having slipped it through the other two (Fig. 1, A) stretch it from thumb tack *c* to *a*. The other elastics are placed around *b* and *d*. The four will not remain above the lines crossing at right angles, but will snap toward the position shown in Fig. 1, *B*, in which they are at rest with angles between them approximating 120°.[7] Professor P. W. Bridgman describes the situation as follows:

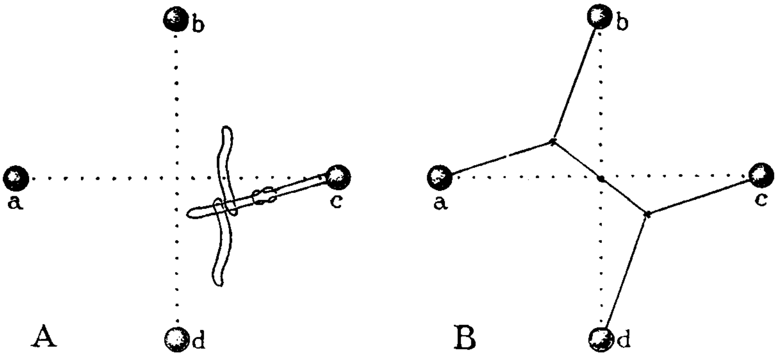

FIGURE 1. Arrangement of elastic loops to show the instability of a 4-rayed intersection.

> Four equal tensions are not in *stable* equilibrium when meeting at a common point. A very simple geometrical calculation shows that the total length of the heavy lines in Fig. 1, *B*, is less than that of the dotted lines. Hence according to the mechanical principle that a system seeks the configuration of

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[7] The same thing may be beautifully shown in a moving picture of soap-bubbles rising over a plate glass surface, such as the author exhibited at the session of the Amer. Assn. of Anatomists in 1930. But the model with the elastic bands seems quite sufficient.

least energy, the configuration with the quadruple point will pass over into that with two triple points, provided the transition can be made. Opportunity for the transition is afforded by the slightest accidental disturbance. In other words, the system with quadruple points is in unstable equilibrium.

The quadruple point or four-rayed intersection, as it exists on the surface of a liquid polyhedron or cell, constitutes a tetrahedral angle. It is a corner of a cell where four of its facets meet at a point, instead of three. Like the 4-rayed intersection in the plane mosaic, it is unstable, and is commonly resolved into two triple points or trihedral angles connected by a line or edge. This has been made very familiar by Plateau's wire cube dipped in soap solution. The directions of the forces involved are more clumsily but perhaps more clearly expressed by arranging elastics as in Fig. 2, *A*. Into each of two drawing-boards insert four thumb tacks at the corners of an 8-inch square, and invert one board over the other at a height of 8 inches. Stretch four elastics from the thumb tacks in the upper square to those in the corresponding corners of the lower square, and tie the four together at the central point, completing the arrangement in Fig. 2, *A*. If the elastics were connected with one another by films extending from the center to the thumb tack squares (but not crossing the squares diagonally) these films would bound six tetrahedral angles meeting at the central point. At the center there is a lateral stress tending to a different configuration. If instead of being tied together at the center, the elastics there pass through another and rather short elastic, they will stretch it into more or less of a square as seen in Fig. 2, *B*. Friction offers some resistance, but all the tetrahedral

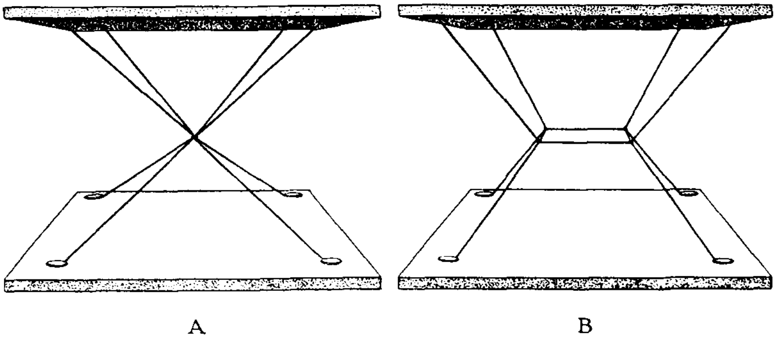

FIGURE 2. Arrangement of elastic loops to show the instability of tetrahedral angles.

angles are replaced by trihedral angles. Since the elastic skins of liquid drops—the cell membranes—act in this way, avoiding as a rule the unstable tetrahedral angles and 4-rayed intersections, it is inevitable that cells in masses shun the rhombic dodecahedral form, but a further condition must be met for an average of 14 facets.

The production of 14-hedra by the avoidance of tetrahedral angles and 4-rayed intersections.—That compliance with this requirement may yield tetrakaidecahedra is shown in a conventional diagram, Fig. 3.

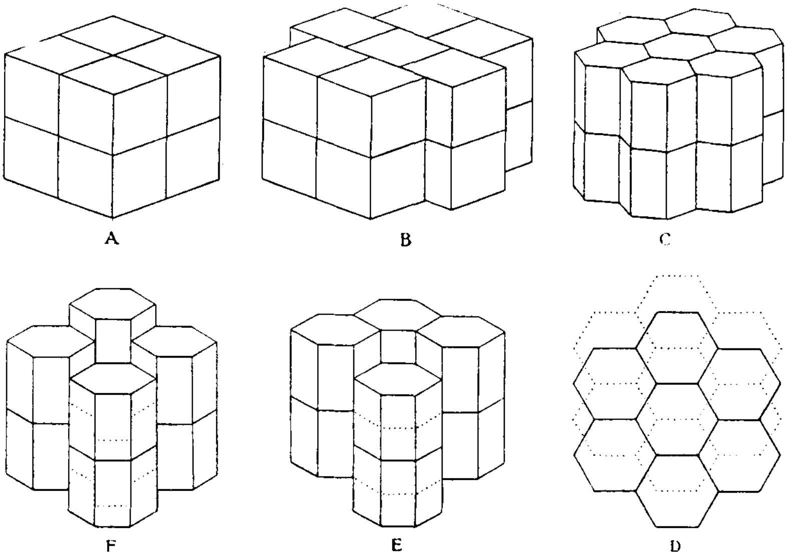

FIGURE 3. Diagram to show transformation of an aggregation of cubes into tetrakaidecahedra by the avoidance of 4-rayed intersections and tetrahedral angles.

Cubes (Fig. 3, *A*) have trihedral angles only, but they can not be arranged so as to avoid 4-rayed intersections without ceasing to be cubes, *i. e.* cells which are in contact with but six of their neighbors. If arranged as in Fig. 3, *B*, they avoid 4-rayed intersections in the horizontal plane but retain them in the vertical planes: they have become hexagonal 8-faced prisms. If the sides and angles of the hexagons are made equal, which is a change toward minimal surface expected in the more plastic animal tissues, Fig. 3, *B*, becomes *C*. There remain the 4-rayed intersections in the vertical planes.

If in an attempt to avoid these 4-rayed intersections, a single column is depressed by half the height of a prism, as in *E*, every 8-

faced prism in the depressed column acquires six additional facets by the bisection of its lateral facets, and becomes a tetrakaidecahedron. It presents, however, in its equatorial plane, six tetrahedral angles (two of which are fully exposed in the diagram). But this depression by half the height can not occur in alternate columns so as to make all the prisms 14-hedra. Depression of alternate rows by half the height makes them all dodecahedra, with two undivided vertical facets, which are on opposite sides of the prism. These two undivided facets retain their 4-rayed intersections above and below. The other four vertical facets are cut in halves transversely, forming eight quadrilaterals, separated by the undivided verticals into two groups of four each. Each group forms at its center a tetrahedral angle, and this angle is at the end of the line of 4-rayed intersection formed by the adjacent cells. This curious pattern was the norm postulated by Mirbel and Kieser.

An entire absence of the unstable tetrahedral angles and 4-rayed intersections, with walls so spaced as to avoid them to the maximum, occurs when the columns of prisms surrounding any given column are depressed alternately by one third and two thirds of the height of the prism (Fig. 3, *F*). There are then 14 faces per cell,—8 hexagonal and 6 quadrilateral. The cells are hexagonal in transverse and in vertical section. With all edges of the same length, the polyhedron becomes Kelvin's *orthic* tetrakaidecahedron, which divides space without interstices into uniform bodies of minimal surface.

The same result may be obtained, starting from the arrangement in Fig. 3, *C*, by shifting laterally the horizontal strata so that the central point in the base of any prism is over the mid-point of one of the sides of the prism beneath. The pattern of the superimposed tops and bases of the prisms becomes that shown in *D*. Each prism is in contact with four others below, four above, and six laterally. If all edges are made of the same length, it is again an orthic tetrakaidecahedron, but instead of presenting a top and basal surface as before, it is now oriented with an edge above and below.

It may be noted that when balls of such material as putty are in a pile, they settle into the hollows between their neighbors, so that when compressed, the relation of the superimposed layers (in a diagram corresponding with Fig. 3, *D*) is that of hexagons centered over the *angles* of the hexagons in the layer beneath. Each hexagon is then in contact with three above, three below, and six laterally. Compression thus yields dodecahedra, with six tetrahedral angles, radically different from the pattern of cells.

Polyhedra, formed around nuclei distributed at random, by enclosing them in mutually contiguous compartments which have trihedral angles only, and along whose edges three compartments invariably meet, will have an average of 14 facets.—After considering the exacting manner in which the cubes of Fig. 3 must be adjusted to yield orthic tetrakaidecahedra,[8] one is surprised to find that a random arrangement of polyhedra varying in their number of facets, but avoiding tetrahedral angles and 4-rayed intersections, will have the same average number of facets, namely 14. It can readily be shown that an assemblage of polyhedra, with the requisite trihedral angles and edges of triple intersection, can be constructed which shall *not* have an average of 14 facets, presenting either more or less as may be desired; yet with a *random* distribution of nuclei, it appears that the theorem, as stated in the caption of this paragraph, is correct. It must be left for others to provide a mathematical demonstration. We reach our conclusion as follows.

With certain qualifications, it has already been shown that the meshes of a net, limited to 3-rayed vertices, and formed around spots scattered at random over a plane surface, will be, on the average, hexagonal. Such patterns may be inscribed, for example, around raindrops, or stars in a simple constellation-chart. To resemble cellular mosaics or emulsions, the polygons should have no internal angles of more than 180°. Further, if a cellular mosaic is to be duplicated, the polygons should be mostly pentagons, hexagons, and heptagons. Greater and more frequent deviations from the hexagon are often seen in inorganic emulsions.[9] Finally, with a given set of points, one may draw triangular meshes or omit them altogether. None will be drawn, since triangular cells are infrequent.

With these considerations in mind, and wishing to know the number of facets in polyhedra of the cellular type formed around random nuclei, a series of 8-inch squares was drawn, in each of which 30 points were placed. These roughly followed some constellation pattern, with a single point in place of any close pair or cluster, and the arbitrary insertion of a point somewhere in any large empty space. The polygons with 3-rayed vertices inscribed around such nuclei ranged from quadrilaterals to octagons inclusive, with an average of 5.96 sides. Then the mosaics, which had been drawn on transparent paper, were successively superimposed, in the manner shown in Fig. 4. Wherever tetrahedral angles or 4-rayed intersections occurred, the boundaries of the polygons were shifted to avoid such instabilities. It was then found that the polygons of a given layer were, on the average, in contact with 4 others in the layer above, and 4 in the layer below, making them altogether 14-hedra. Without reconstruction, but quite as definitely as if modeled in wax, the number of facets of

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[8] See also Proceedings, vol. 61, p. 25, for a diagram where, by *tour de force*, solid spheres are arranged to yield 14-hedra on compression: with comments on the same in Anat. Rec., vol. 50, p. 238–239.

[9] *Cf.* Anat. Rec., vol. 50, p. 244 and 241.

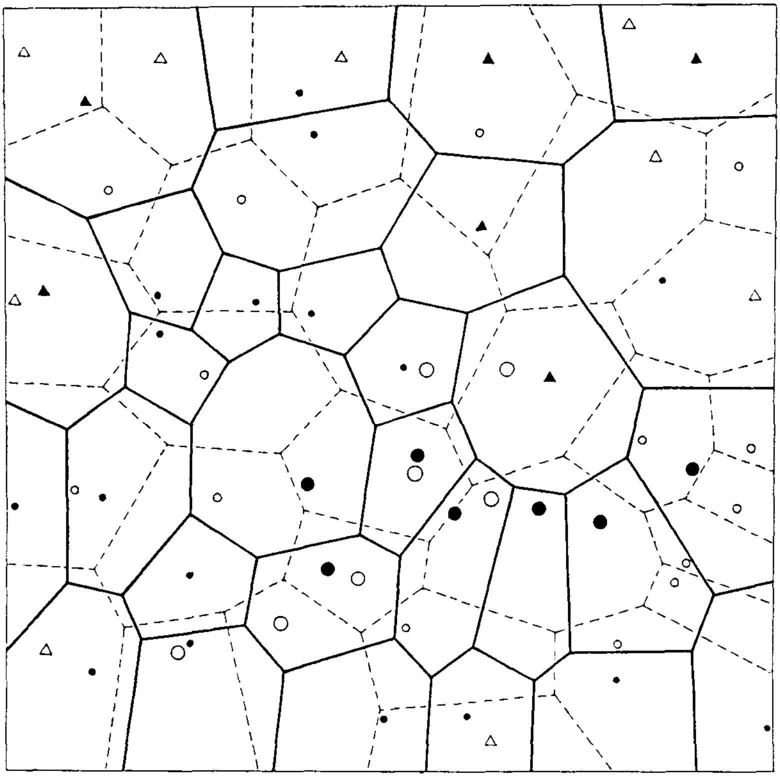

FIGURE 4. In heavy lines, polygons with 3-rayed intersections formed around 30 points distributed in a square at random. In lines of dashes, another set of polygons similarly formed around 30 other random points. (These random points are free renderings of Ursa major and minor, with the triangular points arbitrarily added.) The superimposed patterns, adjusted to avoid 4-rayed intersections and tetrahedral angles, include 27 cells completely within the square, having an average of 5.93 sides, and an average of 4.2 contacts with cells in the adjoining layer.

the individual polyhedra could be counted. It was found that 100 of them had from 9 to 21 facets (Table II) with an average of 14.11. They are quite comparable with the hundred fat cells, repeated from Table I. If, then, a haphazard distribution of nuclei is sufficient to yield a tetrakaidecahedral tissue, we may expect to find it in cartilage, where as Holmes expressed it, the cells are scattered like "plums in a pudding."[10]

TABLE II.

No. of Facets	9	10	11	12	13	14	15	16	17	18	19	20	21
Schematic polyhedra around 100 random centers	1	6	6	10	22	21	11	4	8	3	4	3	1
Human fat, 100 cells	2	4	5	15	13	20	15	17	4	2	2	1	

The form of cells in precartilage.—In view of the foregoing conclusion that cells properly formed around random nuclei will have an average of 14 facets, it seems indeed 'idle' to reconstruct precartilage cells and find that they too are tetrakaidecahedral. But it was a preliminary inspection of the models of such irregular cells that led to the generalization, and there are properties of the cells other than their number of facets which will be revealed by this study.

A single tadpole of the common toad (*Bufo lentiginosus*) supplied all the cartilage cells reconstructed. Preserved presumably in formalin, this tadpole measured 25 mm.: its hind legs were nearly 3 mm. long and the front legs were not yet free. A portion of the head was cut in serial sections 3 μ thick, by Mr. A. G. Mahurin, using a Minot precision microtome, and they were stained with iron haematoxylin and Orange G. Individual cells, readily identified, extended through 7 to 15 sections, usually about 10. They were drawn, with projection apparatus, at a magnification of 1330 diameters; and after their contacts had been determined, certain of them were reconstructed by the wax-plate method. I am indebted to Ethel S. Lewis for making all the models. They have been drawn, also with great skill, by Mr. F. Schuyler Mathews.

The portion of the mandibular cartilage shown at *a* in Fig. 5 was chosen for special study, and all the models were made from it. Its cell pattern is seen in Fig. 6. Avoiding the small cells at the periphery and those in immediate relation with them, a count of the intercellular surfaces of 25 cells was made, and the average number of faces per cell was 14.08. For 25 cells from the interior of the cranial cartilage (Fig. 5, *b*) the average was 13.7, making the average for 50 cartilage cells, as listed in Table I, 13.90.

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[10] Holmes, O. W. The human body, Atlantic Almanac, 1869, p. 48.

The cell pattern of precartilage (Fig. 6), except for higher magnification, has been drawn in the same manner as the fat cells and elder pith of the earlier studies (*cf.*, respectively, these Proceedings, vol. 61, p. 20 and vol. 58, p. 543). On comparing these figures, one thinks of Leydig's comment that cartilage may look like fat 'to a hair.' "Betrachtet man z. B. die Kehlkopfknorpel der Nager (Ratte), so glaubt man nicht Knorpel vor sich zu haben, sondern echtes Fettgewebe; erst genaueres Zusehen belehrt, dass ein Knorpel vorliege. . . ."[11]

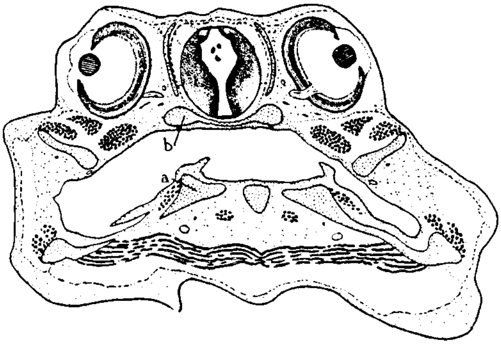

FIGURE 5. Section of the head of a toad tadpole; at *a* and *b*, the precartilages studied. × 15 diam.

But when these two tissues are compared with vegetable parenchyma, even though an eminent cytologist has noted a "striking resemblance," we find a more striking difference. In the elder pith, but not in the other two, there tends to be an orderly arrangement in columns, with a vertical repetition of queerly sectioned corners and a lateral dovetailing of the polygons. When modeled, the elder cells alone are commonly oriented with a surface above and below: frequently they have two tiers of lateral facets, with occasionally the alternation of quadrilaterals and hexagons in a manner suggestive of Kelvin's orthic tetrakaidecahedra. In fat and precartilage, with the same average number of facets, the cells seem piled in lawless confusion. This is shown in the model, Plate I.

The model includes 14 of the cells marked with numerals in Fig. 6. With them are two other entire cells (16 altogether), and a portion of

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[11] Leydig, F. Lehrbuch der Histologie. 1857, p. 34.

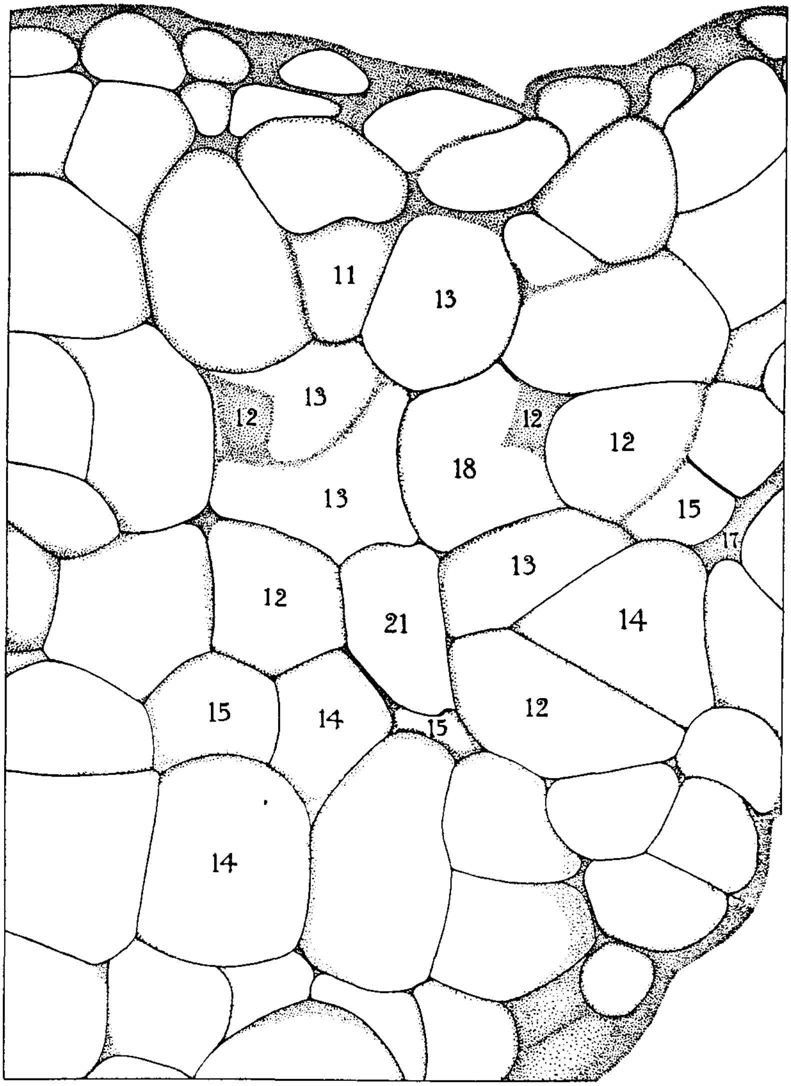
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FIGURE 6. Section of the mandibular precartilage of a toad tadpole of 25 mm. The numerals in certain cells indicate their number of contacts with other cells. × 600 diam.

a seventeenth which was defective in its hidden aspect. The 16 complete cells, with an average of 14.1 facets, range from 12 to 21 facets each. These facets vary from triangles to nonagons. They form 312 trihedral angles and 15 tetrahedral, nearly 5 per cent. being of the unstable form. In such a group one may well ask whether there is any semblance of order. Most decidedly there is, as will appear from the following considerations.

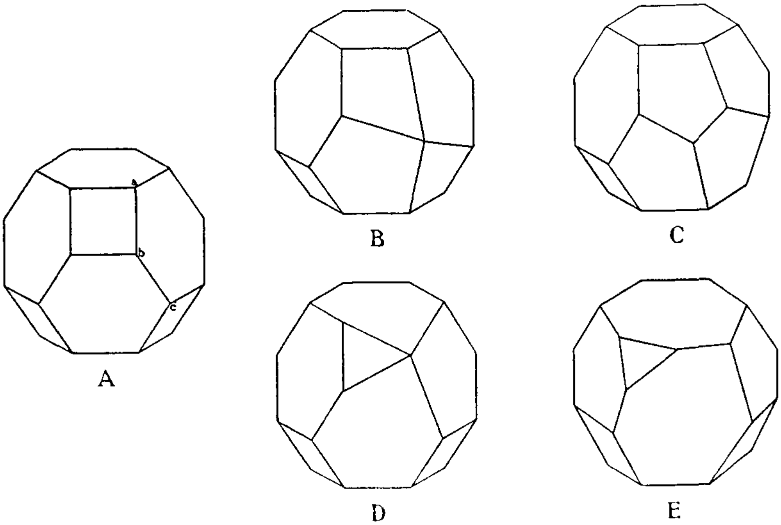

FIGURE 7. Irregular tetrakaidecahedra (*B–E*) derived from the regular or orthic tetrakaidecahedron, *A*.

The orthic 14-hedron (Fig. 7, *A*) has 8 hexagonal and 6 quadrilateral facets, which together have 72 sides (36 edges, each common to two polygons). Let the edge *bc* between two hexagons be reduced to a point, and a tetrahedral angle is formed (Fig. 7, *B*): there are then 14 facets with 70 sides and one tetrahedral angle. Then, if between the irregular quadrilaterals of Fig. 7, *B*, an edge develops, making them pentagons (Fig. 7, *C*), we again have a 14-faced cell with 72 sides and no tetrahedral angles. Or, in Fig. 7, *A*, let the edge *ab*, between a quadrilateral and a hexagon, be reduced to a point (Fig. 7, *D*) and we have another 14-faced body with 70 sides and a tetrahedral angle. If, then, an edge develops extending from the triangular face to the pentagon (Fig. 7, *E*), thereby producing two heptagons and eliminating the tetrahedral angle, there is once more a 14-faced body with trihedral angles only, and facets totaling 72 sides.

Thus it is observed that any combination of polygons which covers the surface of a 14-hedron and makes trihedral angles only, will have a total of 72 sides, and that there will be a deduction of two sides for every tetrahedral angle. The cartilage cells are formed under this law, which as Professor W. C. Graustein informs me, is readily deducible from Euler's famous formula. If F is the number of faces of a polyhedron; V, its number of vertices or corners; and E, the number of edges; then $F + V - E = 2$. Euler's theorem is cited in Thompson's *Growth and Form* (1917, p. 385 and 484), but its wide application to cell shapes and mosaic patterns has generally been unsuspected. In a previous paper,[12] and in what follows, we deal only with the cytological phases of Euler's comprehensive scheme which assures us that the empirical deductions from the study of cartilage cells are mathematically sound.

Plate II, fig. 17, shows a 14-hedral precartilage cell with 3 triangular facets, 3 quadrilateral, 2 pentagons, 3 hexagons, 1 heptagon and 2 octagons, and no tetrahedral angles. There are of necessity 72 sides (36 edges) and 24 corners, as in an orthic 14-hedron. Plate II, fig. 18, is another precartilage cell with 14 facets,—2 triangles, 5 quadrilaterals, 1 pentagon, 5 hexagons and 1 heptagon—but there are two tetrahedral angles, both of which appear in the drawing. This cell then has necessarily four less than the 72 sides of the orthic 14-hedron.

If, in Fig. 7, *E*, the triangular facet should become smaller till it disappeared, three edges would be lost: a 13-hedron would be produced, with facets having 6 less sides than the 14-hedron. Conversely if a triangular facet should be added to the orthic 14-hedron (Fig. 7, *A*) by truncating one of its angles, six sides would be added. In general it may be said that with trihedral angles only, when n is the number of facets, $6(n - 2)$ is their total number of sides. There is a deduction of two sides for every tetrahedral angle. The number of edges is $3(n - 2)$ with a deduction of one for every tetrahedral angle. The number of angles when all are trihedral, is $2n - 4$, and there is a deduction of one for every tetrahedral angle. These being geometrical necessities, actual examples are really superfluous. Pl. II, fig. 19, is a flattened misshapen precartilage cell with 13 facets. Its angles are all trihedral. There must be 22 of them, and the facets must have a total of $6(13 - 2)$ sides, or 66. This cell has 2 triangular facets, 3 each of quadrilaterals, pentagons, and hexagons, 1 heptagon and 1 octagon. Plate II, fig. 20, is a cell with 21 facets, and 2 tetrahedral angles. Its facets have 110 sides, being $6(21 - 2) - 4$. It has 36 angles, 34 being trihedral.

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[12] Anat. Rec., 1933, vol. 55, p. 323.

Volume of precartilage cells.—It has previously been shown that the average area of the polygons in an epithelial mosaic varies with their number of sides, and in definite arithmetical progression.[13] We rather confidently expect the same to be true of the polyhedral cells in the interior of precartilage, yet this can be demonstrated only with far more extensive and more accurate data than are at hand. Modeling in wax at a magnification of 1330 diameters, or × 2,352,637,000, is not very precise. Variation in the size of cells of a given number of facets is clearly so extensive, that for a significant average many should be measured. However, the volumes of the twenty models of precartilage cells, determined by displacement, are shown in Table III.

TABLE III.

No of Facets	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Precartilage, 20 cells, × 1330 diam. Volume in cc.		25		25	30 50 50 55	30 35 45 55 60	50 55 55 80	60		75	60	75		80
Average Volume		25		25	45	45	60	60		75	60	75		80
Elder pith, 30 cells × 500 diam. Volume in cc.	30		55	65 65	45 55 65 75 100 125	60 80 85 90 90 95	70 80 95 140 140	80 95 105 120	120 130		175	175 295		
Average Volume	30		55	65	80	85	105	100	125		175	235		

Thirty cells of elder pith, the volumes of which in relation to their number of facets, have not previously been recorded, have been added in the table. Fragmentary as these data are, it seems clear that the average volume of polyhedral cells, under similar conditions within a mass, increases with the number of facets.

A more decided increment in volume per facet is found in conventional polyhedra, in which all edges are made of equal length. This accords with the fact that in a plane epithelial mosaic, whereas the average area of a polygonal cell increases with its number of sides, the increase is less than that of the corresponding regular polygons with sides of unit length. There is a tendency toward uniformity. The few-sided polygons and few-faceted polyhedra are larger than would be expected: the many-sided and many-faceted are smaller.

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[13] Anat. Rec., 1928, vol. 38, p. 365; 1931, vol. 50, p. 258.

Of relatively little interest is the actual size of the precartilage cells. Calculated from the models, it is found to range from .000011 c. mm. to .000034 c. mm., with the 14-hedron averaging .000025 c. mm. The average human fat cell is 20 times larger; and the cell of elder pith, with an average volume of approximately .001 c. mm., is 40 times larger.

Cell division in precartilage cells.—One of the most significant observations in this ten years' study may be discounted as axiomatic, so patent and obvious it is—after attention has been called to it. In a plane mosaic, the inevitable result of the division of a polygonal cell by a wall which avoids 4-rayed intersections is to produce two daughter cells which together have four more sides than the parent, and to add one side to each of two adjacent cells. Every such division thus adds one cell and six sides to the mosaic. Given a mosaic of hexagons, division must introduce few-sided polygons and create many-sided. Even though subsequent divisions tend to correct this, they never perfectly succeed. The average number of sides of the dividing cells in an epithelial mosaic is found to be seven, but heptagons can not divide into two hexagons; nor are the cells at the end of the division plane always pentagons to be restored to hexagons. A variety of polygons in the mosaic, and of polyhedra in the mass, is the inevitable result of cell division, except under the ideal conditions of ingenious geometrical diagrams.

Self-evident as this is, it is not mentioned in Wetzel's frequently cited paper on the few-sided and many-sided pigmented cells of the retina. His conclusion as reported and apparently sanctioned by Schaffer is as follows:—

> Hätten alle Zellen den gleichen Wachstumsdruck, so müssten sie alle sechsseitige Form annehmen. Das Vorkommen der nicht sechsseitigen Formen kann durch gesteigerten Turgordruck für die siebenseitigen, verminderten für die fünfseitigen, allgemein aus den ungleichen biologischen Zuständen der einzelnen Epithelzellen verstanden werden.[14]

Gray's too cautious statement reads:—

> It must be admitted that there is no general agreement concerning the origin of those cells which exhibit more or less than six sides. . . . Cell division, cell growth, and cell absorption and fusion may well cause irregularities, but in no case has the actual process been seen under the microscope.[15]

Perhaps it can be generally agreed that whereas growth and shrinkage *may* cause changes in the number of cell contacts, they do not do so necessarily. There is the collapsed fat cell figured in these Proceedings (vol. 61, Pl. 2, fig. 22) which retains 17 contacts, and the much shrunken cartilage cell (Pl. 2, fig. 19) with 13. Both of these cells show the markedly concave facets indicative of diminished turgor. An inordinately large fat cell, with decidedly convex walls, due to excessive turgor, has but 12 sides. Thus it appears that after certain labile stages the facets become firmly adherent and considerable changes in cell volume may occur without affecting their number. But in stratified epithelium, as the cells grow, they certainly gain facets at the expense of adjoining cells. To what extent this takes place in other tissues—the pigment layer, for example, or in cartilage—is a problem as interesting as it is difficult to determine. Curvature of the walls does not indicate that the few-faced cells are generally under diminished turgor, as compared with the many-faced. In none of these tissues, however, can cell division take place without altering the number of contacts of several cells. To deny cell division as a cause of differences in the number of facets is to deny its occurrence.

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[14] Schaffer, J. Das Epithelgewebe. Hdb. d. mikr. Anat. d. Menschen, herausgegeben von W. v. Möllendorff, Bd. 2, Tl. 1, 1927, p. 13.

[15] Gray, J. Experimental Cytology, 1931, p. 257–258.

There is an important distinction between division in plane mosaics and in masses, in both cases, of course, avoiding 4-rayed intersections and tetrahedral angles. In the mosaic any division necessarily adds one cell and six sides, so that there is no possible deviation from the general average of six sides. In the mass of 14-hedra, however, the general average of 14 facets can be raised or lowered according to the number of sides in the plane of division. This may be readily visualized.

When a plane of division cuts through a polyhedral cell, it makes a new surface of mutual contact between the two daughter cells, thus creating two new facets. The division plane reaches the surface of the parental polyhedron and encircles it, cutting in two a varying number of facets, normally six. Each of these six is common to a surrounding cell, which exchanges its single facet of contact with the parent cell, for two facets of contact with the daughter cells. Accordingly the total number of facets added to the mass by any division is twice the number of facets in the division plane, plus two. When the plane is hexagonal, a new cell and a total of 14 facets are added. But if the division plane cuts off a trihedral corner of some cell, a tetrahedral cell is introduced, with the addition of 8 facets altogether. Such divisions would lower the general average of 14 facets, just as divisions through octagonal planes, for example, by adding one cell and 18 facets, would raise it. Since there is a tendency for large cells, which are many-faceted, to divide, we are interested in the latter alternative. If the average number of facets in precartilage is 14, either the planes of division must be on the average hexagonal, or, if of higher average, there must be some compensating reducing factor such as 4-rayed intersections. The models include five pairs of cells in which the division planes may be studied.

It now appears that Leydig overstated the case when he said that sections of cartilage may look like fat *aufs Haar*. The cells of precartilage, but not of fat, are frequently in pairs and clusters as the result of division, and the pairs become evident on modeling. Some are separated by newly formed septa, characteristically large and thin. Others show thickening and contraction of the division plane until it becomes an ordinary facet, indistinguishable from the rest. Meanwhile the daughter cells grow unequally, since by divisions of adjacent cells, one may receive more facets than the other, and being diversely pulled about by surface tension, the pairs become unrecognizable.

Plate II, fig. 21 shows a pair of cells—a 14-hedron on the left and 12-hedron on the right—produced by the division of a 16-hedron. Parent and daughter cells have trihedral angles only. It is known that the polygons of a parent cell have a total of $6(n-2)$ sides, and after division each daughter cell will have $6(n-2)$ sides, but n is the same only in case of dividing cubes: usually there are two or three different values for n in this formula as applied to parent cell and the two daughters. The way in which the polygons covering the daughter cells come to have 12 less sides than if they were all hexagons, being in this respect like the parent, is as follows.

The total number of facets in the two daughter cells will be the number in the parent plus the number of sides in the division plane, plus 2. The 16-hedron in Plate II, fig. 21, might divide in a plane having any number of sides between 3 and 8. If the division plane were triangular, the daughter cells would have $16+3+2$ facets: but the plane is octagonal and the daughter cells have $16+8+2$ facets, or 26. With an octagonal division plane the daughter cells may be a pair of 13-hedra, or a 12- and 14-hedron as in the pair figured, or a more diverse combination totaling 26 facets.

From previous studies it is known that when the division plane cuts across a facet, it divides it into daughter facets which have 4 more sides than the parent. Plate II, fig. 21, shows a pentagonal, a hexagonal, and a heptagonal facet so divided. If x is the number of sides of the division plane which transects a polyhedron, $4x$ is the number of sides added to polygons on the exterior of the parent polyhedron, and $2x$ is the number of sides belonging to the two daughter polyhedra in their facet of mutual contact: $4x + 2x$, or $6x$, is the total addition in sides, but it is made by $x + 2$ facets. That is to say, the added facets have 12 less sides than if they were all hexagons. Since the parent cell already lacks 12 sides, there will be a total lack of 24. In the vertical bisection of an orthic tetrakaidecahedron oriented with a hexagonal facet above and below, an even apportionment of this deficit between the daughter cells is readily visualized. But also in the most unequal divisions, this deficit is shared alike. When a corner is cut from an orthic 14-hedron by a triangular division plane, the daughter cells have 4 and 15 facets respectively, with 12 and 78 sides,—in all cases $6(n - 2)$. In the irregular cells of the precartilage it is none the less inevitable, as exemplified in the pair shown in Pl. II, fig. 21. In agreement with the formula $6(n - 2)$ the polygonal facets of the 16-hedral parent must have a total of 84 sides; and the totals for the 14- and 12-hedral daughter cells must be respectively 72 and 60. The formula was established by counting the facets of these and similar cells and finding that such were the actual totals. It should be noted that the division plane cuts off quadrilateral facets, or, as in fig. 22, introduces triangular facets, *without any shrinkage of the adjacent cells*. Similarly division gives origin to heptagonal and octagonal facets of the neighboring cells, which need not enlarge to obtain them.

Plate II, fig. 22, is the model of a great cell with 20 facets, subdivided by a concavo-convex plane into daughters of very unlike shape but similar volume,—on the left, a 14-hedron of 55 c.c.; and on the right, a 15-hedron of 60 c.c. The plane of division is again octagonal. Just below the center of the figure, the plane passes quite exactly through a trihedral angle of the parent, thus forming an unstable tetrahedral angle, at the apex of a triangular facet of the 15-hedron. All other angles of the 15-hedron are trihedral, and its polygons therefore have 76 sides, being $6(n - 2)$ with a deduction of 2 for the tetrahedral angle. As the division plane extends from the triangular facet to the bottom of the figure, it coincides with the line of separation between two slender hexagonal facets of the parent. Since every parental facet is in contact with a separate adjoining cell, 4 cells are made to meet along this segment of the division plane. It will yield a 4-rayed intersection when cut across. This unstable feature accounts for the fact that the 20-faceted parent, dividing in an octagonal plane, produced daughter cells with a total of 29 facets instead of 30. Slight deviation of the plane to one side or the other would avoid the 4-rayed intersection and supply the missing facet. In other cases the division plane coincides with an interfacial boundary through part of its extent and then leaves it, perhaps in process of eliminating the instability.

Three other pairs of recently divided cells have been modeled, in all of which the division plane is hexagonal. The parent cells have 14, 15, and 19 facets respectively, making the average for all five parental cells 16.8 facets. The ten daughter cells, with three lines of 4-rayed intersection, average 12.4 facets.

Tentative scheme of growth and division in precartilage cells.—In the foregoing fragmentary data, and the rough volume determinations, there is no inconsistency with an extension of the remarkable outcome of studies on the simple epithelium of the cucumber. It is here presented as a tentative possibility. In the interior of the precartilage of the tadpole the 14-hedron is then the average cell both in number of facets and in volume. At the enlargement used, 220 cells in section, without regard to their number of facets, occupied 3200 sq. cm., so that, if isodiametric, the average volume of a cell would be 55 c.c. Four of them were the 14-hedra modeled (Table III), which had an average volume of 60 c.c., but included a very large example. If the 14-hedron is the average both in facets and in volume, the average volumes of polyhedra with other numbers of facets must vary in arithmetical progression. The displacement measurements of our few models do not show what arithmetical progression exists, if any· but if it is like that clearly found in the epithelium of Cucumis, the 14-hedron should have an average volume of 12 units, and the average volumes of all other polyhedra should be two units less than their number of facets. In that event, the average end result of the division of any polyhedron in a hexagonal plane will be to add one cell, 12 units of volume, and a total of 14 facets to the mass. Accompanying such a division, six surrounding cells receive an added facet, and together grow 6 units, which is one-half of the volume to be added. The other half is supplied by the growth of the daughter cells, which together come to exceed the volume of the parent by 6 units. There are some reasons to suppose that the average number of facets of the dividing cells is 16. Such cells, if dividing equally in a hexagonal plane, would produce pairs of 12-hedra, and the average daughter cell in our series of ten has 12.4 facets. Not infrequently a large 14-hedron divides into a pair of 11-hedra. But if such a division should occur in a hypothetical mass composed entirely of 14-hedra, six of them would be changed to 15-hedra, becoming larger than the 14-hedra, and hence more likely to divide. But their division, in certain planes, adds to the sides of adjacent 15-hedra, making 16- and 17-hedra, with increasing propensity toward division. The tentative conclusion that the average number of facets of dividing precartilage cells is 16, is by no means established by the data at hand. In the five pairs which were modeled, the average was 16.8; but it is quite possible that some of these cells had received an added side subsequent to their division into daughter cells, by the division of neighbors, and the probable error in so small a group is very high. The arithmetical progression in precartilage cells may well be a different one from that suggested; but that some equally definite ratio exists is the deduction from the foregoing study.

Intercellular spaces.—The resemblance between precartilage and vegetable parenchyma—Schwann's solid foundation for his cell theory—is primarily a matter of nuclear relations; but it depends also on the production of thin-walled 14-hedral units with a tendency to separate from each other at the corners and edges. There, in precartilage, the single line between adjacent cells splits in two, to enclose "a three or four-cornered intermediate space filled with a kind of homogeneous intercellular substance," of which there is not enough to yield gelatine on boiling.[16] In plants Amici had found that these spaces were not a casual laceration of the tissue, but were elegantly formed channels for conveying atmospheric air.[17] In fat, they lodge the vessels.[18] But these spaces are supposed not to exist in an aggregation of soap bubbles, and the physical factors which produce them have been described as quite abstruse.[19] In no other connection is a model made from rubber 'elastics' more instructive.

In Fig. 8, *A*, three elastics, *b*, *b*, *b*, of the same strength, each about 6 cm. long, have been looped around a shorter one, *a*, at equal intervals. In *B*, this combination of elastics has been stretched by slipping the three free loops over the thumb tacks *a*, *b* and *c*. These tacks have been inserted at equal distances from a central point, along lines diverging at angles of 120°. If we wish the three elastic rays, so constituted, to meet at the central point, they must be held there with pins, *d*, *e*, *f*. Remove the pins and a triangular intercellular space results (Fig. 8, *C*), the extent of which depends on the distance from the center that the walls are double. In Fig. 8, *B*, they are double from the center to x, x' and x'', and those points become the corners of the interspace. In regard to the tension in this system, I am indebted to Professor L. J. Johnson for the following memorandum:

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[16] Schwann, Th. Untersuchungen. (Heft 1) 1838, p. 17, 18, 31.

[17] Amici, G. B. Osservazioni. Atti d. Soc. Ital. d. Sc. in Modena, 1823, T. 19, p. 47–48.

[18] Proc. Amer. Acad., 1925, vol. 61, p. 19.

[19] Gray, J. Experimental Cytology, 1931, p. 260–262.

Three 'rays', *R*, originally 120° apart, are subject to the same pull, *P*. Each pull, *P*, is resisted by the combined effect of the other two pulls, *P*. That effect not only balances *P* in line of *R* but at the same time makes a cross-wise or splitting effect on the material of *R* next the joint amounting to 0.866*P*. After the material has opened at the joint under the influence of the pull of 0.866 *P*, and the resistance to the pull *P* in *R* is afforded by two forces 60° apart (instead of 120°), the cross-wise 'splitting' effect drops to 0.289 *P*. This means that if *P* were 100 grams in both cases, the cross-wise or splitting force would be 86.6 grams and 28.9 grams, in the 120° and the 60° cases, respectively.

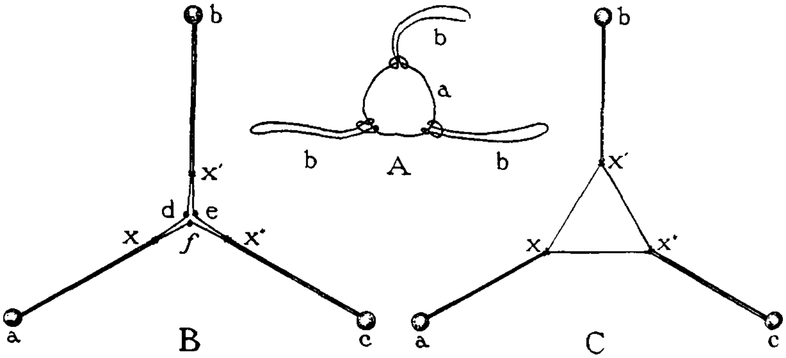

FIGURE 8. Arrangement of elastic loops showing that the tension where three rays meet tends to produce a triangular interspace.

Thus there is good reason for cell walls to split at the corners. Even if the cell is full of incompressible fluid, readjustments during growth allow this differential tension to express itself, producing little triangular interstices of the type seen in elder pith. An interstice of small size reduces the tension as much as a large one, and is made with almost inappreciable increase in the diameter of the cell. Accordingly, unless the substance within the space is under greater pressure than the cell contents, *small* triangular interstices may be expected. If these spaces follow the edges of massed 14-hedra, four of them become confluent at every trihedral corner, since a trihedral angle is obviously a place where four cells meet—the cell itself and three which are adjacent. A triangular prismatic space between the three adjacent cells will join three similar prismatic spaces which bevel the edges of any given polyhedral cell, meeting angle to angle. It is desirable to picture such a junction, though as Gray has said of Gibbs' ring,—"unfortunately the exact form of this area has not yet been shown to be susceptible to geometrical or mechanical definition."

Yet the general relations are absurdly simple, suggesting in our conventional model (Fig. 9) a hip roof and two chimneys.

The roof on the right of Fig. 9, in shape like an inverted V, is part of a quadrilateral facet of an orthic 14-hedron; the ridgepole runs between two hexagonal facets, and with the chimneys marks out three sides of a quadrilateral facet. Although in actual tissues the mutual inclination of the triangular prismatic spaces may differ widely from those of orthic 14-hedra, four of them will ordinarily meet at a corner, with an arrangement essentially like that of the model. There is no marked enlargement of the spaces at their confluence, so that in sections the distinction between a space along an edge and a space at a corner is not evident. Schaffer, in cartilage, calls them all *Zwickeln* or gussets, including under this term larger masses of matrix infiltrating degenerating cells.[20] From triangular intercellular gussets, hyaline matrix spreads over the entire surface of the cartilage cell, which thus loses its average 14-hedral shape, with 24 corners and 36 edges, becoming merely a rounded cell lodged in a spheroidal lacuna. In other tissues the intercellular spaces have more complex and engrossing relations. How some classic examples are related to the primary 14-hedral shape of cells in masses may be reviewed in conclusion.

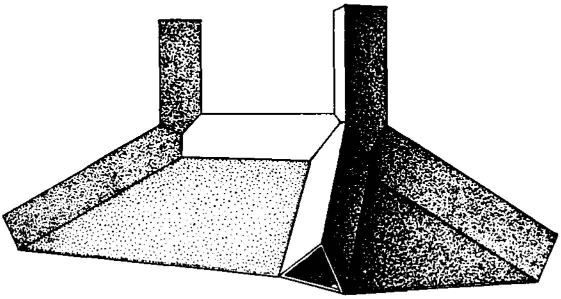

FIGURE 9. Model of triangular prismatic interspaces ("*edge spaces*") following the edges of orthic 14-hedra.

When the contents of the intercellular space, whether air or liquid, is under less pressure than the contents of the cell, it is expected that the triangular intercellular spaces will have concave sides, as in elder pith and countless other examples. But if the pressure in the space exceeds that in the cell, the triangular prismatic interspace becomes cylindrical, and in section may spread out as a lake (Fig. 10, *A*, at *x*). Over the facets of the cell, tension of the walls acts against separation, as seen in the close approximation of the two sides of the elastic loop in each ray of Fig. 8, *C*. But beads of excreted air or other substance may overcome this tension, forcing the walls apart in little blebs (Fig. 10, *A*). They may expand more readily toward the fluid interior of the cell than laterally in the crusta, thus becoming perpendicular ellipsoids, as in Meyen's old drawing of Sagittaria (Fig. 10, *B*). On surface view the cell membrane over the facet of such a cell appears coarsely perforated by circular vacuolar pits, into which, however, it descends, thinned but without interruption. A peripheral condensation of the cytoplasm would account also for the reversed orientation of the triangular prismatic spaces which follow the polyhedral edges. Instead of presenting sides toward the cytoplasm as in elder pith, these sides, after being pushed into the cells, have become the angles of the triangular space.

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[20] Von Möllendorff's Handbuch, Bd. 2, Tl. 2, Abb. 106 and 108, on p. 148 and 150.

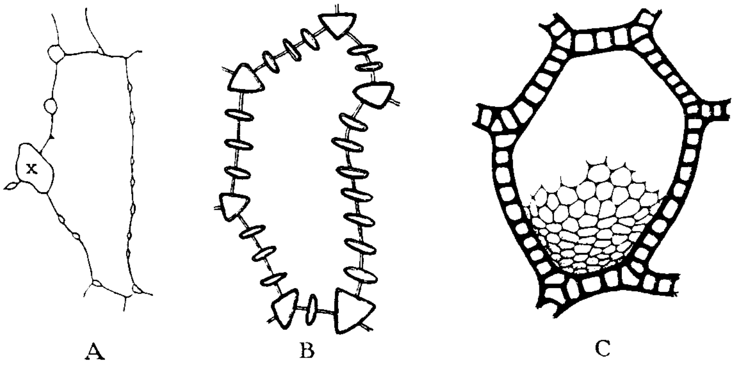

FIGURE 10. *A*, a cell with adjoining interspaces, from the stem of *Scirpus validus*. L. M. Snow (*Bot. Gaz.*, 1914, vol. 58, p. 507, fig. 10). *B*, cell with adjoining interspaces, from the leaf-stalk of *Sagittaria indica*. F. J. F. Meyen (*Pflanzen-Physiologie*, 1837, Bd. 1, Tab. 2, fig. 9). *C*, cell with adjoining interspaces from the stomach of a cow embryo of 30 cm. M. Ide (*La Cellule*, 1888, T. 4, following p. 432, fig. 5).

In the prickle cells of animals (Fig. 10, *C*, from Ide's noted paper), numerous and closely crowded intercellular vacuoles such as Meyen figured in Sagittaria, have become confluent. They coalesce around the vertical strands where three of them meet, and these strands persist as intercellular bridges. The vacuoles communicate also with the triangular spaces which follow the edges of the polyhedral cells. At the base of the bridges or prickles, Ide observed a reticulate cell membrane, shown in surface view in the lower half of his figure. The prickles extend upward from the thickened nodal points in this net. They are only "des dépendances de la membrane cellulaire, des portions de la membrane primitivement simple qui persistent et s'allongent après son clivage."[21] Ide emphatically denies that the bridges are protoplasmic processes withdrawn and sent forth like the pseudopods of an amoeba.

Although the cupped ends of the intercellular vacuoles would produce such a reticular membrane as Ide described, depending chiefly on late enbryos, there is no trace of it in a finely preserved block of tissue from the adult cervix uteri for which I am indebted to Dr. Shields Warren.[22] In this uterine epithelium, with the stain used, nothing is seen of the much studied 'tonofibrils.' But the neglected system of spaces following the polyhedral edges is shown clearly. At *c* and *d* in Fig. 11 one may look down the chimneys of an arrangement of canals like that in Fig. 9. The connecting ridgepole, however, is

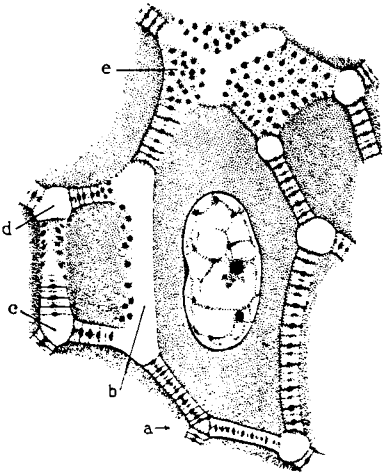

FIGURE 11. Intercellular edge-spaces following the edges of the cell facets, and interfacial spaces crossed by bridges, in the normal epithelium of the cervix uteri from a woman of 55 years. 3 μ section, Mallory's phosphotungstic acid haematoxylin. × 2600 diam.

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[21] Ide, M. La membrane des cellules du corps muqueux de Malpighi. La Cellule, 1888, T. 4, p. 425.

[22] Shaffer figures the reticular membrane in the epidermis from a man of 20 (Abb. 28 in von Möllendorff's Hdb., Bd. 2, Tl. 1, p. 33), yet in two inconsistent ways, without comment: he places the bridges in the centers of the meshes in one part of the figure; and, like Ide, at the nodal points in another part of the same figure.

bounded by transverse bridges across its top, and converging bridges along its sides. For, owing to the development of the interfacial layer of communicating vacuoles, the three simple cell membranes, which met at angles of approximately 120°, have each been split into a pair of parallel membranes meeting at the same angle, the membranes of each pair retaining connection with each other by perpendicular bridges. Hence the triangular interspace, bounded by bridges, has the reversed orientation (Fig. 11, *a*). But many of these spaces are distended and round, the cell bridges preventing so wide a separation over the facets as occurs at the edges and corners. At *b* an edge-space is cut lengthwise and shows no special enlargement at its junction with other canals.

The primary system of channels which follow the edges should be more carefully distinguished from the secondary interfacial spaces than has been done, for example, in Flemming's study of the branchial epithelium of the salamander. Apparently Flemming considers that leucocytes may wander through both sorts of spaces, and Wilson interprets his guarded statements and dubious drawings to mean that "they rupture the plasma-bridges which are then formed anew behind them."[23] In the uterine epithelium—at least in a hundred instances examined for the purpose—I found that the leucocytes had forced their way into the edge-canals, stretching, but not rupturing, the adjacent bridges: they were never in the interfacial spaces.

The bridges are somewhat irregularly distributed as they rise from the surface of a facet (Fig. 11, *e*). Their considerable increase in number as the cell enlarges and moves to the outer strata, comes evidently from the splitting of those already present: no pseudopodial outgrowths were observed. The prickle cell is therefore typically a 14-hedral object, with canals along its 36 edges, and with facets beset with a large number of prickles, or simple short branches, due to the *coalescence of a layer of interfacial vacuoles.*

If the confluent vacuoles should extend further into the cell, decreasing in number as they approach the nucleus, the processes would unite in a ramifying system, converging toward the center of the 14-hedral territory. In sclerenchyma (Fig. 12) the same thing takes place *within* the cell. The cell outlines are certainly those of 14-hedra (on the average), showing minute intercellular spaces at many edges and corners. But the shape of bone cells is more difficult to determine, since the deposit of bone is outside the cells, and there are no cell walls, —the branching processes show no such nodes as conveniently mark the limits of the unbranched processes of the prickle cells. The problem may be approached, however, for it would be solved by combining, in three-dimensional form, two such sections as are shown in Fig. 13, *A* and *B*. *A* is the shape of bone cells in tangential view of the Haversian system; and like cork cells in tangential section of the tree trunk, the average territory of the cell tends to be a regular hexagon. This territory in the bone cell includes the processes of a given cell, and is similar to the branching cell territory in the peach stone. But when the Haversian system—or the tree trunk—is bisected longitudinally, so as to produce a radial section, the average shape of a cell territory is a *rectangular* hexagon (Fig. 13, *B*). Transverse sections of Haversian system and tree trunk are similar to the radial sections. Hence we conclude that the branching bone cells in relation to the Haversian system (and in the periosteal lamellae, in relation to the shaft of the bone) are oriented like the simple cork cells in relation to the tree trunk. Like the latter they are then, on the average, 14-hedral.[24]

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[23] Flemming, W. Über Intercellularlücken des Epithels. Anat. Hefte, Abth. 1, 1895, Bd. 6, p. 10–11; Taf. 1, Fig. 1–2. Wilson, E. B. The Cell. 3d ed., 1925, p. 106.

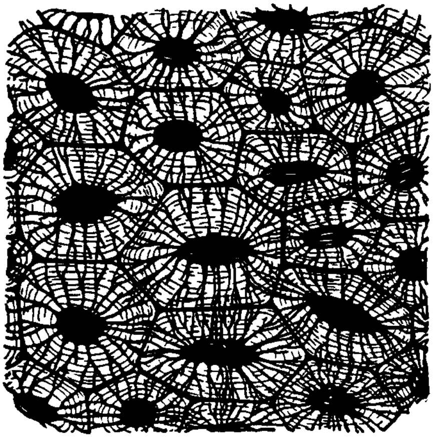

FIGURE 12. Sclerenchyma. Stone cells in a section of a peach stone. E. C. Jeffrey (*Anatomy of Woody Plants*, 1917, p. 4, fig. 3. Courtesy of the Univ. of Chicago Press).

In the development of the bone cells in Fig. 13, *A*, we may assume a corresponding layer of osteoblasts in the plane of the page, each in relation, on the average, with six others laterally. These osteoblasts deposit bone in confluent interfacial vacuoles on one side of the cell only (which is in the plane of the page), and the layers of bone so deposited, penetrated by the lengthening vertical bridges, will lift the osteoblasts above the page. At various levels, an osteoblast becomes imbedded in the matrix, selected presumably by its spatial relations to those already imbedded. It is then flattened out in a tangentially radiating pattern, receiving its parallel vertical processes at right angles. Presumably it is active in the formation of the additional laminae of bone, parallel with the first set, which now form on the opposite or upper side of the cell and contain another set of vertical processes; but unfortunately, as already noted, the processes show no line of nodes to indicate the cell boundary. Otherwise comparison between the bone cells (Fig. 13, *A*) and the prickle cells (Fig. 11) may readily be made, even though the interfacial vacuoles in bone have become very extensive. There are strips of bony matrix free from canaliculi which seem to represent the edge-canals.

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[24] Lewis, F. T. The shape of cork cells. Science, 1928, vol. 68, p. 625–626.

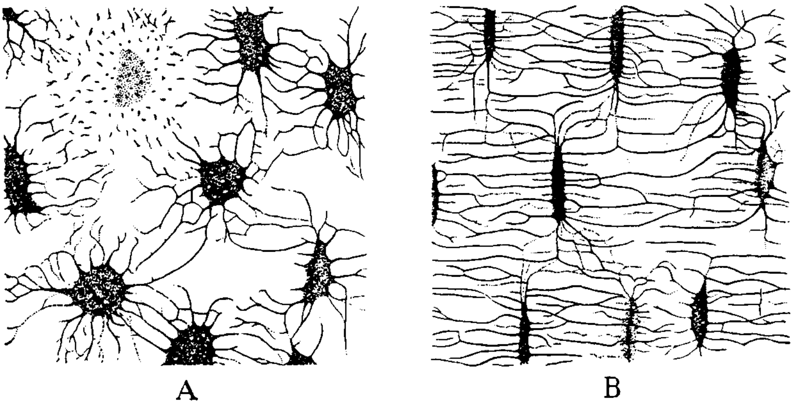

FIGURE 13. *A* and *B*, two longitudinal sections of bone cells, from one and the same Haversian system in a human humerus. × 500 diam. The Haversian canal is vertical in relation to the drawing. *A*, tangential section: *B*, radial section. These patterns are consistent with an average 14-hedral cell territory, in which case the orientation of the bone cells in relation to the Haversian canal is like that of cork cells in relation to the tree trunk.

In mesenchyma there are both extracellular and intracellular vacuoles, the latter apparently breaking through to the exterior so that the 14-hedral cells become very irregularly radiate. Yet the territory controlled by a nucleus is presumably on the average 14-faced, as is certainly true of the stellate parenchyma of Juncus. In the pith of Juncus no intracellular vacuoles split its cytoplasm into shreds, and there are usually no extracellular vacuoles other than those which follow the edges of the cell. Hence the 14-hedral cell becomes primarily a 14-rayed star. Two of these rays which, on connecting with others, form continuous vertical strands up and down the pith, early disappear.[25] Suspecting that they were mechanically superfluous in resisting the lateral swaying of the stem, and otherwise unable to suggest why these two rays should sever and retract, whereas all the others are well represented, I submitted the question to the civil engineer, and received from Professor L. J. Johnson the following reply:—

> The presence of the firm vertical casing of Juncus—the casing being a stiff ring in cross-section—seems to me to leave no mechanical reason for vertical strands in the pith. The zig-zag strands of webbing, together with the horizontal discs, form a sufficiently firm sponge to support the relatively slight vertical load and carry it to the external walls of the Juncus.

But the fact that the vertical rays are functionless is only further incentive to study the mechanics of their elimination. If growth were equal in all directions, there would be no reason why one ray should rupture rather than another (except that the rays are of two diameters and lengths, six of the fourteen being somewhat longer and thinner than the other eight). But the rush does not expand equally. It grows tall and slender, and the special tension on the vertical strands accounts for their rupture. Their stumps are then withdrawn into the central nucleated discs which are stretched laterally by the remaining rays, all of them coming to rest at angles approaching 120°. This deduction, however, rests on computations, quite beyond the author's ability, generously supplied by Professor Johnson and his assistant Mr. F. J. Mardulier. Their memorandum is as follows:—

> Since vertical growth is greater than lateral growth, the main stress on the cell is that due to vertical forces which can be indicated by the force T in the diagram, Fig. 14. This force T must be resisted by the pull on the top face and the vertical components of the pulls on the six adjacent oblique faces. Hence we obtain the statical equation:
>
> $$T = P + 3Q \sin \alpha + 3R \sin \beta$$
>
> The magnitudes of P, Q and R can be determined by the aid of the principle of least work, *i. e.* the magnitudes of P, Q and R must be such as will make the work done in elongating their respective rays a minimum.

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[25] Proc. Amer. Acad., 1925, vol. 61, p. 10–11.

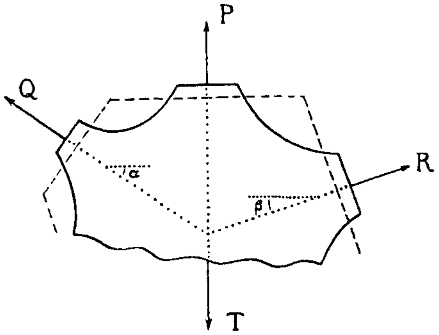

FIGURE 14. Diagram (drawn by F. J. Mardulier) of forces which rupture the vertical rays of the cells in Juncus pith. In heavy lines, a portion of a developing stellate cell; in dashes, the corresponding part of the orthic 14-hedron from which it was derived.

Hence, using Ap, Aq, Ar for the respective areas of the cross sections of the rays P, Q and R, and Lp, Lq, Lr for the respective lengths of the rays P, Q and R, the magnitudes of the forces P, Q and R are found to be as follows:

$$P = \frac{T}{1 + 3\frac{Lp}{Lr}\cdot\frac{Ar}{Ap}\cdot\sin^2\beta + 3\frac{Lp}{Lq}\cdot\frac{Aq}{Ap}\cdot\sin^2\alpha}$$

$$Q = \frac{T\sin\alpha}{\frac{Lq}{Lp}\cdot\frac{Ap}{Aq} + 3\frac{Lq}{Lr}\cdot\frac{Ar}{Aq}\cdot\sin^2\beta + 3\sin^2\alpha}$$

$$R = \frac{T\sin\beta}{\frac{Lr}{Lp}\cdot\frac{Ap}{Ar} + 3\frac{Lr}{Lq}\cdot\frac{Aq}{Ar}\cdot\sin^2\alpha + 3\sin^2\beta}$$

If we take dimensions from the schematic wax reconstruction of the developing stellate cell (such as is figured in these Proceedings, vol. 61, p. 8, fig. 6), which we are justified in doing since the various dimensions enter the equations in the form of ratios only, we get the values: $P = .250T$, $Q = .204T$, and $R = .145T$. Hence $P = 1.23Q$, and $P = 1.73R$. In other words, the vertical rays of the cell bear the brunt of the vertical attack, and the other rays bear smaller tensions as their angle of inclination to the horizontal decreases. Accordingly the vertical rays break.

This eminently satisfactory outcome in connection with a tissue of diagrammatic simplicity gives reason to believe that a further study of bone and mesenchyma by the same methods will be correspondingly illuminating. For the present we can give only a tentative negative answer to the fundamental question whether protoplasmic processes in mesenchyma branch out through the intercellular spaces and make connections with cells beyond the 14-hedral limits. Whether leucocytes in leaving the endothelial tubes elect the points where three cells meet, which are mechanically the places of least resistance, and whether the transformation of endothelium into reticular tissue occurs by the enlargement of spaces at such points, must be subjects for further study. The morphology of reticular tissue is unfortunately unknown.

A surmise concerning nerve cells and neuroglia.—Out of the primitive uniformity of the walls of the medullary tube there arise two types of branching cells;—the morphological problem of their origin is the subject of this surmise. Since cells formed around nuclei distributed at random are on the average 14-hedral, it may be assumed that the primitive cells of the medullary tube are of this character. Though the preceding study of cartilage has shown what diverse forms may be expected under that generalization, a diagrammatic model, in which the primitive cells of the medullary tube are all orthic 14-hedra, is nevertheless serviceable. As in other tissues, intercellular spaces will doubtless develop most readily along the edges of the facets. Further, I would utilize the *verdienstvolle Untersuchungen* of two of my early Harvard teachers,—Davenport, who found that in tadpoles growth is chiefly the result of the "differential imbibition of water," and Schaper, who recognized that there are two cytological types of this imbibition,—intracellular, and intercellular. The former gives to precartilage cells the turgor requisite for their functional rigidity, and incidentally a plump polyhedral form: the latter loosens mesenchyma and makes its cells stellate.[26] If this is true, I would suggest that in the wall of the medullary tube both processes occur simultaneously. Nerve cells imbibe and grow by sending out processes, perhaps at the expense of the neuroglia cells. The latter become relatively shrunken, behaving like cells of Juncus pith. The processes of the nerve cells would grow out where the resistance of the surrounding cells is least, and in making their way between the neighboring cells they would follow the paths of least resistance. That is to say, they would grow out from the corners of the nerve cells, and would extend through the little intercellular spaces following the edges of the facets. It is easy to make a hypothetical model on these specifications and see if it at all resembles nerve tissue.

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[26] Davenport, C. B. The rôle of water in growth. Proc. Boston Soc. Nat. Hist., 1897, vol. 28, p. 73–84. Schaper, A. Beiträge zur Analyse des thierischen Wachsthums. Tl. 1. Arch. f. Entw.-Mech., 1902, Bd. 14, p. 376–381. Schaffer in his admirable presentation of cartilage already cited commends these papers.

In a mass of orthic tetrakaidecahedra, let one become a nerve cell, sending out a primary process or neuraxon from one corner, and secondary processes or dendrites from all other corners, and let them follow the interstitial spaces. Rather than bifurcate, the dendrites would trifurcate, or have opportunity so to do. There is, however, no

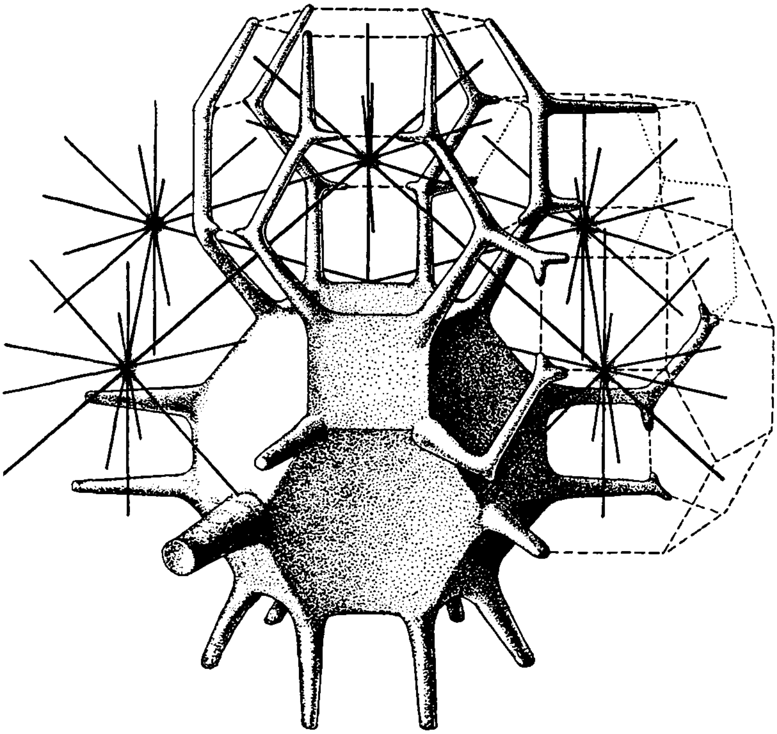

FIGURE 15. Schematic model based upon six orthic tetrakaidecahedra in contact with one another. One of the six, in the manner of a nerve cell, has sent out processes from all its corners and they extend through intercellular edge-spaces. The other five, perhaps like neuroglia, are reduced to radiate form by the enlargement of the intercellular edge-spaces. Since the ray to the nerve cell has been omitted, each of the five has 13 rays.

necessity for taking advantage of all these opportunities, nor for the nerve cell to produce its full quota of $2n - 4$ processes. It may be unipolar or bipolar; but in the model, Fig. 15, the shaded body is a 14-hedron which has sent out processes from every corner,—one neuraxon and 23 dendrites.

The relative abundance of nerve cells and neuroglia in the medullary tube, and their relation to one another, should be known before proceeding with the schematic model. If a nerve cell is not completely surrounded by neuroglia, it would be expected to retain interfacial contacts with one or more nerve cells, and such contacts could be drawn out into thick processes extending from nerve cell to nerve cell. Luys figured this sort of thing plainly enough in '65, but neurologists of later date do not recognize it.[27] Certainly no process of a nerve cell goes out to join a process from a glia cell, end to end. Thus it appears that neuroglia withdraw from their original interfacial connection with nerve cells, while retaining such connection with other neuroglia. Then, if the primary intercellular spaces along the edges of the facets greatly enlarge with the growth of the tissue, the glia cells make a net-work comparable with that in Juncus pith, though differing in detail. Five such cells are shown in the model. If nerve cells are completely surrounded by glia cells, there will be five per nerve cell in what seems the most compact arrangement.

On the right of the model the outlines of the original 14-hedra are shown in dashes; and within them, in solid lines, are the 13-rayed neuroglia to which they have given rise. The 14th ray, which would run to the nerve cell, has been omitted. Each ray of the glia cell extends from the center of the primitive 14-hedron which produced it, to the center of one of its facets, and thence is continued directly by the ray of an adjoining glia cell. Fibers developed along or in these rays may pass straight from one cell territory to another. Many rays doubtless rupture and disappear; others may split secondarily, so that the pattern postulated is consistent with as great a degree of irregularity as that actually observed in nerve tissue.

On the left of the model, there are two glia cells, quite like those on the right, except that the 14-hedral territory from which they were derived is not indicated. With the stellate regression of the neuroglia, the intercellular spaces become so large that the dendrites would not preserve the angular kinks, which are perhaps the most unnatural feature of the model.

---

[27] Luys, J. Recherches sur le système nerveux. Paris, 1865. Atlas, Pl. 19, fig. 1.

The nerve cell is formed for a purpose which would be defeated by cell division, and division does not occur. Although the size of the motor cells and of their nuclei is such that if found in other tissues division would surely be expected, the dendrites are not to be separated from their neuraxon and the mechanism wrecked by division. In stellate neuroglia, division would have the effects previously described in Juncus pith, reducing the radiations per cell. It is presumably very infrequent. Proliferation yielding unbranched polyhedra, which subsequently become stellate, would account for the new formation of neuroglia of more normal pattern. In the growth of the medullary tube, cell division in both nerve cells and neuroglia is presumably in the unbranched polyhedral stage of development.

The model shows, at least, the possibility of deriving the nervous system from an aggregate of cells which on the average are 14-hedra. It indicates a more radical difference between the aggressive nerve cells and the regressive neuroglia than the stellate shape of both might suggest. But it is at present a surmise, awaiting the actual morphological reconstruction of a block of nerve tissue by some one who can carry Cajal's studies a step further.

EXPLANATION OF PLATES.

PLATE I.

FIGURE 16. Wax plate reconstruction of a group of 17 precartilage cells. From the mandibular cartilage of a 25-mm. tadpole of *Bufo lentiginosus*. × 800 diam.

PLATE II.

Wax reconstructions of individual precartilage cells included in the group in Plate I (with an additional cell in Figure 17). × 900 diam.

FIGURE 17. Typical 14-hedral cell.
FIGURE 18. 14-hedron with two tetrahedral angles.
FIGURE 19. Laterally compressed cell with 13 facets.
FIGURE 20. Large cell with 21 facets
FIGURE 21. 14-hedron (left) and 12-hedron (right) produced by the division of a 16-hedron.
FIGURE 22. 14-hedron (left) and 15-hedron (right) produced by the division of a 20-hedron.

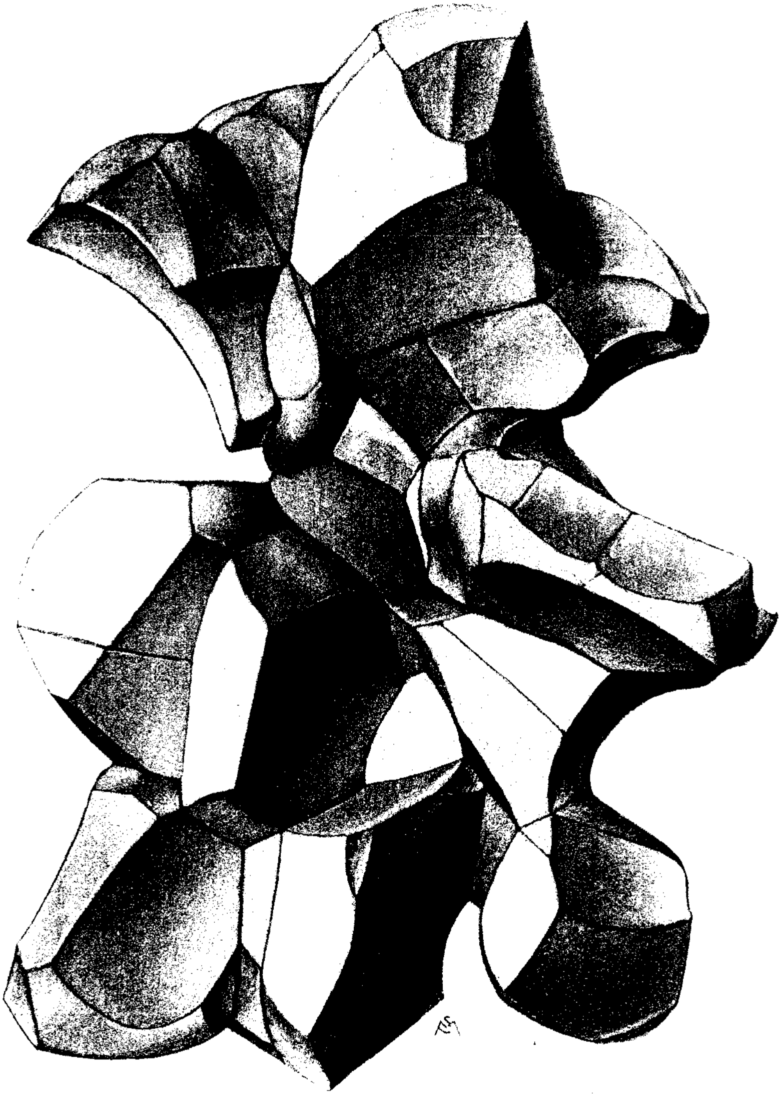

FIG. 16

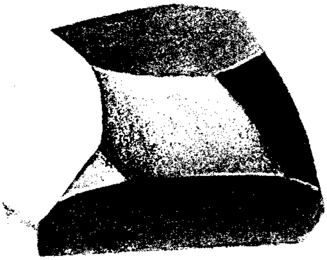

FIG. 17

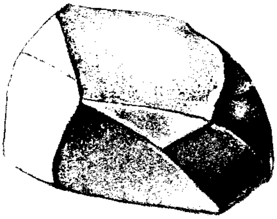

FIG. 18

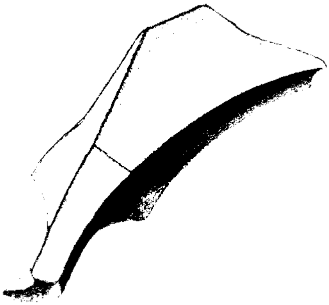

FIG. 19

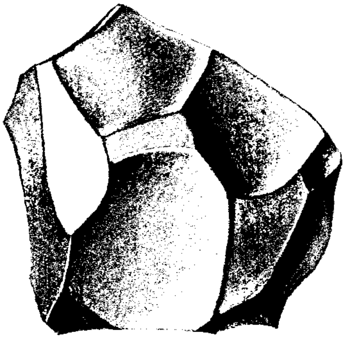

FIG. 20

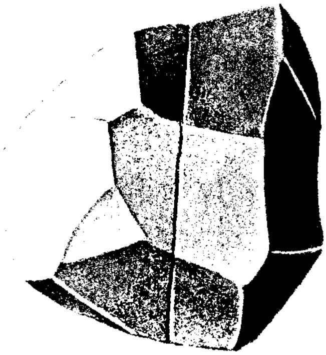

FIG. 21

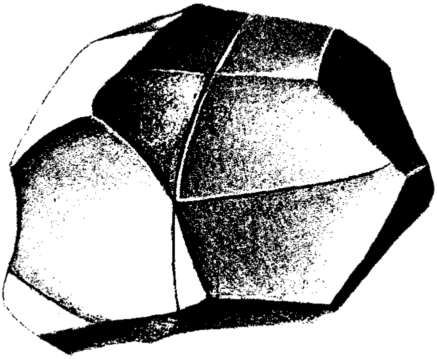

FIG. 22

F. S. M. Del.

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LEAST SQUARES AND LAWS OF POPULATION GROWTH.

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I. GENERAL INTRODUCTORY DISCUSSION.

1. *The Malthusian Law.* Although various empirical expressions have been used to fit census data,[2] few of them have been dignified with the name of "laws" of growth and few have been considered to have any rational basis. The simplest rational law which has been proposed is the so-called Malthusian Law[3] that population tends to increase in geometric ratio, the rate of increase being proportional to the population as $dP/dt = nP$ or $P = Ce^{nt}$, where n is the survival rate. This law may be rationalized by pointing out that under constant conditions the birth rate and the death rate should remain constant and consequently also the survival rate which is their difference. The law is irrational in two respects: First, with indefinite increase of time the population increases indefinitely, as is manifestly impossible. Second, although the populations of two different districts (such as an urban and its suburban region), being mere enumerations, must add to form the population of the combined metropolitan district, $P_1 = C_1e^{n_1t}$ and $P_2 = C_2e^{n_2t}$ do not add to give an expression of the form $P = Ce^{nt}$ unless n_1 and n_2 are equal. The first of these difficulties was much emphasized by Malthus and has been recently stressed by the neo-Malthusians; the second stares any practical person in the face if he tries to discuss the growth of partial and combined populations by the device of plotting on semi-log paper.

2. *The Finite and the Differential Equation.* In discussing laws of growth, and indeed laws of any kind, it is convenient to distinguish between the finite equation which expresses the phenomenon as a function of the time and the differential equation which results from the elimination of some of the parameters in the finite equation. Thus if from $P = Ce^{nt}$ the constant C be eliminated by differentiation, the differential equation $dP/dt = nP$ results, and its integration reintroduces the constant C or its equivalent. The constant C is determined for any particular population by using the observed value of P, say P_0, at a given time t_0 and noting that $P_0 = Ce^{nt_0}$ must hold, so that $C = P_0e^{-nt_0}$ and $P = P_0e^{n(t-t_0)}$. It is possible also to eliminate the constant n by a second differentiation to obtain the differential equation

$$\frac{d}{dt}\frac{d\log P}{dt} = 0 \quad \text{or} \quad \frac{d^2P}{dt^2} - \frac{1}{P}\left(\frac{dP}{dt}\right)^2 = 0 \tag{1}$$

To determine the two constants which arise in the integration of this equation we should need to know from observation not only the value

P_0 of P at the time t_0 but also the value of dP/dt at that time—or some equivalent conditions, such as the values of P at two specified times. If consideration be had for the fact that statistical fluctuations from any theoretical law must be expected it may be preferred to determine both the constants by some method of curve fitting which uses all the observed values of P and t.

3. *Integration Constants and Constants of Nature.* In most problems involving natural law one does not seek to eliminate all the constants or parameters which enter into the finite equation but one divides the constants into two categories, one the group of disposable constants which will be determined empirically from the observations and one the group of natural constants which refer to intrinsic properties of the system studied; indeed some have gone so far as to state that the search for natural law is a search for constants of nature. In the Malthusian law $P = Ce^{nt}$, C is indubitably of the nature of a constant of integration arising from the intrinsic differential equation $dP/dt = nP$, or arising from the finite equation without recourse to the differential equation by virtue of the consideration that for the law $P = Ce^{nt}$ a change of the origin of time which is arbitrary changes the value of C; but n could well be regarded as a natural constant, namely, the natural survival rate of a species,[4] to be determined once for all and not to be fittable separately in each case, and on this hypothesis we should not proceed to the differential equation (1).

4. *The Logistic.* Shortly after Malthus' time Verhulst[5] proposed that the effect of the pressure of the population on the food supply and the reaction upon the survival rate of the population might be expressed by the formula $dP/dt = nP - \phi(P)$, where ϕ is some function of P increasing with P and he gave especial attention to the simple case[6] $\phi(P) = naP^2$ so that[7]

$$\frac{dP}{dt} = n(P - aP^2) \quad \text{or} \quad P = \frac{1}{a + be^{-nt}} \qquad (2)$$

Thus he defined a curve which he called the logistic. Here n is the survival rate when P is small, i. e., before there is perceptible pressure of the population on the food supply, and a is the reciprocal of the limiting population, and b is a constant of integration. This law has been proposed again by Pearl and Reed who have fitted it to a goodly number of populations.[8] They have considered all three constants a, b, n to be disposable in each particular case, thus leaving no place for natural constants in the population problem. As they have been interested largely in forecasting population growths and in emphasizing the impossibility of an unlimited population they have naturally laid their chief emphasis on the interpretation of the constant $1/a = K$ which is the value to which the population tends asymptotically in infinite time. Yule,[9] however, has treated the problem mainly for the light it throws on the constant n, which measures the natural survival rate in an unlimited environment and for the course it would determine for the actual survival rate which is $n - aP$. That Yule finds practically the same value of n for England (.160) and for France (.183) which have had such different economic situations and have been in such different phases of their growth cycle is perhaps some evidence that there is a natural survival rate for man of about $n = .17$ corresponding to a tendency to double the population (in unlimited environment) in about four decades.

5. *The Exhaustion of Available Land.* If we are to think in very general biological terms of the problem of the growth of human or other (animal or plant) populations we should expect to find two natural constants involved, first the natural species survival rate n and second the saturation density of population. If A be the area over which the population ranges and α be the area just necessary on the average to support the life of one individual of the species, the limiting stationary population must be $K = 1/a = A/\alpha$. (East[10] estimates $\alpha = 2.5$ acres of tillable land as the amount necessary to support one human being.) The differential equation could then be written

$$\frac{d(P/A)}{dt} = n\left[\frac{P}{A} - \alpha\left(\frac{P}{A}\right)^2\right] \quad \text{or} \quad \frac{dD}{dt} = nD(1 - \alpha D) \qquad (3)$$

and express the rate of increase of density of population as an autocatalytic equation in the density with the two natural constants n and a of which the second is the reciprocal of the saturation density. It is interesting, though perhaps not very significant, that the equation may equally well be written

$$\frac{d}{dt}\left(\frac{A}{P} - \alpha\right) = -n\left(\frac{A}{P} - \alpha\right) \qquad (4)$$

stating that the per capita area of range available at any time in excess of the necessary minimum α diminishes geometrically at the rate n. In this form the constant n appears not as a survival rate for the population but as a rate of exhaustion of land per capita available above the necessary minimum.

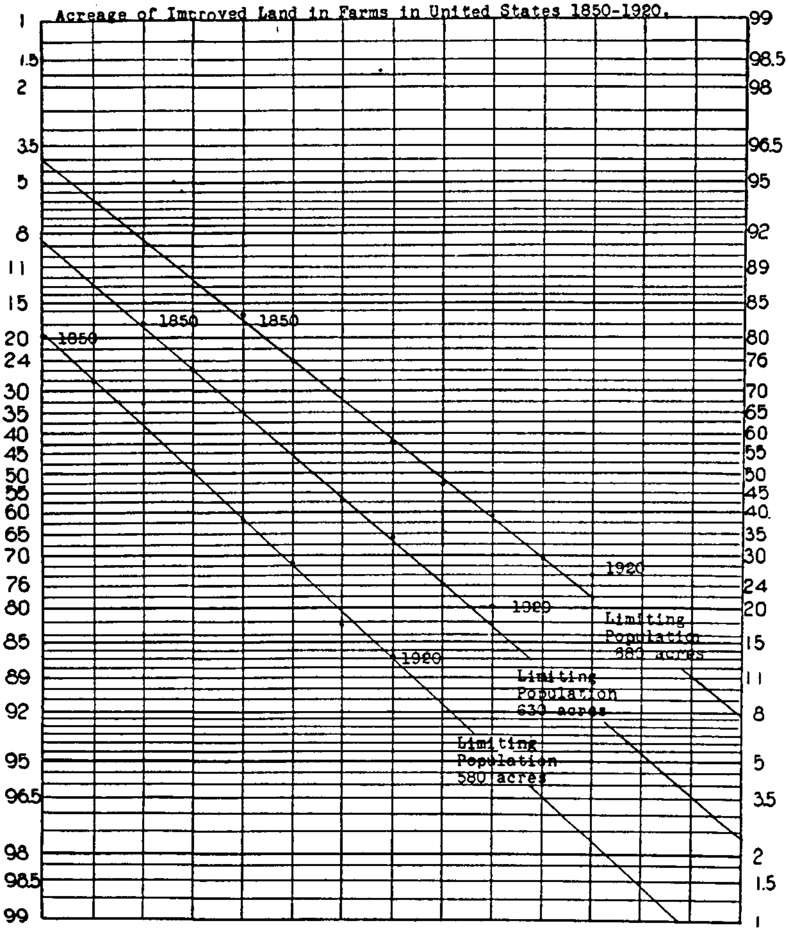
6. *Available Land in the U. S. A.* It may be remarked parenthetically that growth curves are applied empirically not alone to populations but to anything that grows such as automobile production or body weight. To illustrate with a matter germane to the immediate discussion of saturation density (or its reciprocal) as a natural constant in the population problem, we may ask what are the prospects for the "acreage of improved land in farms in the United States" as indicated by fitting the logistic to the census figures of such acreage[11] in millions, namely,

Year	1850	1860	1870	1880	1890	1900	1910	1920
Acreage	113	163	189	285	358	414	478	503

It will be noted that the rate of increase dropped greatly between 1860 and 1870 and again between 1910 and 1920. If the figure for 1920 be given much weight, it will bring the asymptotic value down under 600, but disregarding 1920 and 1870 we estimate graphically an asymptotic value between 620 and 640 (see plot on growth paper[12]). East estimated 800 as the most generous value which could be assigned to the acreage (in millions) tillable in the United States. To get so large a figure he allowed only 360 million acres for forest and woodland, or 19%, whereas France, as he points out, retains 25% of her area in forest and Germany even more; our present figure is 600 or 32%. If we retain ultimately something in excess of 25% of our area in forest and woodland, we should have to increase East's figure of 360 to 480+, and decrease his figure of 800 for tillable land to 680−. As he purposely stretched all his estimates in favor of increasing the available improved area in farms, our graphical indication of an asymptotic value around 630 million acres is corroborative of his calculation and is a value he might accept as entirely fair. This acreage might support 250 million persons at a Japanese or Chinese standard of living but scarcely more than 160 million on the American standard[13] (which would allow 4 acres per capita instead of the minimum of 2.5 and instead of our present value of about 5).

7. *Indeterminability of the Natural Constants.* Although from a theoretical point of view one may be very desirous of having some constants of nature in the formulation of any natural problem, and may sincerely hope that at some future time, one may so treat n and α as above adumbrated, there are from the practical point of view great present difficulties in assigning values to such constants partly

because of their dependence upon standards of living which some students of population problems regard as likely to determine the



THE LOGISTIC GRID. Ordinates: Percentage of Saturation

Limits	1- 2	2- 8	8-16	16-30	30-50	percent
Limits	99-98	98-92	92-84	84-70	70-50	percent
Intervals	0.25	0.5	1.0	2.0	2.5	percent

limiting population more effectively than any value of the minimum area necessary barely to support life, partly because of the complications due to trade which make it difficult to determine the area from which a population really derives its sustenance (the British may live in England but surely not off it!), and partly because of complications due to movements of population which have occurred both internationally and intranationally during the periods for which the censuses of population are available. It is at present a far cry from the problem of describing the growth of various populations to that of constructing a theory of population growth for humans or for species seriously upset by human activity in taking up and modifying the aspect of the land. Hence it is practically a necessity to regard the functional forms adopted for fitting curves to population data as empirical expressions and a justification for treating all the constants as disposable in the fitting.

8. *Populations Not Always Increasing.* One may go further and doubt whether there is any general rational law of population growth. To give expectation that there be such a law we should have to postulate that conditions governing the growth remain constant during at least that length of time which interests us in a practical way for the problem at hand. A study of the history of populations in the past shows that populations in considerable districts not only rise but fall. A noteworthy case of fall in recent times is found in Ireland which reached a maximum population of a bit above eight million about 1841 and has subsequently steadily declined to about four million. Studies by Beloch[14] on populations of ancient and medieval times indicate that the numbers in the region occupied by the Roman Empire at its greatest extension were presumably greatest around 300 A. D. (estimated at 100 million) and that when again the figure was reached for the region as a whole the distribution of the population as between Europe, Asia and Africa was quite different. He further estimates that Egypt contained about 8 million persons under Nero and did not subsequently reach that figure until within recent decades. We are familiar with the fall as well as rise and with the alternating rise and fall of populations in the animal kingdom. It is far from certain that a part of a rational population law for humans would be a steady approach to an asymptote as indicated by the logistic.[15]

9. *The Law of Superposition.* If we wish to proceed to set up the possibilities under the most general terms we meet little encouragement. The population forecaster who desires to treat districts which may be combined into larger districts would desire a strictly additive law. This would mean that after the elimination by differ-

entiation of all the truly fittable (as distinguished from the natural) constants he would desire a linear differential equation to express intrinsically the law of growth of population. For most practical cases it would be necessary to assume that there was no natural origin of time, i. e., that the change of t into $t + t_0$ in the finite equation should merely change the values of the fittable constants (as is the case with the logistic) and in the differential equation should make no change at all. These conditions seem equivalent to assuming a linear differential equation with constant coefficients. Such equations are familiar in applied mathematics and are often specifically stated as a consequence of the "law of superposition" (or additivity); they have as solutions sums of exponential or trigonometric terms in the time. No such expression seems to be satisfactory as a general law of population growth—it seems qualitatively wrong over any sufficiently extended period of time.

10. *Discontinuities in Rate of Growth.* Thus we are pretty surely thrown back on empiricism and the question is whether any law is reasonably satisfactory over a reasonable range of time and for a reasonably large number of cases. Knibbs[16] has maintained that the census facts show rather clearly that populations do not grow at a continuously diminishing rate as assumed by Verhulst, but at a constant rate for a period and then at another constant rate for a subsequent period, and so on. This is well illustrated by the case of Sweden, which appears to have had three periods of Malthusian growth;[17] first from 1750 to 1800 at the rate $n = .052$, second from 1810 to 1860 at the rate $n = .099$, third from 1870 to 1920 with $n = .068$. It should, however, be remarked that even if populations do grow according to the Malthusian law with values of n changing from time to time so that the semi-log plot is a polygon, one may be justified in fitting a logistic or other smooth curve as a graduation of the data provided the time interval over which the smooth curve holds is long compared with the intervals of time over which the individual rates of Malthusian growth are maintained. The forecasting value of such a smooth curve might, however, be much less than if the growth were really along the curve. Thus in the period 1750 to 1870 Sweden showed on the whole an acceleration rather than retardation of the rate of growth and a logistic fitted to this period would become infinite in finite time. On the other hand from 1820 to 1920 Sweden showed a retarded growth rate and the logistic would give a finite asymptote. So Connecticut is fitted nicely by two Malthusians,[17]

one from 1790 to 1840 with $n = .055$, the other from 1840 to 1930 with $n = .188$ indicating an accelerated growth over the whole period for which censuses are available.

11. *Limitations of the Logistic.* As the logistic has been used so freely recently as a law of growth and as it was very early proposed as a modification of the Malthusian law, we shall close this general discussion by a few comments upon it, some of which have been alluded to in the descriptive material above. The logistic (2) is a three-constant generalization of the two-constant Malthusian. It has the same difficulty that it is not additive, becomes in fact additive[18] only when the natural growth rates n are the same and the ratios $a : b$ are also the same. As the rational assignment of values to n and to a is practically impossible, the curve is fitted by treating all three constants as disposable in adjusting the curve to the observed populations. Under these conditions we are really considering that the law of population growth satisfies the differential equation obtained by eliminating the three constants, namely,

$$\frac{d^2}{dt^2} \log \left(\frac{d}{dt} \frac{1}{P} \right) = 0 \tag{5}$$

Furthermore it then becomes impossible to regard the logistic as symbolizing a retarded growth rate resulting in a finite saturated value for the population because through the process of fitting we may find opposite values for the constants a and b which will result mathematically in forecasting an infinite population at the finite time $t = -[\log(-a/b)]/n$. This would not be a significant difficulty were it not for the fact that this situation actually arises in not a few cases when attempts are made to fit the logistic to actual data.[19] In such instances although the logistic, having an extra constant, will fit the observations better than the Malthusian, and may (or may not) forecast better the populations which will be observed at a few censuses following the last one available for fitting, it surely will give a long time forecast which is worse than that given by the Malthusian.

12. *Types of Logistic.* To bring the matter clearly into the light we may classify logistics into five types on the basis of fitting them to three equally spaced observations by the usual method. Let the value of P be P_0, P_1, P_2, let time be measured from the first observation, and let $r = e^{-nt}$ where t is the interval of time between the observations; then

$$1/P_0 = a + b, \quad 1/P_1 = a + br, \quad 1/P_2 = a + br^2.$$

These equations will have a definite solution for a, b, r, namely,

$$r = \frac{1/P_1 - 1/P_2}{1/P_0 - 1/P_1}, \quad a = \frac{1/P_0 \cdot 1/P_2 - (1/P_1)^2}{1/P_0 + 1/P_2 - 2/P_1},$$
$$b = \frac{(1/P_1 - 1/P_0)^2}{1/P_0 + 1/P_2 - 2/P_1} \tag{6}$$

with one exception, namely, when the populations advance in harmonic progression with $1/P_0 - 1/P_1 = 1/P_1 - 1/P_2$. In this exceptional case $r = 1$ and $a = \pm\infty$ and $b = \mp\infty$ and the logistic becomes indeterminate. This will be called the critical case. The five types are

1. Hypomalthusian (with saturation), $\left(\frac{1}{P_1}\right)^2 < \frac{1}{P_0}\cdot\frac{1}{P_2}$ and $\frac{2}{P_1} < \frac{1}{P_0} + \frac{1}{P_2}$
2. Malthusian (no saturation), $\left(\frac{1}{P_1}\right)^2 = \frac{1}{P_0}\cdot\frac{1}{P_2}$ and $\frac{2}{P_1} < \frac{1}{P_0} + \frac{1}{P_2}$
3. Hypermalthusian (no saturation), $\left(\frac{1}{P_1}\right)^2 > \frac{1}{P_0}\cdot\frac{1}{P_2}$ and $\frac{2}{P_1} < \frac{1}{P_0} + \frac{1}{P_2}$
4. Critical (no saturation), $\left(\frac{1}{P_1}\right)^2 > \frac{1}{P_0}\cdot\frac{1}{P_2}$ and $\frac{2}{P_1} = \frac{1}{P_0} + \frac{1}{P_2}$
5. Hypercritical (no saturation), $\left(\frac{1}{P_1}\right)^2 > \frac{1}{P_0}\cdot\frac{1}{P_2}$ and $\frac{2}{P_1} > \frac{1}{P_0} + \frac{1}{P_2}$

Four of the five cases indicate no saturation, three of them are hypermalthusian.

13. *The Critical Logistic.* In the critical or indeterminate case, it is not the curve which is indeterminate as a geometrical object but the constants in the logistic form. There is no difficulty in taking the limit, as r approaches 1 and a becomes infinite, in such a manner as to show that the limiting curve has the form

$$\frac{1}{P} = \alpha - \beta t \quad \text{or} \quad P = \frac{1}{\alpha - \beta t}, \tag{7}$$

with

$$\alpha = \frac{1}{P_0}, \quad \beta = \frac{1}{t}\left(\frac{1}{P_0} - \frac{1}{P_1}\right),$$

which becomes infinite at $t = \alpha/\beta$. It may be remarked that as

$$n = -\frac{\log r}{t} = \frac{1}{t} \log \left(\frac{1/P_1 - 1/P_0}{1/P_2 - 1/P_1} \right)$$

the interpretation of n as the rate of growth when the population is small is equally possible in the first three cases. In the critical case n becomes zero, which may still be interpreted; but in the hypercritical case n becomes negative. The following table gives a numerical example. The reason for the discussion of the critical case is partly because population data have been found which seem to be critical if not hypercritical and partly because of curve fitting difficulties which arise in that case and to which we have to allude (Art. 48).

	P_0	P_1	P_2	a	b	r	nt
Hypermalthusian	1	1.5	2.5	$-2/3$	$5/3$	$4/5$	.223
"	1	1.5	2.97	-32	33	$98/99$	.0102
Critical	1	1.5	3.0	$\mp\infty$*	$\pm\infty$*	1	0
Hypercritical	1	1.5	3.03	$104/3$	$-101/3$	$102/101$	−.0099
"	1	1.5	3.5	$10/3$	$-\ 7/3$	$8/7$	−.134

* $\alpha = 1$ and, assuming $t = 1$, $\beta = 1/3$. Thus $1/P = 1 - t/3$.

II. Fitting the Logistic.

14. *Simple Methods of Fitting.* We have given the formulas (6) for finding a, b, $r = e^{-nt}$ for the logistic from three values of P separated in time by equal amounts t, the origin of time being at the first of the three observations. Naturally, in practice one would space the three points well apart. If the number of censuses, supposed themselves equally spaced, were odd we could use the first, middle and last censuses or the second, middle and next to the last. If we wished to throw more weight into the later censuses as likely to be more determinative of the subsequent course of population we could ignore the first few altogether. If statistical fluctuations seemed to be rendering the individual points inadequately reliable we could use a graphical method of trial based on growth paper, or in the absence of such paper we could assume various values of the constant a and plot on semi-log paper $1/P - a$ to ascertain what value of a seemed to give the best straight line

$$\log_{10}(1/P - a) = \log_{10} b - n \log_{10} e \cdot t \tag{8}$$

and by this formula we could equally well make the study on ordinary plotting paper. Another method would be to follow Yule's suggestion[9] of summation of the reciprocals of the populations in equal equally spaced groups of k, the formulas being

$$\Sigma_1 \frac{1}{P} = ka + b\,\frac{1 - r^k}{1 - r}, \quad \Sigma_2 \frac{1}{P} = ka + br^p\,\frac{1 - r^k}{1 - r},$$

$$\Sigma_3 \frac{1}{P} = ka + br^{2p}\,\frac{1 - r^k}{1 - r}$$

where the initial times of the groups are p units t apart. These equations with

$$a' = ka, \quad b' = b\,\frac{1 - r^k}{1 - r}, \quad r' = r^p \tag{9}$$

are identical in form with those for three points and thus permit the use of (6) to find a', e', r' from which a, b, r immediately follow.

15. *Instability of the Constants.* The difficulties into which one comes when trying to fit a logistic by these methods may be illustrated by taking the case of New York City (present area). The census figures are in millions:[20]

For	1850	1860	1870	1880	1890	1900	1910	1920
P =	0.696	1.175	1.478	1.912	2.507	3.437	4.767	5.620

If now we fit to 1860, 1890, 1920 taking three points as far apart as possible we get $K = 1/a = -30$ (millions) for limiting population showing that the logistic is hypermalthusian with no indication of saturation. If we take 1870, 1890, 1910, we find $K = -5$ and again hypermalthusian growth. If we take 1850, 1880, 1910 we find $K = 21.5$ showing hypomalthusianism but a well-nigh incredibly large asymptotic value,—and why favor 1850, 1880, 1910 as over against 1860, 1890, 1920 for forecasting? If we apply Yule's method of summation (9) using the three pairs 1870–80, 1890–1900, 1910–20 we find $K = -38$ again indicating no saturation. By the same method using 1860–70, 1880–90, 1900–10 we find $K = -4.4$. Most of the estimates therefore fail to give any evidence of saturation, and furthermore the value of K varies all the way from 21 through infinity to -4. If we discard all the earlier data and use 1900, 1910, 1920 we find $K = 6.4$, which was already exceeded in 1930. The value of K found by the graphical method (8) using all the points would depend largely on the judgment of the person using that method.[21] In other words these methods as a whole fail to give consistent answers and thus leave the problem of fitting unsolved.

16. *Cases of Stability of the Constants.* In the case of the United States the growth of population has followed the logistic with extraordinary fidelity[22] so that almost any method of fitting will lead to approximately the same results for the constants. It may not be amiss to mention that the United States has been far from a fixed area since 1790 and that as a matter of fact most of its present area was unsettled and even held under foreign sovereignties at the time of the earlier censuses, and further that we have had a large immigration.[23] It is a little difficult to see why, despite these accessions of area and of population from without, our growth should be so regularly according to law unless that law be rather a law of the call of the land than a law of population growth based on reproduction. Yule's discussion of England and of France shows reasonable stability in the constants for those countries. To cite England as an illustration we may point out that using the three censuses 1811, 1861, 1911 we find $K = 108$ (million); Yule by three methods finds $K = 91$, 97, and 100. Such consistency indicates that the constant K is actually mathematically determinate; whether it implies any comparable accuracy as a forecast may be doubted. But the instability shown above in the case of New York City is by no means rare.

17. *Logistics Infinite in Finite Time.* It may be worth while to tabulate the values of the constants obtained by various methods for a case or two. Take Connecticut and New Jersey and the censuses from 1790 to 1920; we find (using millions as the unit, as we shall systematically),

	Connecticut			New Jersey		
	a	b	n	a	b	n
1a	−5.11	9.44	.0370	− .343	6.02	.170
1b	11.96	− 7.76	−.0296	−1.158	6.59	.121
2a	−2.19	7.31	.0719	− .253	7.08	.199
2b	36.54	−32.11	−.0090	−1.082	6.85	.133
3	−1.64	6.64	.0797	− .026	8.17	.245

(The origin of time affects the value of b, only, and is uniformly 1790 in this table. The time unit is, as always in this paper, the decade.) The first values 1a are based on the three censuses 1800, 1860, 1920; the second set 1b on 1790, 1850, 1910. The third set 2a is figured by Yule's summation method applied to the groups 1810–1840, 1850–

1880, 1890–1920, the fourth (2b) is based on the groups 1800–1830, 1840–1870, 1880–1910. In the fifth line (3) are given the values derived by minimizing the sum of the squares of the deviations between the observed census figures and the logistic (Art. 39). None of the methods applied to either State show saturation, the logistics become infinite in finite time. The values of $K = 1/a$ are highly variable as well as non-significant. Even the constant n which gives the rate of growth when the population is small has no stability for Connecticut, and not much for New Jersey. It is to be noted that for Connecticut the methods 1*b* and 2*b* based on omitting the last census (1920) give such acceleration of growth as to make the logistics hypercritical ($a > 0$, $b < 0$, $n < 0$).

18. *Necessity of Definite Methods of Fitting.* If to the indetermination resulting from specified arithmetical methods of fitting there be added the possibilities for variation which arise from graphical methods it becomes clear that the logistic which any one person may get must often be largely arbitrary and personal. By a sufficient exercise of wishful thinking one may even obtain a hypomalthusian logistic showing saturation for a series of populations which by practically every test show a hypermalthusian growth. The fitting of the curve becomes an art in which one gets what he wants, and the values forecast from the curve become personal estimates with perhaps no greater validity than would attach to estimates obtained in any other way and probably less validity than would attach to estimates based on an analysis of the age distribution, of deaths, and of fecundity.[24] For scientific purposes we must have a method of fitting which is agreed upon and the results of which can be duplicated by anybody. For such purposes one naturally thinks of the method of least squares, whether or no this method should logically be applied to such time series as arise in problems of growth. Pearl and Reed have advocated this method and have claimed to practice it[8, 20, 31, 34]. There are interesting mathematical problems which arise in the application of the method to logistics, and we shall turn our attention to some of them.

19. *Conditions for a Least Squares Fit of a Logistic.* In 1806 Legendre stated the principle that the most satisfactory solution of the problem of determining the unknowns from supernumerary equations is to render the sum of the squares of the "errors" or deviations a minimum.[25] In the case in hand this means that such values of the parameters a, b, and n of the logistic shall be chosen as will minimize the function

$$S(a, b, n) = \Sigma_i \left(E_i - \frac{1}{a + be^{-nt_i}} \right)^2 = \Sigma_i (E_i - P_i)^2 \qquad (10)$$

where E_i represent the actual observed enumerations at the times t_i and where P_i are the values computed from the logistic.[26] The conditions for a minimum are[27]

$$\frac{1}{2}\frac{\partial S}{\partial a} = 0 = -\Sigma(E_i - P_i)\frac{\partial P_i}{\partial a} = \Sigma(E_i - P_i)P_i^2, \qquad (11a)$$

$$\frac{1}{2}\frac{\partial S}{\partial b} = 0 = -\Sigma(E_i - P_i)\frac{\partial P_i}{\partial b} = \frac{1}{bK}\Sigma(E_i - P_i)P_i(K - P_i), \qquad (11b)$$

$$\frac{1}{2}\frac{\partial S}{\partial n} = 0 = -\Sigma(E_i - P_i)\frac{\partial P_i}{\partial a} = -\frac{1}{K}\Sigma(E_i - P_i)P_i(K - P_i)t_i, \qquad (11c)$$

where $K = 1/a$. These conditions are immediately interpretable. They state that we must have the sum of the residuals $E_i - P_i$ equal to zero when those residuals are weighted,[28] before summing, in three different ways (1) by the squares of the calculated populations P_i, (2) by the product of the populations and their differences from the asymptotic population $K = 1/a$, and (3) by the continued product of the times and of the populations and of their differences from their asymptotic value.[29]

20. *Other Conditions that Might be Applied.* Such conditions are, of course, different from those to be used if we were to fit by the method of moments which would require that the zeroth, first and second moments of the residuals with respect to the time should vanish, viz.,

$$\Sigma(E_i - P_i) = 0, \quad \Sigma(E_i - P_i)t_i = 0, \quad \Sigma(E_i - P_i)t_i^2 = 0, \qquad (12)$$

and which are the conditions that would arise in fitting the parabolic form $P = a + bt + ct^2$ by least squares.[30] They are also different from the conditions which correspond to Pearl's solution of the problem of fitting the logistic, with a relatively simple computation form.[31] His equations xxxvi–xxxviii are equivalent to the conditions

$$\Sigma\frac{E - P}{P} = 0, \quad \Sigma E\left(\frac{E - P}{P}\right) = 0, \quad \Sigma tE(K - P)\left(\frac{E - P}{P^2}\right) = 0 \qquad (13)$$

whereas, by combining the first two of our correct conditions (11) we have for direct comparison the equivalent conditions

$$\Sigma P(E-P)=0,\quad \Sigma P^2(E-P)=0,\quad \Sigma Pt(K-P)(E-P)=0 \qquad (14)$$

It will be noticed that his equations are similar to ours except that they have been divided respectively by P^2, P^3/E and P^3/E, the values of P^3/E being very nearly equal to those of P^2. Thus his fit makes the mean residual vanish when weighted by the reciprocal of the computed population or by that reciprocal multiplied by the observed population E or by the continued product of that reciprocal squared and by the time and by E and by the difference between the asymptotic population and P. As Schultz[32] has pointed out Pearl's least squares solution is not a least squares solution.

21. *Least Squares by Relative Residuals.* If we should ask what would be the least squares solution under the condition of minimizing the sum of the relative residuals we should have to minimize

$$\text{(a)}\ S=\Sigma\left(\frac{E-P}{E}\right)^2 \quad \text{or} \quad \text{(b)}\ S=\Sigma\left(\frac{E-P}{P}\right)^2 \qquad (15a, b)$$

The conditions for the first are

$$\Sigma\frac{P}{E^2}(E-P)=0,\quad \Sigma\frac{P^2}{E^2}(E-P)=0,$$
$$\Sigma\frac{Pt(K-P)}{E^2}(E-P)=0 \qquad (16a)$$

and for the second

$$\Sigma\frac{E}{P^2}(E-P)=0,\quad \Sigma\frac{E}{P}(E-P)=0,$$
$$\Sigma\frac{Et(K-P)}{P^2}(E-P)=0 \qquad (16b)$$

These two solutions will be slightly different and both of them will be slightly different from Pearl's but in so far as P and E may be treated as equivalent in weighting the residuals prior to summing,[33] all three solutions will be the same. Indeed as

$$\frac{P}{E^2}=\frac{1}{P}\frac{P^2}{E^2} \quad \text{and} \quad \frac{P^2}{E}=\frac{1}{P}\cdot\frac{E}{P}$$

it is clear that the weights in Pearl's first condition (13) are intermediate between those in the first conditions in the least squares solution by relative residuals taken the two ways. The weights in his second and third conditions are identical with those of the least squares fit by relative residuals based on P. It is therefore to be presumed that though Pearl's fit is not a least squares fit it is very close to a least squares fit by relative (not absolute) residuals based on P but shaded a little toward that based on E.

22. *Least Squares With Weights.* We could weight the observations. The primal conception of a weight is that of a repeated observation. An observation of weight w is the equivalent of that observation repeated w times. We should therefore naturally define the problem of least squares with weights as that of minimizing

$$S(a, b, n) = \Sigma_i w_i(E_i - P_i)^2 \tag{17}$$

inserting $(E_i - P_i)$ into the sum S as many times as its weight w_i indicates. When the weights do not involve the unknowns, as is usual, and when the expression P is linear in the unknowns, as is also usual, the conditions for minimizing $\Sigma w(E - P)^2$ lead to the usual normal equations for least squares solutions with weights.[34] The question might fairly be raised as to whether the problem of minimizing S as written is the proper definition of the problem of determining the least squares fit when the expression P is non-linear in the constants and the weights do not involve them. The answer would be indubitably affirmative. But the answer is not so sure when the weights involve the unknown constants. The differentiation of S would then yield equations like

$$\frac{1}{2}\frac{\partial S}{\partial a} = -\Sigma_i w_i(E_i - P_i)\frac{\partial P_i}{\partial a} + \frac{1}{2}\Sigma_i(E_i - P_i)^2\frac{\partial w_i}{\partial a} = 0 \tag{18}$$

as those for the minimum. We might prefer not to differentiate the weights. We shall not attempt here to determine just what the formulation of problem of fitting by least squares should be when the weights involve the constants; we shall simply *assume* that when we say least squares we mean that the sum of the squares is to be least, and leave the critical discussion to the Appendix. The problem of fitting by relative differences (15a) can be stated as minimizing the sum of the squares of the absolute residuals weighted inversely as the squares of the observed populations, and by analogy we would interpret (15b) as minimizing the sum of the squares of the absolute residuals weighted inversely as the squares of the computed populations and we differentiate with respect to the parameters wherever they occur as we did in deriving (16b).

23. *What Weights Should be Used?* The question of the assignment

of weights is always troublesome. It ought to be discussed for the problem of fitting growth curves. Is a variation of 1 lb. in weight of a baby to be treated like that of a variation of 1 lb. in the adult in fitting a growth curve to the body weight? Is a deviation from the logistic of 100,000 when the population is half a million to be considered as equivalent to the same deviation when the population is ten million? Shall we minimize the sum of the squares of the absolute residuals or the sum of the squares of the relative residuals[35] or something else? In frequency function problems where χ^2 is an appropriate test of goodness of fit we do not minimize the sum of the squares of the absolute or relative residuals but the sum of the squares of the absolute residuals divided by the calculated value,[36] i. e., $\Sigma(E - P)^2/P$. Here we have from the sampling theory an indication of what form χ^2 should take. In some problems we know that variations of the same proportionate amounts are equally significant and then we minimize the sum of the squares of the relative residuals. We know of no theoretical or empirical considerations which show what weights should be applied in fitting logistics to population figures; we shall fit them mostly without weights by minimizing the sum of the squares of the absolute errors but give a few cases of fitting by minimizing the sum of the squares of the relative errors for purposes of comparison.[37]

24. *Fitting the Malthusian; First Approximation.* As this matter of fitting with and without weights is of importance and as the formulas and calculations with the logistic are unnecessarily long for the purpose of illustrating principles and methods, we shall choose first the simple problem, proposed to us by a distinguished astronomer and of some importance in itself, of fitting the two-constant Malthusian $P = Ce^{nt}$, a special case of the logistic with $a = 0$, to a series of values E so as to minimize the sum of the squares of the errors. The conditions to minimize

$$S = \Sigma(E - Ce^{nt})^2 \quad \text{are} \quad \Sigma(E - P)P = 0 \quad \text{and}$$

$$\Sigma(E - P)tP = 0 \qquad (19)$$

If we consider the example

$$t = -1,\ E = 0.5; \quad t = 0,\ E = 2.5; \quad t = 1,\ E = 3.875$$

we see that with $P = 1, 2, 4$, equations (19) are satisfied and that consequently $P = 2 \times 2^t$ or $2e^{(\log 2)t}$ and $C = 2$, $n = \log 2 = .6931$. The value of S is 0.515625. The usual method of getting a first approximation to the solution, and one which not infrequently is good enough for practical purposes is to take logarithms and fit the linear expression[38] $\log P = nt + \gamma$, where $\gamma = \log C$, to the values $\log E$, i. e., to minimize

$$S = (\log E - nt - \gamma)^2 \quad \text{by} \quad \Sigma(\log E - \log P) = 0,$$
$$\Sigma(\log E - \log P)t = 0 \qquad (20)$$

Owing to the linearity this may be done at one stroke. For the example above the method gives

$$P = 1.6920\, e^{1.02384\,t}, \quad S = 1.3621$$

and it is seen that the fit is bad. The value 1.692 for C is well removed from the true value $C = 2$ and the value $n = 1.02384$ is not near .6931. The value $S = 1.3621$ is more than two and a half times the true value .515625. Extrapolation would be extremely incorrect.

25. *The Malthusian; Second Approximation.* Now as $d \log P = dP/P$ one may write $dP = P \log P$ and if we consider $E - P$ as a sort of differential of P we should expect that minimizing $\Sigma(E - P)^2$ would be approximated by minimizing

$$\Sigma(\log E - \log P)^2 P^2 \quad \text{or} \quad \Sigma(\log E - \log P)^2 E^2, \qquad (20')$$

since E and P are nearly equal. That is, in fitting logarithmically we should weight the observations as the squares of their values.[39] The first form gives no advantage in calculation over minimizing $\Sigma(E - P)^2$ but the second form is quadratic in the unknowns $\gamma = \log C$ and n, and leads to linear normal equations. The solution is

$$P = 2.3087\, e^{.53445\,t} \quad \text{with} \quad S = .7682$$

and is much better, though by no means very good, nor safe for any considerable extrapolation. As, however, the actual solution of the problem of minimizing $\Sigma(E - P)^2$ is that of solving the two equations of condition

$$\Sigma(E - Ce^{nt})e^{nt} = 0 \quad \text{and} \quad \Sigma(E - Ce^{nt})te^{nt} = 0 \qquad (19')$$

which are non-linear in the constant n (though, as a matter of fact, linear in C), so that the solution must proceed by successive approximations, it is clearly to be desired that the start of the work be from values as nearly correct as may be, and hence this second solution should be preferred as a starting point.

26. *The Approximation Equations for the Malthusian.* There is another item connected with the fitting of logistics by least squares

which may be illustrated by the simpler case of the Malthusian, and that is the derivation of the equations. We have

$$S(a, b, n) = \Sigma\left(E - \frac{1}{a + be^{-nt}}\right)^2$$

to minimize. We may expand $S(a, b, n)$ about any set of values, which in practice should be near the minimum. In the simpler case

$$S(C, n) = \Sigma(E - Ce^{nt})^2 = S_0 + \left(\frac{\partial S}{\partial C}\right)_0 \delta C + \left(\frac{\partial S}{\partial n}\right)_0 \delta n$$

$$+ \frac{1}{2}\left(\frac{\partial^2 S}{\partial C^2}\right)_0 \delta C^2 + \left(\frac{\partial^2 S}{\partial C\,\partial n}\right)_0 \delta C\,\delta n + \frac{1}{2}\left(\frac{\partial^2 S}{\partial n^2}\right)_0 \delta n^2 + \cdots \quad (21)$$

In the usual case of least squares where the constants enter linearly in P, S is a quadratic function of the constants and the expansion has no higher terms. The conditions for a minimum are[40]

$$\begin{aligned} \frac{1}{2}\frac{\partial S}{\partial C} + \frac{1}{2}\frac{\partial^2 S}{\partial C^2}\delta C + \frac{1}{2}\frac{\partial^2 S}{\partial C\,\partial n}\delta n &= 0 \\ \frac{1}{2}\frac{\partial S}{\partial n} + \frac{1}{2}\frac{\partial^2 S}{\partial C\,\partial n}\delta C + \frac{1}{2}\frac{\partial^2 S}{\partial n^2}\delta n &= 0 \end{aligned} \quad (22)$$

and can be solved definitively for δC and δn no matter what values are taken for C_0, n_0,—and it is usual to take the values 0 for both so that we really solve for C and n directly. When the constants do not enter linearly in P, S is not quadratic but may be expanded into a series in the neighborhood of any values of the constants and if the terms higher than the second be neglected we have a quadratic which will represent the function in the neighborhood of those values. If the values are near enough to the minimum sought, the solution of the equations for δC and δn should give a better value,[41]—and the process may be repeated.

27. *Modified Approximation Equations.* Another method of reaching equations to be solved is to expand $E - P$ before squaring

$$S = \Sigma\left[E - P - \frac{\partial P}{\partial C}\delta C - \frac{\partial P}{\partial n}\delta n - \frac{1}{2}\frac{\partial^2 P}{\partial C^2}\delta C^2 - \frac{\partial^2 P}{\partial C\,\partial n}\delta C\,\delta n - \frac{1}{2}\frac{\partial^2 P}{\partial n^2}\delta n^2 \cdots\right]^2 \quad (23)$$

In the case where the constants enter P linearly there are no terms of the second or higher orders. If in the more complicated case we stop the expansion with the linear terms and square we have a quadratic in δC, δn which leads to the equations

$$\begin{aligned} -\Sigma(E-P)\frac{\partial P}{\partial C}+\Sigma\left(\frac{\partial P}{\partial C}\right)^2\delta C+\Sigma\frac{\partial P}{\partial C}\frac{\partial P}{\partial n}\,\delta n&=0\\ -\Sigma(E-P)\frac{\partial P}{\partial n}+\Sigma\frac{\partial P}{\partial C}\frac{\partial P}{\partial n}\,\delta C+\Sigma\left(\frac{\partial P}{\partial n}\right)^2\delta n&=0 \end{aligned}\tag{24}$$

and it is these equations which are ordinarily suggested for solution for δC and δn to obtain the next approximation.[42] It should be noted that when we square as above after rendering the expression $E-P$ linear in the variation of the parameters, we are really neglecting a part of the quadratic terms, namely, those arising from the product of the finite (though presumably small) term $E-P$ and the quadratic terms in the expansion of P. Really the quadratic expression in δC and δn is

$$\begin{aligned} &\Sigma(E-P)^2-2\Sigma(E-P)\left(\frac{\delta P}{\partial C}\,\delta C+\frac{\partial P}{\partial n}\,\delta n\right)+\Sigma\left(\frac{\partial P}{\partial C}\,\delta C+\frac{\partial P}{\partial n}\,\delta n\right)^2\\ &\qquad-\Sigma(E-P)\left(\frac{\partial^2 P}{\partial C^2}\,\delta C^2+2\,\frac{\partial^2 P}{\partial C\,\partial n}\,\delta C\,\delta n+\frac{\partial^2 P}{\partial n^2}\,\delta n^2\right) \end{aligned}$$

That which is upon the first line leads to the equations (24) just written whereas the total expression gives

$$\begin{aligned} &-\Sigma(E-P)\frac{\partial P}{\partial C}+\Sigma\left[\left(\frac{\partial P}{\partial C}\right)^2-(E-P)\frac{\partial^2 P}{\partial C^2}\right]\delta C\\ &\qquad\qquad+\Sigma\left[\frac{\partial P}{\partial C}\frac{\partial P}{\partial n}-(E-P)\frac{\partial^2 P}{\partial C\,\partial n}\right]\delta n=0\\ &-\Sigma(E-P)\frac{\partial P}{\partial n}+\Sigma\left[\frac{\partial P}{\partial C}\frac{\partial P}{\partial n}-(E-P)\frac{\partial^2 P}{\partial C\,\partial n}\right]\delta C\\ &\qquad\qquad+\Sigma\left[\left(\frac{\partial P}{\partial n}\right)^2-(E-P)\frac{\partial^2 P}{\partial n^2}\right]\delta n=0 \end{aligned}\tag{25}$$

These two approximation equations for δC and δn are, of course, different in form only from those (22) written for S but they do differ from (24).

28. *May One Modify the Approximation Equations?* The question might be raised as to whether it is proper, in the sense that it leads to the right answer, to use the simpler equations (24) obtained by reducing the expression $E - P$ to linear form before squaring rather than the more complicated ones. A simple illustration will show that there are sums of squares for which it is not proper. Consider

$$S = (xy - 1)^2 + (x + y)^2$$

The minimum is found from

$$\frac{1}{2}\frac{\partial S}{\partial x} = (xy - 1)y + (x + y) = x(y^2 + 1) = 0$$

$$\frac{1}{2}\frac{\partial S}{\partial y} = (xy - 1)x + (x + y) = y(x^2 + 1) = 0$$

Obviously there is just one solution (0, 0), which is a true minimum and the expansion of S about (0, 0) is $1 + x^2 + y^2 + \cdots$. If we now try to obtain this solution by (22) the equations are

$$x(y^2 + 1) + (y^2 + 1)\delta x + 2xy\,\delta y = 0$$

$$y(x^2 + 1) + 2xy\,\delta x + (x^2 + 1)\delta y = 0$$

and

$$\delta x = -\frac{x(x^2 + 1)(1 - y^2)}{1 + x^2 + y^2 - 3x^2y^2}, \qquad \delta y = -\frac{y(y^2 + 1)(1 - x^2)}{1 + x^2 + y^2 - 3x^2y^2}$$

If we start from x, y near (0, 0) we get a very rapid convergence to (0, 0) from equations which are practically $\delta x = -x$ and $\delta y = -y$. On the other hand if we expand before squaring, retaining only the linear terms, we have

$$\begin{aligned} S' &= (xy + y\delta x + x\delta y - 1)^2 + (x + y + \delta x + \delta y)^2 \\ &= (xy - 1)^2 + (x + y)^2 + 2x(y^2 + 1)\delta x + 2y(x^2 + 1)\delta y \\ &\qquad + (y^2 + 1)\delta x^2 + (x^2 + 1)\delta y^2 + 2(xy + 1)\delta x\,\delta y \end{aligned}$$

The equations for δx, δy are

$$x(y^2 + 1) + (y^2 + 1)\delta x + (xy + 1)\delta y = 0$$

$$y(x^2 + 1) + (xy + 1)\delta x + (x^2 + 1)\delta y = 0$$

and

$$\delta x = \frac{x^2 + 1}{y - x}, \qquad \delta y = \frac{y^2 + 1}{x - y}$$

from which no convergence to (0, 0) is possible. The function S', quadratic in δx, δy, has a minimum for any assigned values of x and y provided x and y are not equal, yet that minimum gives no indication of the minimum of S.

29. *The Standard Errors of the Constants.* Whether such a case can arise in a least squares solution to the problem of fitting a curve we do not know; we have never met such a case.[43] On the contrary we have found that in most of the cases in which we have tried comparatively the two sets of equations (24) and (25), the simpler equations actually converge to the answer somewhat more quickly than the more complicated ones.[44] But one thing does appear reasonably clear, and that is that if we desire to discuss the standard errors of the unknowns by using the ordinary formulas we should logically use the true quadratic expansion (25) of S about the minimum. The standard errors of C and n being respectively[45]

$$\sqrt{\frac{\Sigma(E-P)^2}{k-2}}\sqrt{\frac{\frac{1}{2}\partial^2 S/\partial n^2}{H}}, \quad \sqrt{\frac{\Sigma(E-P)^2}{k-2}}\sqrt{\frac{\frac{1}{2}\partial^2 S/\partial C^2}{H}} \qquad (26)$$

where $4H = \frac{\partial^2 S}{\partial C^2}\frac{\partial^2 S}{\partial n^2} - \left(\frac{\partial^2 S}{\partial C\,\partial n}\right)^2$ and k is the number of observations.

(It is understood that all the quantities are to be evaluated for those values of C and n which determine the minimum.) In evaluating the values (26) it is important not to reject any constants from the equations (24) or (25) such as may have been rejected in discussing the conditions for the minimum.[29]

30. *The Forms of the Logistic.* In fitting the logistic the problem is to satisfy the three conditions (11) or their equivalent, no matter how. If in addition we desire the standard errors of the unknowns we have to be careful to carry the calculation in such fashion that we can apply the appropriate formulas[46] analogous to (26). The form of the approximation equations which one will use depends somewhat on the form in which the logistic is written and of these there are several,

$$\frac{1}{a+be^{-nt}}, \quad \frac{K}{1+me^{-nt}}, \quad \frac{B}{e^{-nt}+C}, \quad \frac{1}{a+e^{\beta-nt}}, \text{ etc.}$$

The relations between the constants in the various cases can readily be obtained so that from the solution in any form that in any of the other forms can be obtained. There is one exception. As $\beta = \log b$ the last form cannot be used when b turns out to be negative, for β

would have to be imaginary and it is impossible to obtain an imaginary number through the convergence process set up for a solution by successive approximations on the assumption that the solution is real. As b is negative only in the hypercritical case there is not much danger of β being imaginary. If however, we write the second form as

$$\frac{K}{1 + me^{-nt}} = \frac{K}{1 + e^{\mu - nt}}, \quad \mu = \log m$$

there is real danger of failing of a solution because of imaginary μ, for the ordinary hypermalthusian but subcritical case arises rather frequently and, in this, both K and m are negative, i. e., a is negative and b is positive. We have used consistently the first form which produces no discontinuity in the transition from the hypo- to the hypermalthusian case; the second form requires that both K and m pass through ∞ in this transition; the third form is free from this difficulty.[47] We started our calculations with the first form and have used it consistently throughout; but we incline to believe that the third form is somewhat to be preferred at least unless one needs the standard error of $K = B/C$.

31. *The Approximation Equations for the Logistic.* The expression to minimize is

$$S = \Sigma(E - P)^2 = \Sigma\left(E - \frac{1}{a + be^{-nt}}\right)^2 = \Sigma\left(E - \frac{B}{e^{-nt} + C}\right)^2 \tag{27}$$

then

$$\frac{\partial P}{\partial a} = -P^2, \quad \frac{\partial P}{\partial b} = -e^{-nt}P^2, \quad \frac{\partial P}{\partial n} = bte^{-nt}P^2 \tag{28}$$

or

$$\frac{\partial P}{\partial B} = \frac{P}{B}, \quad \frac{\partial P}{\partial C} = -\frac{P^2}{B}, \quad \frac{\partial P}{\partial n} = \frac{t}{B}e^{-nt}P^2 \tag{28'}$$

Hence

$$\begin{gathered} \frac{1}{2}\frac{\partial S}{\partial a} = \Sigma(E - P)P^2, \quad \frac{1}{2}\frac{\partial S}{\partial b} = \Sigma(E - P)P^2e^{-nt} \\ \frac{1}{2}\frac{\partial S}{\partial n} = -\Sigma(E - P)P^2bte^{-nt} \end{gathered} \tag{29}$$

or

$$\begin{gathered} \frac{1}{2}\frac{\partial S}{\partial B} = -\Sigma(E - P)\frac{P}{B}, \quad \frac{1}{2}\frac{\partial S}{\partial C} = \Sigma(E - P)\frac{P^2}{B} \\ \frac{1}{2}\frac{\partial S}{\partial n} = -\Sigma(E - P)P^2\frac{t}{B}e^{-nt} \end{gathered} \tag{29'}$$

$$\frac{1}{2}\frac{\partial^2 S}{\partial a^2} = \Sigma[P^4 - 2P^3(E - P)], \quad \frac{1}{2}\frac{\partial^2 S}{\partial a\,\partial b} = \Sigma[P^4 - 2P^3(E - P)]e^{-nt}$$

$$\frac{1}{2}\frac{\partial^2 S}{\partial b^2} = \Sigma[P^4 - 2P^3(E - P)]e^{-2nt}$$

$$\frac{1}{2}\frac{\partial^2 S}{\partial a\,\partial n} = -\Sigma[P^4 - 2P^3(E - P)]e^{-nt}tb \tag{30}$$

$$\frac{1}{2}\frac{\partial^2 S}{\partial b\,\partial n} = -\Sigma\left\{[P^4 - 2P^3(E - P)]e^{-2nt} + \frac{1}{b}[P^2(E - P)e^{-nt}]tb\right\}$$

$$\frac{1}{2}\frac{\partial^2 S}{\partial n^2} = \Sigma\left\{[P^4 - 2P^3(E - P)]e^{-2nt} + \frac{1}{b}[P^2(E - P)e^{-nt}]t^2b^2\right\}$$

$$\frac{1}{2}\frac{\partial^2 S}{\partial B^2} = \frac{1}{B^2}\Sigma P^2, \quad \frac{1}{2}\frac{\partial^2 S}{\partial B\,\partial C} = -\frac{1}{B^2}\Sigma[P^3 - P^2(E - P)]$$

$$\frac{1}{2}\frac{\partial^2 S}{\partial C^2} = \frac{1}{B^2}\Sigma[P^4 - 2P^3(E - P)]$$

$$\frac{1}{2}\frac{\partial^2 S}{\partial B\,\partial n} = \frac{1}{B^2}\Sigma[P^3 - P^2(E - P)]te^{-nt} \tag{30'}$$

$$\frac{1}{2}\frac{\partial^2 S}{\partial C\,\partial n} = -\frac{1}{B^2}\Sigma[P^4 - 2P^3(E - P)]te^{-nt}$$

$$\frac{1}{2}\frac{\partial^2 S}{\partial n^2} = \frac{1}{B^2}\Sigma\{[P^4 - 2P^3(E - P)]e^{-2nt} + BP^2(E - P)e^{-nt}\}t^2$$

With these expressions one may set up the normal equations for δa, δb, δn, or δB, δC, δn. So far as the solution of those equations goes one may discard a factor b or $1/B^2$, but if one wishes to obtain the standard errors of the unknowns, the factors must be kept or adjustment must be made in the formulas for those errors. The equations are generally to be simplified by neglecting the terms in $E - P$ in the coefficients of the differentials of the unknowns, which corresponds to reducing to linear expressions before squaring and thus using equations (24) extended to three variables.

32. *Manner of Judging Convergence to the Solution.* As Schultz[32] has pointed out, in working by successive approximations one must watch what is happening. He examines the size of the terms of next higher order in the expansions. This is a very troublesome procedure

which takes more time than to repeat the ordinary least squares approximation. We have found it sufficient to watch the behavior of the variations computed for the unknowns in successive approximations, of the expressions which we are trying to make vanish, namely, the derivatives of S, and of the sum of the squares S we are trying to minimize. The sort of thing that happens may be illustrated by the case of one unknown. Let S have a minimum C for $x=0$; its expansion is

$$S = C + \tfrac{1}{2}ax^2 + \tfrac{1}{3}bx^3 + \cdots, \quad \partial S/\partial x = f(x) = ax + bx^2 + \cdots$$

If x be an approximate solution of $f(x) = 0$ the correction to x is

$$\delta x = -\frac{f(x)}{f'(x)} = -\frac{ax + bx^2 + \cdots}{a + 2bx + \cdots} = -x\left(1 - \frac{b}{a}x + \cdots\right)$$
$$= -x + \frac{bx^2}{a} + \cdots$$

and

$$x + \delta x = x_1 = bx^2/a + \cdots, \quad f(x_1) = bx^2 + \cdots$$
$$S(x_1) = C + \tfrac{1}{2}b^2x^4/a + \cdots$$

In words, if we regard our departure from the ultimate solution as an infinitesimal, the expression which we are trying to make vanish will usually be an infinitesimal of the same order, and the expression to be minimized will depart from its minimum by an infinitesimal of the second order; the correction to the departure will be nearly equal to the negative of the departure, differing from that value by an infinitesimal of the second order, the new value of the expression which is to vanish will be of the second order, and the new value of the expression to be minimized will depart from the minimum by an infinitesimal of the fourth order.

Naturally, with three unknowns and three simultaneous equations methods are algebraically and arithmetically more complicated. Still the general phenomena of convergence remain the same. What we should observe as we make successive applications of the least squares method is that the corrections δa, δb, δn, (or δB, δC, δn) tend to become more and more nearly equal to the negatives of the departures of a, b, c from their correct values, and the derivatives of S tend more and more rapidly to vanish, and S tends to become constant (the departures from the minimum being of the fourth order, they cannot well be followed unless a great many arithmetic places are carried). The real difficulty as in other cases of successive approximations is to find a set of values from which to start which are near enough right to make the first step converge to better values. In practice we have taken trial values of the constant a and plotted $\log (1/P - 1/a) = -nt + \log b$ to a large scale, read off the values of n and b, and computed S for these values. Often it has appeared better to make several such trials in an effort to reduce S before starting in on the long calculation by least squares. Even at that, we have found at times that the least squares solution gave a worse value of S than the one from which we started and it has seemed better further to explore the values of the constants by graphical methods before again proceeding to a least squares calculation.[62]

33. *Illustration of Convergence for a Malthusian.* To illustrate these matters we may return to the case of the exponential $P = Ce^{nt}$ and fit it to the five values 1, 1.7, 3, 6, 13 for $t = 0, 1, 2, 3, 4$. The formulas for the exponential are

$$A = \frac{1}{2}\frac{\partial S}{\partial C} = -\Sigma(E - P)e^{nt}, \quad B = \frac{1}{2}\frac{\partial S}{\partial n} = -\Sigma(E - P)Cte^{nt}$$

$$L = \frac{1}{2}\frac{\partial^2 S}{\partial C^2} = \Sigma e^{2nt}, \quad M = \frac{1}{2}\frac{\partial^2 S}{\partial C\,\partial n} = \Sigma[Ce^{2nt} - (E - P)e^{nt}]t$$

$$N = \frac{1}{2}\frac{\partial^2 S}{\partial n^2} = \Sigma[C^2e^{2nt} - (E - P)Ce^{nt}]t^2$$

and we shall denote by M' and N' the values of M and N in which $E - P$ is put equal to 0; when near the minimum M and M' are nearly equal since $M - M' = 0$ is one of the conditions which must vanish, but $N - N'$ does not vanish at the limit though in most cases it is small compared with N or N'. To get a start we fit $\log P$ to $nt + b$ where $b = \log C$ (without weights). We find $C = .922179$ and $n = .639103$ and $S = 1.419636$. We write the linear equations for least squares as

$$A + L\delta C + M\delta N = 0, \quad B + M\delta C + N\delta n = 0.$$

The values for the next approximation are

$$A = -11.36717, \quad B = -45.663384, \quad L = 229.89432$$

$$M = 718.4296, \quad N = 2469.5106$$

$$M' = 767.9464, \quad N' = 2661.6705$$

	C	n	S	A	B	L	M	N
1	.922179	.639103	1.419636	−11.367170	−45.663384	229.89432	718.4296	2469.5106
1′							767.9464	2661.6705
2	.7050711	.7188990	.405617	−8.911186	−23.590136	412.20661	1039.67917	2778.99722
2′							1073.13698	2873.60640
3	.8303975	.6842949	.351956	1.696641	1.711828	319.56020	974.35331	3047.85461
3′							972.29185	3053.28184
4	.6880415	.7292423	.270896	−5.067610	−12.492042	444.96221	1114.70733	2913.07975
4′							1132.86327	2963.84615
5	.7036532	.7275566	.212009	−.068085	.108425	439.44661	1143.96597	3056.58991
5′							1143.81188	3059.79450
6	.7139411	.7237956	.210991	.117927	.301050	427.39303	1128.24818	3056.66059
6′							1127.82651	3059.80521
7	.7133192	.7239269	.210958	−.000071	.000060	427.80895	1127.97172	3053.31725
8	.7133326	.7239218	.210957	.000125	.000204	427.79188	1127.94672	3053.30387
8′							1127.94643	3057.56746

	δC	δn	C	n	S	$C-C_m$	$n-n_m$	$S-S_m$
1	−.0917815	.0451919	.8303975	.6842949	.351956	.117070	−.039628	.140999 to line 3
1′	−.2171079	.0797960	.7050711	.7188990	.405617	−.008257	−.005024	.194660 to line 2
2	.0036847	.0071102	.7087558	.7260092	.211659	−.004572	.002086	.000702
2′	.0088700	.0048968	.7139411	.7237958	.210991	.000613	−.000127	.000034 to line 6
3	−.1423560	.0449474	.6880415	.7292423	.270896	−.025286	.005319	.059939 to line 4
3′	−.1158234	.0363223	.7145741	.7206172	.235439	.001246	−.003306	.024482
4	.0156117	−.0016857	.7036532	.7275566	.212009	−.009675	.003634	.001052 to line 5
4′	.0245012	−.0051502	.7125427	.7240921	.211010	−.000785	.000169	.000053
5	.0096134	−.0036334	.7132666	.7239232	.210958	−.000061	.000000	.000001
5′	.0091528	−.0034569	.7128060	.7240997	.210962	−.000522	.000177	.000005
6	−.0006219	.0001311	.7133192	.7239269	.210958	−.000009	.000004	.000001 to line 7
6′	−.0006085	.0001260	.7133326	.7239218	.210957	.000005	−.000001	.000000 to line 8
7	.0000084	−.0000031	.7133276	.7239238	.210957	.000000	.000001	.000000
8	−.0000045	.0000016	.713328	.723923	.210957	Taken as the final solution C_m n_m		
8′	−.0000043	.0000015	.713328	.723923	.210957	Taken as the final solution C_m n_m		

It is observable that the equations involving M', N' are considerably different from those involving M and N. The solutions are different:

$$\delta C = -.0917815, \quad \delta n = .0451919;$$
$$\delta C' = -.2171079 \quad \delta n' = .0797960$$

With the correct equations, S is reduced to .351956 and with the primed equations only to .405617. The new values of the constants are

$$C = .8303975, \quad a = .6842949; \quad C' = .7050711, \quad a' = .718899$$

The results of the various approximations by least squares are given in the prceeding Table. (As there are sixteen columns in the Table it has had to be broken in printing but the lines have been numbered so as to guide the reading.)

34. *Discussion of the Convergence.* In this table we have two chief lines of descent from the original estimate. One goes to line 3 thence to line 4, thence to line 5 where it stops with evidently a pretty good fit. In this line of descent we have used the equation in M and N, i. e., the equations in which $E - P$ is not neglected in the expressions for the second derivatives, the equations we have called correct or complete. In the other line of descent we go from the original estimate to line 2, thence to line 6 with a recalculation in lines 7, 8, 8′ for check. In this line of descent we have used M', N', the incomplete values. In other words, in this case we got a better solution out of two applications of least squares with the incomplete form than out of three with the complete form, at least in the sense that $C - C_m$, $n - n_m$, $S - S_m$, where the subscript m denotes the values at minimum, are all better at the end of line 2′ than at the end of line 4. But it must not be inferred that the simpler equations give invariably the better values for if we compare lines 5 and 5′ we find that the departures in 5′ are much larger than those in line 5. The method of least squares attempts to find the right values of the constants, namely, the values determined by that method, those which we accept, those from which we measure variations. Until the answer has been found to as many places as one determines to find it (in this case six) we cannot use departures from the right values as a criterion of nearness to the final solutions. We cannot use the value of S with any assurance. Thus in line 1 the value of S is considerably less than that in line 1′; yet the solution is not so good. Also if we look at A and B, which are to vanish, as tabulated in lines 2 and 3 we find that, like

S, they are less for the poorer solution. Thus criteria of excellence based (1) on the departures of the unknowns from their values at minimum and (2) on the values of the quantities we are trying to minimize or make vanish are here inconsistent.[48]

35. *The Quadratic Expansion at Minimum.* These matters may be discussed from the point of view of the quadratic in δC, δn at the minimum

$$\begin{aligned} S - S_m &= L\delta C^2 + 2M\delta C\delta n + N\delta n^2 \\ &= 427.79\delta C^2 + 2255.9\delta C\delta n + 3053.3\delta n^2 \end{aligned}$$

For a given value of $S - S_m$ small enough so that the terms of the third order are negligible, the locus in δC, δn, is an ellipse with its axes at an angle to the axes of C and n and with very dissimilar major and minor axes. The ratio of the axes is 18 or 19 to 1. The linear equations $427.79\delta C + 1127.95\delta n = 0$ and $1127.95\delta C + 3053.3\delta n = 0$ which we have to solve for $\delta C = \delta n = 0$ are inclined to one another at a very small angle and the major axis of the ellipse lies between them. Along one $\delta C/\delta n = -2.64$, along the other $\delta C/\delta n = -2.71$. If then $\delta C/\delta n$ is in the neighborhood of -2.7 the values of δC and δn for given departure of S from S_m may be very much larger, possibly 18 times larger, than if the ratio $\delta C/\delta n$ is about $+.37$. The ratio $C - C_m : n - n_m$ in line 1 of the table is -3.0 and $S - S_m$ is .14 whereas in line 1′ the ratio is $+1.6$ and $S - S_m = .19$ although the values of $C - C_m$ and $n - n_m$ are both decidedly smaller than in line 1, and in line 2 the ratio is -2.2 and $S - S_m$ is only .0007 although $C - C_m$ and $n - n_m$ are more than half their magnitude in line 1′. In the case of the three constant logistic we shall find ordinarily a much elongated ellipsoid so that one set of values near to the correct ones may have a much larger value of $S - S_m$ than another set which is much further away.[49]

36. *England and Wales, 1801–1911.* After this discussion of method we may take some fits beginning with the case of England and Wales.[50] Yule has given three arithmetic solutions and by feeling around with the graphical method we obtained one which had a smaller sum of squares. The constants and the quantities which have to vanish are given for these four solutions in the table and below them the constants arising in the first least squares solution from the graphic and the constants for a second application of least squares.[51]

	a	b	n	S
Yule 1)...........	.010964	.10151	.16358	.44
Yule 2)...........	.010277	.10169	.16006	.35
Yule 3)...........	.0099946	.10137	.15764	.39
Graphic)..........	.010000	.10186	.15888	.327
Least Sq. 1).......	.0093274	.101603	.155317	.30792
Least Sq. 2).......	.00931790	.1016265	.1553480	.30739

	$\Sigma P^2(E-P)$	$\Sigma P^2(E-P)e^{-nt}$	$\Sigma P^2(E-P)te^{-nt}$
Yule 1)...........	-417	-109	-941
Yule 2)...........	-100	-26	-272
Yule 3)...........	572	124	1153
Graphic)..........	-55	-6	-162
Least Sq 1)........	$+48$	$+11$	$+103$
Least Sq 2)........	$-.03$	$-.008$	$-.074$

The corrections to a, b, n of the graphic solution found in the first least squares were $\delta a = -.0006726$, $\delta b = -.000257$, $\delta n = -.003563$ whereas for the second least squares solution they were $\delta a = -.00000950$, $\delta b = +.0000310$, $\delta n = +.0000235$. There seems to be no doubt but the solution found for a, b, n would be improved exceedingly little by another application of least squares. Assuming that solution can be treated as a true least squares solution for the evaluation of the standard errors of the constants we have[52]

$$\sigma_a = .001065, \quad \sigma_b = .000894, \quad \sigma_n = .00488$$

Yule's best solution judged by the value of S or of its derivatives is his second. Our graphic is better by these criteria. His third solution, however, has values of a and n nearer to the correct ones than either his second or our graphic solution. It may be noted that the limiting populations forecast for England and Wales by the least squares solution is $1/a = 107.3$ million.

37. *Germany, 1816–1910.* It will not do to think that every case will be as plain sailing as England and Wales. Let us consider Germany.[53] The censuses are not equally spaced in time and one cannot apply any of the three methods of Yule without interpolating for some of the intercensal populations. We may however begin by trying the graphic method. After considerable experimenting we found that $a = -.03$, $b = .068754$, $n = .043749$, populations being in millions and time in decades, gave $S = 6.3162$. A least squares pro-

cedure (simpler equations), led to a very large increase in S, which was quite unsatisfactory. Further exploration showed that $a = -.05$, $b = .088308$, $n = .03177567$ gave $S = 5.8279$. By the least squares procedure the sum of the squares again increased, becoming 6.1518. This was not a large increase and as there is always the possibility that the values of the constants actually improve despite an increase in S, the least squares procedure was repeatedly applied to the result. We found successively

a	b	n	S	$\Sigma P^2(E-P)$	$\Sigma P^2(E-P)e^{-nt}$	$\Sigma P^2(E-P)e^{-nt}tb$
−.05	.0883	.0318	5.83	−885	−656	−558
−.0554	.0936	.0293	6.15	4004	3155	2386
−.05537	.09356	.02951	5.78	−12.6	−10	−7.1
−.05565	.09384	.02941	5.78	6.7	5.3	4.1
−.05566	.09385	.02940	5.78	− 1.5	−1.2	− .9

This is about as far as one can go carrying seven places; using 10 places one comes to the result

$$a = -.055666575, \quad b = .093856745, \quad n = .029400635, \quad S = 5.7799491,$$

with the three semi-derivatives of S equal to something numerically less than .001. There is no doubt about the convergence and no doubt that the logistic fit to Germany from 1816 to 1910 was hypermalthusian, showing less than no saturation.

38. *Germany, 1861–1910.* The standard errors were however considerable, being about .046 for a and b and .017 for n. This raises sharply the question as to whether the fit is worth anything insofar as throwing light on the "law" according to which the population of Germany was growing. Fitting the period from 1861 to 1910 to a logistic by graphical methods we found[54]

$$P = \frac{1}{.08 - .0537 \times 10^{.01641t}}$$

whereas

$$P = 33.587 + \frac{82.944}{1 + 297.546e^{-.0472t}}$$

is the fit Pearl gives for this period using the logistic augmented by an additive constant. The following table gives the comparative results:

Year	E	Pearl	$E-P$	Ours	$E-P$
1861	38.140	38.281	− .141	38.02	.12
1871	41.059	40.867	.192	41.27	− .21
1880	45.230	44.230	1.000	44.84	.39
1890	49.430	49.430	.000	49.80	− .37
1900	56.367	56.370	− .003	56.27	.10
1910	64.930	64.930	.000	65.02	− .09
Sum of the differences.			1.048		− .06
Sum of absolute diffs.			1.336		1.28
Sum of squares of diffs.			1.057		.366
1855	36.114	37.173	−1.059	36.38	− .27

The fit offered by Pearl does not seem to be anything other than arbitrary. The fact that a four constant generalization of the logistic fits so much worse than the logistic is sufficient to disqualify the result, and incidentally it does not give so good a figure when extrapolated back to 1855. Of course our fit cannot be extrapolated forward very far because of its hypercritical character which carries it to infinity in 106 years from 1861.

39. *Connecticut, New Jersey, New York and the Three States, 1790–1920.* We have pointed out (Art. 17) that Connecticut and New Jersey were hypermalthusian in their growth from 1790 to 1920, Connecticut so much so as to be almost hypercritical. The values we find by least squares for the constants of each of these states and of New York, and of the three states together are[55] (with the addition of the sum of squares S and of the standard deviation σ of the residuals).

	a	b	n	S	σ
Connecticut.	*−1.638*337	*6.637*899	*.07972*561	*.006720*	.022
New Jersey.	− *.02569*654	*8.173*653	*.2448*360	*.02342*1	.041
New York	*.03941*499	*1.361*848	*.2437*437	*.798*846	.239
The Three States	*.01293*182	*.9538*168	*.2209*441	*.639*388	.214

This illustration shows that the logistics when fitted to parts of a region and to the total region may be quite non-additive, when extrapolated as forecasts. Connecticut and New Jersey being hypermalthusian will become infinite, New York remains finite, and the three states remain finite. The asymptotic population forecast for New York is 25.4 million, that for the three states is 77.3 million. If we subtract these we have 51.9 million to divide up between Connecticut and New Jersey which, being more than a quarter of the asymptotic population forecast for the whole United States by the logistic, should be rather too much to be consistent with that forecast, but is unhappily not enough to satisfy the logistic forecast for Connecticut or New Jersey individually!

40. *Reed and Pearl on Connecticut.* The table gives the data E for Connecticut (in thousands), the figures P obtained by Reed and Pearl[34] to fit this data, and the deviations; then, as P', the populations calculated from the first graphic straight line obtained by plotting E directly on semi-log paper and reading off the constants to get $P' = 10^{2.09+.080t}$, and the deviations $E - P'$; the next two columns contain the sums of the squares of the deviations cumulated from the end; and the last column the values of $E - P$ for our least squares logistic of Art. 39.

Year	E	P	$E-P$	P'	$E-P'$	$(E-P)^2$	$(E-P')^2$	$E-P''$
1790	238	116	122	123	115	47810	39590	38
1800	251	142	109	148	103	32926	26365	28
1810	262	172	90	178	84	21045	15756	13
1820	275	210	65	214	61	12945	8700	−4
1830	298	255	43	257	41	8720	4979	−16
1840	310	309	1	309	1	6871	3298	−45
1850	371	374	− 3	372	− 1	6870	3297	−33
1860	460	453	7	447	13	6861	3296	− 3
1870	537	547	−10	537	0	6812	3127	2
1880	623	659	−36	646	−23	6712	3127	− 2
1890	746	792	−46	776	−30	5416	2598	7
1900	908	948	−40	933	−25	3300	1698	18
1910	1115	1131	−16	1122	− 7	1700	1073	18
1920	1381	1343	+38	1349	+32	1444	1024	−15

It is seen that the two-constant Malthusian fits better than their logistic (three-constants) and that judged by the sum of squares it fits decidedly better over any period terminating in 1920; indeed as the differences $E - P'$ are practically all less than $E - P$ it would be difficult to find any considerable period for which this Malthusian fit failed to be better than their logistic.[56]

41. *Reed and Pearl on Additivity.* Wilson and Luyten illustrated the non-additivity of the logistic by citing the simple example

$$E_1 = 1, \quad E_2 = 3, \quad E_3 = 8; \quad E_1' = 1, \quad E_2' = 4, \quad E_3' = 9$$

$$E_1 + E_1' = 2, \quad E_2 + E_2' = 7, \quad E_3 + E_3' = 17$$

$$K = 33, \quad K' = 12.57, \quad K + K' = 45.57$$

but with K'', the asymptotic population for the summed data, equal to 30.33 which is not only less than the sum $K + K'$ but actually less than K. In the effort to show the practical additivity of the logistic, Reed and Pearl developed formulas for the addition of logistics based on fitting logistics through fitting their derivatives (not the logistics themselves) by the method of moments throughout the whole range of time $-\infty$ to $+\infty$. This method forces the sum of the asymptotes to equal the asymptote of the sum. It is not a method which would seem directly applicable to the data, and, even if it were, it is not a method which can meet the point raised by Wilson and Luyten, and mentioned several times in the present paper (Arts. 11, 38), that if logistics are fitted to the data for two or more regions and for the combined regions, the forecasts will not in fact be additive. Wilson and Luyten did not say and we would not say that given two or more regions, constants for their logistics and for that of the combined region could not be so selected as to make a tolerably good fit to the regions and their combination both over the time covered by the enumerations and even over the forecast[57] to $t = \infty$. We do not consider their fit to Connecticut as tolerable; it certainly is not a least squares fit, the sum of the squares is 47890 instead of 6720 and the constant a has the wrong sign so that the equation is of the wrong type. We do not consider that anything from p. 737 to and including the first paragraph of page 742 of the paper by Reed and Pearl[34] is tenable, either in theory or in practice, but that, naturally, is for others to judge.

42. *The Three States by Relative Residuals 1790–1920.* We fitted a logistic by relative residuals based on minimizing the sum of $[(E - P)/P]^2$ starting from the values of the constants given by Wilson and Luyten. We will give three sets for the constants (1) those from which we started, (2) those obtained by least squares and (3) those obtained by applying Pearl's method previously discussed (Art. 20).

	a	b	n	S
Wilson and Luyten[58]	.045	1.264	.302	.0656
Least Squares	.040163	1.157186	.2781681	.0412
Pearl's Method	.041861	1.180323	.282974	.0435

The asymptotic population forecast by least squares with relative residuals is 24.9 million as contrasted with the figure of 77.3 forecast by the method using absolute residuals. The values of the constants found by a repeated application of Pearl's computation form are no nearer to those given by the least squares fit with relative residuals than might be expected considering the sensitivity of the values to even moderately different weights of the residuals in the equations which should vanish, but they are so far from the values obtained in the solution by absolute residuals as to be an illustration and confirmation of our analysis of his method as giving a result near to the relative rather than to the absolute fit.

43. *New York City and Environs 1790–1920.* We have least squares solutions for a logistic to New York City and Environs, as defined by Pearl and Reed,[20] by absolute and by relative residuals and a fit not carried quite to a finish for Area III (1850–1920) by absolute residuals. The values of the constants are:

	a	b	n	S
New York City and Environs (Absolute)	.03664116	6.207466	.3414512	.2326
New York City and Environs (Relative)	.00266301	4.946915	.2989057	.1146*
Area III, 1850–1920 (Absolute)	−.41059	4.1451**	.14739	.000552

* $S = \Sigma(E - P)^2/P^2$ in this case as the relative residuals are minimized.

** The origin of time is here 1850 instead of 1790 as it is for the others.

The asymptotic population of Greater New York is 27.3 by absolute and 375.5 by relative residuals. Thus the relative residual solution makes the city ultimately larger than the three states (Art. 42) in which it lies (24.9) and larger than the whole United States.[59] The absolute residual method makes the suburban area III hypermalthusian though the total area is hypomalthusian by this method. We see here other illustrations of the non-additivity of the logistic and of the really serious clashes in forecasts which arise therefrom.

We see also a good illustration of the instability of the method. If we use relative residuals we get for greater New York City the very large value 375 but for the three states a figure of 25 million which is not inconceivable,—indeed they had 15 million in 1920 out of a total of 106 for the United States and if the United States is ultimately to have about 197, the three states would in proportion have about 25 million. On the other hand absolute residuals give for Greater New York 27 million which appears rather high and for the three states 77, which is very high.

44. *A Method Which Might Replace Least Squares.* The method of least squares is obviously very tedious to apply. The method of using three points or Yule's summation of reciprocals or his other schemes are somewhat indefinite in that different groups may be applied, and experience with such methods has given us the impression that they do not give solutions very near to the least squares solution. We have pointed out that this solution can be expressed by the vanishing of the sum of the residuals weighted in various ways, and so can Pearl's solution and the method of moments. It seems as though it might be well to set up a standard method of fitting the logistic based directly on such equations using all the data. Now

$$\frac{1}{P} = a + be^{-nt}, \quad \frac{1}{P} - \frac{1}{E} = a + be^{-nt} - \frac{1}{E}$$

and

$$\Sigma w\left(\frac{1}{P} - \frac{1}{E}\right) = \Sigma w\left(a + be^{-nt} - \frac{1}{E}\right) = 0$$

with any system of weights w which can be calculated directly from the data will give an equation linear in a and b and algebraic in $r = e^{-nt}$, where t is a common divisor of the periods between the censuses. Three such equations with different systems of weights w, w', w'' will enable one to eliminate a and b and have an algebraic equation for r. Now the weights are for

Absolute Residuals	P^2E,	P^3E,	$P^2E(K - P)t$
Relative Residuals on P	E^2/P,	E^2	$E^2(K - P)t/P$
Relative Residuals on E	P^2/E	P^3/E	$P^2(K - P)t/E$
Pearl's Method	E	E^2	$E^2(K - P)t/P$
Moments	PE,	PEt,	PEt^2

In none of these systems is w calculable from the observations but it would become so calculable for the method of moments if we replace

P by E, choosing as weights E^2, E^2t, E^2t^2; similarly if we place $P = E$ in the other weights, except for those in the final column where the asymptotic population enters. In those cases in which P remains much smaller than K, we might replace $K - P$ by K in the last set of weights and reject the K from the equation in which it enters, but this is precisely the case in which the asymptote is so far above the observed values as to make it futile to determine K. It would seem that one might adopt a standard set of weights corresponding to no least squares solution, say E^2, E^3, E^4 and make

$$\Sigma(E - P) = 0, \quad \Sigma(E - P)E = 0, \quad \Sigma(E - P)E^2 = 0 \tag{31}$$

the last two equations being identical with the two first arising in fitting by absolute residuals insofar as P may be put equal to E in the weights, and the first expressing the fact that the sum of the residuals shall vanish. This should give a solution fairly near to that for absolute residuals by least squares. We should correspondingly take as weights 1, E, E^2 for the case of relative residuals.[60]

45. *Application of this Method.* We have obtained some solutions for r (and hence n) and for a and b based on the equations

$$\Sigma(a + be^{-nt} - 1/E)E^2 = 0, \quad \Sigma(a + be^{-nt} - 1/E)E^3 = 0$$

$$\Sigma(a + be^{-nt} - 1/E)\,E^4 = 0$$

leading to

$$\begin{vmatrix} \Sigma E^2 & \Sigma E^2e^{-nt} & \Sigma E \\ \Sigma E^3 & \Sigma E^3e^{-nt} & \Sigma E^2 \\ \Sigma E^4 & \Sigma E^4e^{-nt} & \Sigma E^3 \end{vmatrix} = 0 \tag{32}$$

It is interesting to compare them with the least squares solutions

	a	b	n	S
Greater New York	.031380	5.7282	.33056	.2368
Least Squares	.036641	6.2075	.34145	.2326
Connecticut	−1.654	6.603	.07882	.00676
Least Squares	−1.638	6.638	.07973	.00672
The Three States	.017597	1.0002	.23125	.6651
Least Squares	.012932	.9538	.22094	.6394
Chicago[61] (1840–1920)	.18477	9.5231	.59559	.01331
(12 Counties)	.18142	9.2043	.58671	.01305
Chicago (1860–1920)	.16962	2.4576	.55087	.009435
(15 Counties)	.17049	2.4749	.55334	.009414

The results run a good deal better than those obtained by the summation method of Yule or by the use of three points or by the graphical-exploration method,—as they must do if the extra work of applying this method is to be justified. Yet in the case of Chicago (12 counties) near as the solution seems to be to the final least squares solution, it was far enough off so that the application of the least squares procedure led to worse values for S and for the constants[62]—so great is the sensitivity.

III. The Augmented Logistic.

46. *The Cycle Theory.* Pearl and Reed very early abandoned the logistic in favor of the formula[63]

$$P = d + \frac{1}{a + be^{-nt}} = d + \frac{B}{C + e^{-nt}}, \tag{33}$$

or equivalent, wherein the logistic is augmented by an additive constant. Whenever one of the constants is additive one condition for the least squares fit is that $\Sigma(E - P)$, the sum of the residuals, shall be zero,—provided that by a least squares fit one means the fit by absolute residuals. Although Edgeworth[9] is very doubtful of the utility of the added constant and Yule does not use it and apparently does not favor it,[9] we consider it worth while to discuss the matter somewhat. Although Pearl and Reed talk about a cycle theory in connection with the added constant, they do not make it clear that this is anything more than a method of interpreting the equation. It should be pointed out that the use of this constant introduces a discontinuity into the law of population growth. The differential equation of the first order is

$$\frac{dP}{dt} = \frac{d}{dt}(P - d) = n(P - d)[1 - a(P - d)]; \tag{34}$$

the excess population over the lower asymptote d grows on a logistic and the law must change every time that d changes. It is not clear why it should be the excess population which grows in this way, i. e., why the effect of the conditions in the earlier "cycle" should persist for generations in the new in the form specified.[64] The constant n is no longer the natural rate of growth of a sparse population but the rate of growth of one scarcely above its lower asymptote.[65] In the case of Germany where Pearl claims two cycles it is noteworthy that the lower asymptote of the second cycle is well below the upper asymptote of the first so that the change takes place without the realization of either asymptote—the population jumps from one cycle considerably before it is finished into another already well begun; such a change is entirely analogous to the simpler change suggested by Knibbs.[2b]

47. *The Conditions for a Minimum; the Imaginary Case.* It would seem unnecessary to give the long equations which arise from applying the methods of the calculus to the minimization of

$$S = \Sigma(E - P)^2 = \Sigma\left(E - d - \frac{B}{C + e^{-nt}}\right)^2 = \Sigma\left(E - d - \frac{1}{a + be^{-nt}}\right)^2$$

or any of the equivalent forms.[66] The conditions which must be satisfied at the minimum are

$$\begin{gathered}\Sigma(E-P) = 0, \quad \Sigma(E - P)(P - d) = 0, \quad \Sigma(E - P)(P - d)^2 = 0 \\ \Sigma(E - P)(P - d)t(1/a - P + d) = 0\end{gathered} \tag{35}$$

It has been pointed out that in the simple case of determining the four constants to pass the curve through four enumerations equally spaced in time, the equations for the constants are quadratic and their solution may lead to imaginary numbers.[61] A solution involving imaginaries we take to be useless, and in any case it could not be found by successive approximations based on the usual formulas involving real quantities only. The types of augmented logistic include the hypomalthusian, Malthusian, hypermalthusian, critical and hypercritical (optionally) previously discussed, each with an added constant, and further the "imaginary" case. As in the form involving B and C, two of the constants, d and B, enter linearly they may be eliminated by the usual procedure and the problem of minimization may be reduced to one involving two variables only, namely,[67]

$$S = \Sigma(E - E_m)^2 - \frac{k\left(\Sigma \dfrac{E - E_m}{C + e^{-nt}}\right)^2}{k\Sigma\left(\dfrac{1}{C + e^{-nt}}\right)^2 - \left(\Sigma \dfrac{1}{C + e^{-nt}}\right)^2} \tag{36}$$

where k is the number of observations and E_m is the mean value of E.

48. *Convergence in the Imaginary Case.* An interesting mathematical question is: What if anything does one get by applying the ordinary least squares procedure to the case in which the constants are imaginary? We may investigate this for the simple case of four observations supposed to be

$$E_0 = 2, \quad E_1 = 3, \quad E_2 = 4, \quad E_3 = 7, \quad \text{at } t = 0, 1, 2, 3$$

Here the imaginary solution is

$$e^{-n} = r = \tfrac{1}{3} \pm \tfrac{1}{3}\sqrt{-8}, \quad d = 3 \mp \tfrac{1}{2}\sqrt{-8}, \quad C = \tfrac{1}{3} \pm \tfrac{1}{3}\sqrt{-8}$$
$$B = -\tfrac{8}{3} \pm \tfrac{1}{3}\sqrt{-8}$$

We assume a value of r, substitute it and various trial values of C into S in the form (36)to find for C that value which minimizes S. The values of B and d, the eliminated constants, may then be had from[67]

$$B = \frac{k\Sigma A(E - E_m)}{k\Sigma A^2 - (\Sigma A)^2}, \quad d = E_m - \frac{\Sigma A(E - E_m) \cdot \Sigma A}{k\Sigma A^2 - (\Sigma A)^2} \qquad (37)$$
$$\text{if} \quad A = \frac{1}{C + e^{-nt}}$$

In this way we obtain the following table (converted to our usual variables a, b).

r	a	b	d	S
1/3	.0689	3.45	1.90	.124
1/2	− .0250	1.665	1.534	.081
.7	− .181	1.007	.896	.059
.9	− .777	1.263	.039	.0525
.99	− 8.427	8.824	−.4198	.05202
.999	−84.8868	85.2765	−.46668	.052012
.999000	−84.8872	85.2768	−.46697	.052012
1.001001	+84.8868	−84.4989	−.47846	.052012*

* The next to the last line was obtained from the preceding by least squares. In the last line we are over on the hypercritical side using the reciprocal of $r = .999$ which results in a curve identical with that for $r = .999$ but with a different equation.[65]

49. *The Augmented Critical and the Imaginary Case.* What is happening is now apparent. A least squares procedure will take us very slowly (only with infinite repetition) to the critical solution $r = 1$ with $a = \mp \infty$ and $b = \pm \infty$ and the sum of $a + b$ finite and equal to about 0.4, with d equal to about .470 and S equal to 0.052012 and hence with the standard deviation of about 0.115. Although we have four disposable constants and only four points the curve will not pass through them and the minimum of S is not zero (See footnote

47). Indeed the least squares procedure is a very awkward one with which to evaluate an indeterminate form. We should fit at once the augmented critical form

$$P = d + \frac{1}{\alpha - \beta t} = d + \frac{\gamma}{\varepsilon - t} \tag{38}$$

and for this we can proceed with (36) and (37) using $A = 1/(\varepsilon - t)$ and replacing B by γ, thus arriving at a minimizing problem in the single variable ε or we may set up the method of least squares for (38). In either case the problem is readily solved, an approximate result being

$$P = -.4703 + \frac{11.76}{4.577 - t} \tag{39}$$

to which we could apply the standard procedure with rapid instead of infinitely slow convergence. We thus may obtain a least squares solution but is it of any mathematical significance? Theoretically our problem is to solve the problem of determining d, a, b, n and mathematically they can be perfectly determined, but imaginary. The procedure gives not those imaginary values but real values with $d = -.4703$, $n = 0$, and a and b infinite with opposite signs. The result is certainly no approximation to the correct imaginary solution though the graphic (39) has been found (with very different values of the constants) which fits tolerably well!

We have used the term "imaginary" case when the constants turn out imaginary because the solution cannot then be reached by the ordinary least squares procedures based on reals; but the augmented logistic curve defined by the imaginary constants is perfectly real, namely,

$$P = 3 + \sqrt{2} \tan [\tfrac{1}{2}(t - 1) \tan^{-1} \sqrt{8}]$$

And in general since $r = e^{-nt}$ must be a root of an equation of the form $r^2 + ar + 1 = 0$, the two values of r must be of the form $e^{\pm i\varphi}$ so that n is pure imaginary and the augmented logistic takes the general form

$$\alpha + \beta \tan \tfrac{1}{2}n(t - \gamma) \tag{38$'$}$$

which may be regarded as a sort of hypercritical case since the critical case (38) is in a way intermediate between (38′) and the ordinary form (33). This hypercritical form has neither upper nor lower asymptote, the population being positively or negatively infinite in finite time, namely, when $t = \gamma \pm \pi/n$.

50. *Germany, 1816–1910.* Interesting as such results are, they are annoying when the conditions represented by them arise in practice. If we undertake to fit to the enumerations of Germany a single augmented logistic, obtaining trial values of the constants by picking out four points, we obtain imaginary solutions with such large pure imaginary parts that it seems impossible that they arose from fluctuations in the stated enumerations. The graphic approximations obtained in exploring to reduce S indicated very small values of n (less than .01) and the application of the least squares procedure moved the constant toward zero very slowly, reducing S very little. It was obvious that we had truly an imaginary case and should fit the limiting form (38). The least squares result found was:

$$P = -\ 7.6785 + \frac{599.36}{17.717 - t}, \quad S = 5.7045$$

This is not a biologically satisfactory solution; the lower asymptote is negative, and the population becomes infinite in finite time but it is undoubtedly the mathematical solution of the problem of fitting the German censuses 1816–1910 to an augmented logistic by least squares procedures. We may regard the answer as a limiting form of the augmented logistic and it is possible, though actually of infinitesimal probability, that such a form should arise in fitting a set of points which lay near an augmented logistic with real constants (a and b large and of opposite sign, n small but not zero, d arbitrary), but the curve seems sure to arise when the augmented logistic fitted to four points is imaginary and it is extremely doubtful that it should then be regarded as an appropriate solution.[68] It may be noted in passing that the value of $S = 5.70$ based on a four-constant fit is in a sense not so good as the value $S = 5.78$, which we had (Art. 37), based on a three-constant fit because the "errors to be feared" in a determination whose weight is unity[25] have to be compared on the basis of S divided by the number of observations diminished by the number of constants. We have 5.70 to divide by $11 - 4 = 7$ whereas 5.78 need be divided only by $11 - 3 = 8$.

Although a least squares procedure operating on the ordinary logistic form would lead (infinitely slowly) to the critical form discussed above, the application of a least squares procedure based upon the form (38′) leads directly to a solution for Germany, 1816–1910, near

$$P = 32.285 + 18.497 \tan\,(0.86961\,t + 158.837), \quad S = 1.26$$

with a value of S much smaller than that (5.70) found for the critical case. This was to be expected in view of the large pure imaginary parts of the values of r found from four equally spaced points. Although the good fit shows the high degree of fittability of the augmented logistic, the result is not satisfactory biologically as the population would have been zero in 1771 and would become infinite in 1944. As a mere matter of curve fitting, however, it may be pointed out that the fit to the single four-constant augmented (imaginary or hypercritical) logistic giving $S = 1.26$ is closer than that given by Pearl's two logistics[69] involving eight constants with $S = 1.44$.

51. *Germany, 1816–1855 and 1861–1910.* Now Pearl[31, 63] broke up the series of eleven enumerations for Germany into a first series of five to which he fitted one four-constant augmented logistic and a subsequent series of six to which he fitted another. This is using a great many constants (eight) to fit eleven points besides leaving a certain amount of additional freedom in the choice of the place where one would place the break between the two series. The graph of the German population on semi-log paper or the "divided differences"[25] between the observations show decided irregularities in the rate of growth. The arcs for 1840–1880 and 1880–1910 are concave up indicating that the logistic fit to these periods individually would show no saturation. The augmented logistic fitted to the four censuses 1880–1910 gives $d = 41.43$, $a = .02487$, $b = .2383$, $n = .8670$. The lower asymptote is impossibly high for extrapolation back even to the preceding census nine years earlier, the upper asymptote is 81.64, the value of n is extremely high but the demographic interpretation of n offers serious difficulties in the case of the augmented logistic. To fit augmented logistics to other series of four points would require interpolation of the enumerated populations and the irregularities of the differences would make the values obtained for the populations depend somewhat on the method used to estimate them. But if we take 37.81 for 1860 and 40.68 for 1870 and use 1880 and 1890 we find $d = 35.88$, $a = .05770$, $b = .4611$, $n = 1.1181$; if now we try to take this forward we find an asymptote of 53.21, a population which is exceeded by 1900, the very next census after the last we used. These two sets of values figured from two sets of four points show the instability of the solution. If we fit to the middle four censuses the results are imaginary! The stability of the series of six points for fitting by the augmented logistic is small. How Pearl obtained his "fit" to the period 1861–1910 we do not know; we have shown (Art. 38) that with the simple logistic

we could make the sum of squares less than his and certainly with the four-constant augmented logistic one could make it still less.[69]

52. *England and Wales, 1801–1911.* The augmented logistic fitted by absolute residuals to England and Wales gives $d = -4.682588$, $a = .002579344$, $b = .07124426$, $n = .10722407$ with $S = .23982$. Note that the lower asymptote comes out negative which is a corroboration of Yule's judgment[9] in not trying the augmented logistic and in contrast to Pearl's[31] positive value of $d = 2.373$. The value of the limiting population is 383.0 million instead of Pearl's value of 73.0. The value of S for Pearl's solution is .5998, well over two and a half times ours, and not so good as our three-constant fit nor any of Yule's three solutions. The standard errors are[70]

$$\sigma_d = 4.7, \quad \sigma_a = .0065, \quad \sigma_b = .019, \quad \sigma_n = .039$$

These show that the values of d and a, statistically speaking, are without significance. Now the population curve for England and Wales plotted on semi-log paper shows a distinct curvature which may be due to the saturation found in the logistic or to the presence of a value of d or to both. We have given the logistic fit (Art. 36) and we now give the fit to an augmented Malthusian (graphical, not least squares)

$$P = d + Ce^{nt}, \quad d = -6.520, \quad C = 15.35, \quad n = .093035$$

which makes $S = .2435$. This value of S is far nearer to the minimum for the four-constant augmented logistic than is Pearl's, and is lower than our three-constant least squares logistic. We have also fitted graphically an augmented critical case ($n = 0$) of the type (38) with

$$d = 33.507, \quad \gamma = 1178.4, \quad \varepsilon = 27.89, \quad S = .2997$$

The value of S is lower than for Pearl's four-constant fit or any of Yule's three solutions, or our logistic.

Now if we are to fit any type of curve by any assigned method to any set of points, we solve our problem of fitting correctly only when we do in fact fit that type of curve by that method to those points. Our solution may be without physical significance, it may be without statistical significance, but it has mathematical validity. The problem of finding a "satisfactory" solution, namely, one which suits us, all things considered, is a different problem, one indeterminate mathematically, one determinable only personally. The results we have obtained for England and Wales will bear study for the light they may throw on the various laws of population growth under discussion. We can obtain most excellent fits judged by the smallness of S, the sum of the squares of the absolute residuals, either by a simple logistic ($S = .3079$), or by an augmented Malthusian ($S \leqq .2435$), or by the augmented critical (or limiting) type of the augmented logistic ($S \leqq .2997$), or by the standard augmented logistic ($S = .2398$), and there is Pearl's solution ($S = .5998$). Taking these five we have the following comparative table of the values of the constants:

S	d	a	b	n	$d+1/a$
.2398	-4.68	.0026	.071	.107	383
.2435	-6.52	0	.065	.093	∞
.2997	-33.51	$\pm\infty$*	$\mp\infty$*	0*	∞*
.3097	0	.0093	.102	.155	107
.5998	2.37	.0141	.135	.195	73

* $bn = 1/\gamma = .00085$, $a + b = \epsilon/\gamma = .024$; population infinite in finite time.

With these facts he would be a rash prophet who would assign a limiting value to the population on the basis of using the augmented logistic. Arithmetically the trouble is that in the usual formulas for the standard errors, the denominator which is the determinant of the normal equations nearly vanishes whereas the first minors which occur in the numerator do not so nearly vanish. The standard errors are large. This is itself a warning, but it would give no adequate intimation, we think, of the very great range of which the values of the unknowns are capable without largely increasing S and thus no adequate picture of the real instability of the problem of determining those unknowns or the unreliability of any particular determination of them. If the problem is to select from these five solutions that which is the best forecast we should probably select Pearl's because it gives the lowest asymptote,[71] but we should arrive at a forecast in a totally different way which would lead to a considerably smaller value of the asymptote.[24]

53. *Development About the Minimum.* As we have spoken of the instability of the whole problem of fitting logistics, particularly augmented logistics, and as the fact that the unknowns enter non-linearly makes the locus $S =$ const. non-quadratic, it is worth while to illustrate these matters by an example. Consider the locus $S = .6$ in the case of England and Wales for which the minimum value of

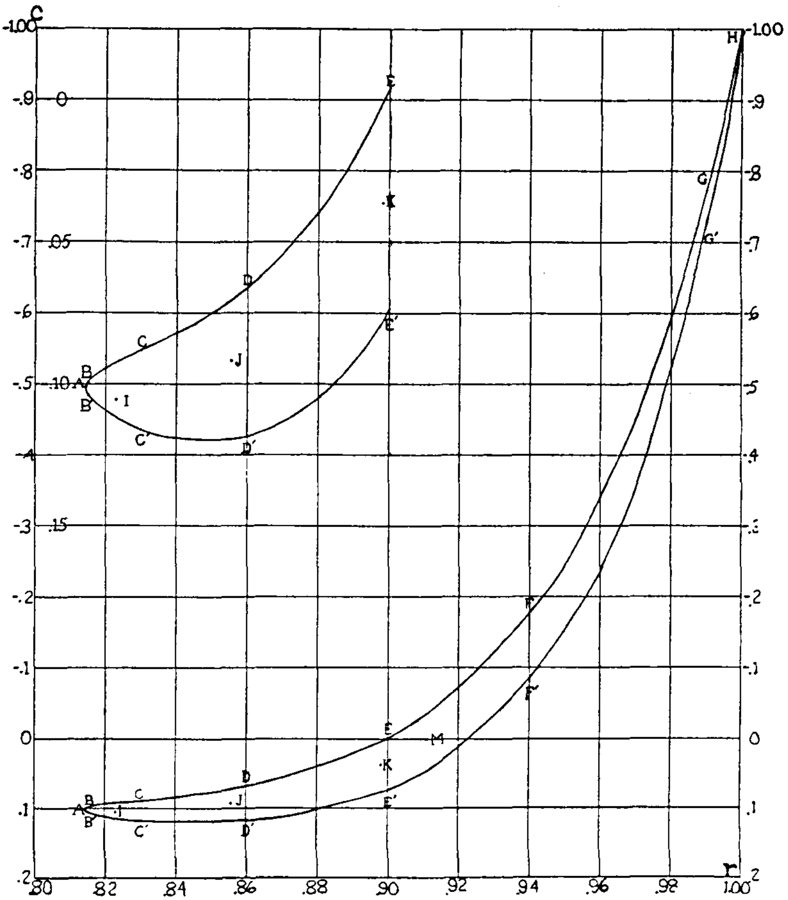


Fig. 2. Plot of the limits of C and r for $S = .6$. The scale of C on both margins is from 0.2 to -1.0 and the scale for r is from 0.8 to 1.0. That part of the figure for which $C > 0$ is redrawn in the upper lefthand corner with the scale for C enlarged four times as indicated by the figures .10, .05, 0. For any values of C and r within the curve $H\ E\ A\ E'\ H$ the value of S will be less than 0.6 if B and d are properly chosen (37). The values for the lower and upper asymptotes for the respective points on the contour are:

	H	*G*	*F*	*E*	*D*	*C*	*B*
Lower	−27	−23	−9.4	−3.3	.60	2.5	3.1
Upper	−27	−33	−95	−4000	140	85	71

S is .2398. There will be a surface or spread of three dimensions in the four-dimensional space of d, a, b, n or d, B, C, r or other four unknowns which we choose for representing the analytical form of the augmented logistic. The minimum point will lie inside this locus; so will all the points listed in the table next above. In the space of d, a, b, n the locus $S = .6$ will go to infinity for a and for b. As we cannot well represent the three dimensional locus in four-dimensional space, we may make a plot on the plane of r and C of the projection of the spread $S = .6$ upon that plane and we may give the values of d and $K = d + 1/a$ around the plot (see legend to fig. 2). One way to make the plot is to use (36) to find the values of C which for a given value of r make $S = .6$ and compute the values of d and B from (37) and the value of K from $d + B/C$. The figure shows the results. The value of r may vary between .8145 to 1.0, C may vary from 0.12 to $-$ 1.0 (which represents augmented critical cases); K may vary from 69.2 to infinity (for $C = 0$) and run down from $-\infty$ to small negative values ($-$ 41.8 or $-$ 27.2). The values of d range from about + 3.0 to about $-$ 42 or $-$ 27 on the two sides of the locus. The locus itself is nothing like quadratic, but is rather crescent shaped. Of course in the immediate vicinity of the minimum such as $S = .2400$ the locus would be essentially quadratic. The standard error of d is 4.7 so that $d = -4.7 \pm 4.7$ but the actual range of d for $S = .6$ is from + 3 to $-$ 42 which is far from symmetric on $-$ 4.7 as a center; and as K runs from 69.2 to $+\infty$ and from $-\infty$ down to $-$ 27 the value $K = 383$ plus or minus whatever the standard deviation of K might be would give no picture of the range of K when $S = .6$. Owing to the double way in which the same augmented logistic may be represented we could make a continuation of the plot for $r > 1$, but with no advantage.

54. *Connecticut and the Three States 1790–1920.* The values we find for the constants in fitting the augmented logistic by least squares are:

	A	B'	C'	D'	E'	F'	G'	H
Lower	3.0	2.8	1.4	−1.6	−6.8	−14.5	−35	−42
Upper	70	69	74	96	220	−260	−51	−42

The part of the horizontal line $C = 0$ intercepted by the curve is the locus $K = \infty$ corresponding to augmented Malthusian solutions. The part of the curve above this line corresponds to hypermalthusian solutions. The points I, J, K, M within the curve correspond respectively to Pearl's solution, the least squares logistic, the least squares augmented logistic, and the graphical augmented Malthusian.

	d	a	b	n	S
Connecticut	.1983387	.1374734	29.48463	.2865332	.001793
Standard Error	.016	.110	8.12	.0310	
Three States	−2.202367	−.04003093	.3821285	.1047145	.363807
Standard Error	1.82	.0561	.140	.0660	
(Relative)	−1.373437	−.0113281	.4832953	.1456240	.008721*

* $S = \Sigma(E - P)^2/E^2$ as the relative residuals are minimized to the base E, not P.

These solutions have not been carried through to completion due to the large amount of work necessary to get a graphic solution near enough to the minimum to obtain satisfactory results by the application of least squares procedure. It is noticeable that Connecticut which was largely hypermalthusian with the simple logistic has shown saturation with the augmented logistic. The sum of squares has been reduced from .0067 to .0018. The limiting population is 7.33 million. The standard errors are however, rather large, but d, judged by its standard error, is well established as positive. The Three States (Conn., N. Y., N. J.) taken together have a negative lower asymptote and have become hypermalthusian, the population becoming infinite in finite time. Thus neither d nor a make sense. The value of S has been considerably reduced by the use of the additional constant. Yet the constants, judged by their standard errors are not well established. The relative fit also shows good improvement in S but is hypermalthusian with negative d like the absolute fit. In fitting the simple logistics we found that both the absolute and the relative fits gave finite asymptotes. It may be mentioned that we have found two graphical fits which were better than the logistic (judged by the value of S), one the augmented Malthusian, the other the augmented critical[72]—as in the case of England and Wales.

55. *New York City and Environs, 1790–1920.* New York City (Areas I, II, III and the total area) were discussed by Pearl and Reed[20] for the Plan of New York and Its Environs in 1923; in this paper (p. 21) they said they fitted augmented logistics by least squares to the population counts of the four areas. The populations of the separate areas I, II, III were available only from 1850 to 1920, but that of the total area was used from 1790 to 1920. Our augmented logistics with the value of S and of the asymptotic population $d + 1/a$ are given below with theirs for comparison—origin of time the initial time, whether 1850 or 1790.

	d	a	b	n	S	$d+1/a$
Area I	.448368	.09519626	2.082730	.4980710	.1244	11.0
P and R	"small"[73]	.050279		.349348	.1567	19.9
Area II	.1057562	.2782966	17.97468	.5115934	.002812	3.7
P and R	.087	.093500		.428817	.003766	10.8
Area III	.1358433	.2650746	7.011994	.2804549	.0004625	3.9
P and R	.150	.24518		.290181	.0005574	4.2
Total Area	.0787729	.0428006	7.618335	.2626669	.2245	23.4
Stand. Error	$\sigma = .127$	$\sigma = .0138$	$\sigma = 2.96$	$\sigma = .0423$		
P and R	"small"	.028653		.32300	.2500	34.9
(1850–1920)	.712831	.06396383	1.533368	.4733573	.1761	16.3
Relative (1790–1920)	.1430515	.06229789	11.22922	.412555	.02924	16.2

In regard to these results we may note the great difference we find if we compute the curve for the total area on the basis of 1850–1920 which is that on which the others must be computed, and on the basis of the total time 1790–1920; all the constants are markedly different and the terminal population is 16.3 instead of 23.4. One may also note the difference between the limiting population 23.4 we find and the figure 34.9 found by Pearl and Reed. In view of the statistical indetermination which we have found in other problems we do not claim that 34.9 departs from 23.4 by more than the standard error, but it is nearly 50% greater and that might be regarded as a substantial difference in city planning. We note further that all our limiting populations are less than theirs, that of Area I being only 55% as much and that of Area II only 34% as much.[74] Our indicated limiting populations in the three areas add up to 18.6 whereas that of the total area based on the same period of time is 16.3 and on the whole time is 23.4. If we compare these results with those obtained by the logistic we see that with $d = 0$ the limiting population of the whole area is 27.3 (by absolute residuals) whereas with d disposable it is 23.4, not a great difference; on the other hand with relative residuals we have 375 with $d = 0$ but 16.2 with d disposable which shows a great difference and furthermore the variations are in the opposite direction. On Area III the straight logistic gave (by absolute residuals) a hypermalthusian growth whereas with d disposable it gave a hypomalthusian growth with an asymptote at 3.9. This shows how sensitive the value of the forecast may be to the method of fitting and how necessary it is to specify the method. The great differences between our values and those of Pearl and Reed show how necessary it is to carry the work through correctly. We are not addicted to logistics or to augmented logistics as a method of forecasting populations but we believe that the values we have found for the three areas and the total area by applying the method of least squares with absolute residuals are not only more correct arithmetically than those of Pearl and Reed but more reasonable as forecasts. We doubt if Area I will ever have 19.9 million persons, we doubt if Area II will ever have 10.8 million; we believe that 11 million and 3.7 million, respectively, are much more reasonable figures,—but in view of the various results which we have found for different populations throughout the course of this paper we attribute this reasonableness to accident rather than to the infallibility of the method as an expression of natural law.

56. *Chicago, 1840–1920.* Monk and Jeter[61] worked up Chicago for 12 counties (1840–1920) and for 15 counties (1860–1920) and came to the conclusion that there was too much instability to put much dependence on the use of the logistic or augmented logistic. We give a table of the constants obtained by least squares for the augmented logistics and the simple logistic

	d	a	b	n	S	$d+1/a$
12 Counties.........	−.074429	.1653623	6.811656	.5348088	.010229	5.97
Stand. Error........	.0676	.0184	1.88	.0507		
15 Counties.........	.0547	.17901	2.8009	.58257	.009114	5.64
12 Counties.........	0	.1814158	9.204304	.5867093	.013054	5.51
Stand. Error........		.0110	1.09	.0275		
15 Counties.........	0	.1704878	2.474853	.5533423	.009414	5.87

Whether there be or be not stability the results for the limiting populations by the logistic with and without d are reasonably consistent, more so than the results for New York City, and the standard errors are no greater fractions of a and n than in the case of New York. It is perhaps a slight inconsistency that the limiting population of 12 counties (with d) should be less than that of 15 counties which includes them, but this lack of additivity seems no greater than that found in other cases.[75]

57. *The Weights of the Observations.* The weights of the observations are generally determined a priori from a knowledge of the reliability of those observations individually. In the case of fitting a growth curve this would mean not merely the precision of determination of the individual observations but their inherent variability due to

irregularities of growth. This is in itself a difficult statistical problem and we know of no way to settle it for population growth; it perhaps has not yet been adequately discussed for growth (e. g., in body weight) of individuals. We may, however, now that there are available a considerable number of populations fitted by least squares to logistics and to augmented logistics, give a brief statistical discussion of the a posteriori evidence as to the variability of the departures of the enumerated figures from the calculated figures. We may discuss this question: Having fitted the curves by absolute residuals and thus without weights, do the variations of the census figures from the fitted curve show any significant trend in time, i. e., with increase of population? One simple way to analyze the material is to consider the sum of the absolute errors for the first half of the censuses (when the populations were smaller) and the sum of the absolute errors for the second half of the censuses (when the populations were larger). We tabulate below, for the augmented logistic and for the logistic the percentage of the total arithmetic deviations which was constituted by the sub-total of these deviations occurring in the first half of the censuses (omitting entirely the middle census in case their number is odd).

Percentage of Total Deviation Falling in First Half

	Augmented Logistic	Simple Logistic
U. S. A., 1790–1910	24.5	18.4
Connecticut, 1790–1920	53.3	73.1
New Jersey, 1790–1920	—	58.7
New York State, 1790–1920	—	50.3
The Three States, 1790–1920	16.7	40.1
England and Wales, 1801–1911	39.6	59.5
Germany, 1816–1910	—	64.7
Chicago (15 counties), 1860–1920	43.9	51.2
Chicago (12 counties), 1840–1920	40.3	48.4
New York City Area I, 1850–1920	43.9	—
New York City Area II, 1850–1920	40.5	—
New York City Area III, 1850–1920	32.1	28.3
New York City and Environs 1790–1920	19.4	25.6
New York City and Environs 1850–1920	43.9	—
Sweden, 1750–1920	53.7	—
Mean of the percentages	37.6	47.1

In the 12 cases for the augmented logistic only 2 have percentages greater than 50 whereas 10 have them less than 50 and the mean is 37.6; in the 11 cases for the simple logistic 6 have percentages greater than 50 and 5 less than 50 with the general mean 47.1; furthermore, in almost every case (6 out of 8) in which we have both solutions the percentage of error in the first half is less for the augmented logistic than for the logistic. From these facts we make the tentative inference that in fitting the simple logistic by absolute residuals we find a posteriori that there is no evidence that the variations in the smaller populations of earlier times were numerically less than the variations in the larger populations of later times and hence that there is no evidence that we should not fit by absolute differences, but that in fitting augmented logistics by absolute residuals there is considerable evidence that there is a trend in the residuals with time (or with the population) and that a given variation is more significant if on a smaller population indicating that probably we should not fit by absolute residuals but use weights which decrease with increasing population.[76] Considerable further investigation would, however, be necessary to establish such inferences with reasonable assurance.

58. *Forecasts and Fulfilment.* This study would be incomplete if reference were omitted to a comparison of the enumerations of the census of 1930 with the forecasts from the fits based upon periods terminating in 1920. We have already referred to the results for the whole United States.[22] It would perhaps be unfair to use the 1921, 1931 figures for England and Wales for comparison with fits based on 1801–1911 because of the gross disturbance due to the war; suffice it to say that all the forecasts were naturally far too high. For the same reason, and in greater degree, we should not make comparisons in the case of Germany. For Greater New York we have not been able to identify any of the regions but Area I. The table gives the comparisons where we have been able to make them.

In respect to these forecasts and their fulfilment it may be noted that those for Chicago were worthless as Monk and Jeter surmised. The best forecast for Connecticut is that of Reed and Pearl on a curve type that is wrong for a least squares fit. The same is true for New Jersey. The least squares fit and the fit by Reed and Pearl for New York State are about equally good as might be expected from the nearness of the two solutions. For the Three States the least squares fit gives a better forecast than the Reed-Pearl fit, and the augmented logistic is better than the logistic. However, the relative augmented

	E	L	$E-L$	A	$E-A$	P	$E-P$	σ
Connecticut...	1.607	1.866	−.259	1.688	−.081	1.587	.020	.020/.011
New Jersey...	4.041	4.173	−.132			3.931	.110	.041
New York....	12.588	11.862	.726			11.907	.681	.239
3 States......	18.236	17.796	.440	18.553	−.317	17.363	.873	.214/.161
(Relative)....		15.694	16%	18.009	1.2%			5.4/2.5
Area I (NYC)	8.063			7.914	.149	8.441	−.378	.125
Chicago (12)..	4.979	4.381	.598	4.457	.522			.038/.034
Chicago (15)..	5.058	4.506	.552	4.471	.587			.037/.036

E = Census enumeration, L = least squares logistic (Arts. 39, 42, 54, 55, 56), A = least squares augmented logistic, P = Pearl and Reed figures. When two figures for σ are given the first is for the logistic, the second for the augmented logistic.

logistic is better than the absolute. For Area I of New York City the least squares augmented logistic gives a better fit than the figures of Pearl and Reed. From these results we can only infer that the least squares logistics or augmented logistics fitted by absolute residuals are not very satisfactory as forecasters, probably not so satisfactory on the whole as the Pearl-Reed fits, which though claimed to be by least squares, are not least squares fits.[77] It is worth noticing that the errors in the forecast of the curves fitted by least squares are on the whole much larger than the average errors σ of fitting although we are not so far removed from the portion of the curve actually fitted as to make a theoretical increase of importance in the deviation. Indeed for the least squares fits the ratio of error to σ are − 13, − 7, − 3, + 3, + 2, − 2, + 3, + 0.5, + 1.2, + 18, + 14, + 15, + 16.

IV. SUMMARY AND CONCLUSIONS.

59. *Summary.* We have tried by topical headings and frequent summary to make clear the general nature of the results of this investigation throughout the course of this paper, but a general brief summary may be added: (I) There does not appear to be any law of population growth which will rigorously satisfy the law of superposition or additivity, the logistic (simple or augmented) may satisfy the law reasonably well or not at all. The logistic has a plurality of types and as fitted empirically to the data cannot be affirmed to contain the law of saturation of population. (II) The determination of the constants in the logistic, simple or augmented, by various simple methods indicates a high degree of unreliability in the determination. The

method of least squares though yielding definite results is extremely laborious in its execution after a proper trial solution has been obtained from which to start the successive approximations and often involves a long process of trial before the proper trial solution can be found. (III) The approximation equations for the least squares procedure given in standard treatises will in some exceptional cases fail to give the convergence at all and must be amended, though for most cases they are satisfactory. The constants in the form of the logistic should be taken so as to avoid imaginary values or we may be forced very slowly to limiting forms, i. e., the solution may depend on the analytic form of the formula. For the augmented logistic conditions may arise where we are forced to such limiting forms no matter how we take the constants. (IV) The method of least squares though affording definite results does not fail to indicate, through the standard errors of the unknowns, a considerable degree of instability in the values of the constants, yet the actual variability in them for fits which make the sum of the squares of the residuals not very much larger than its minimum may be out of all proportion greater than indicated by the standard errors. (V) A considerable number of actual fittings by least squares have been made so that for the first time one may see how such solutions behave.

60. *Tabular Summary.* In the table are given in summarized form the values of the essential constants which we might desire to interpret, as they have been determined. Where the entry under *d* is blank the fit has been by a logistic. All fits are by absolute residuals unless marked "relative." When the entry under "limit" is "infinite" it means "infinite in finite time." All figures for *d* and for the "limit" are in millions.

We regret that we cannot include in the table a greater amount of data, but the calculations are heavy. Generalization from so few cases is hazardous but some observations may be risked. The value of *d* is negative in 6 out of 16 cases. For five of these cases in which we have standard errors, and presumably for the sixth, the value of *d* is not significant. However, in the five cases for which we have fitted also a simple logistic, there are three in which the upper limits of the population are totally different according as we keep the insignificant negative *d* or set *d* rigorously equal to zero. We regret that we cannot give a similar discussion for *d* positive because of insufficient time to calculate standard errors. The logistic becomes infinite in four out of thirteen cases, as it is for Canada.[19] The aug-

	d	*a*	*n*	*limit*	σ†	σ'†
England and Wales, 1801–1911	—	.0093±.0010	.115±.0045	107	.160	.185
England and Wales	−4.7±4.7	.0026±.0065	.107±.039	383	.141	.173
Germany, 1816–1910	—	−.0557±.046	.029±.017	infinite	.725	.850
Germany (imaginary*)	$32-18.5i$	$-.027i$	$17.4i$	infinite	.338	.424
Germany (Critical)	−7.7±5.0	$(\epsilon=599,\ \gamma=17.7)$	0	infinite	.720	.844
Germany, 1816–1855	14.9	.0399	.545	40	.109	.244
Germany, 1861–1910	19.7	−.0246	.107	infinite	.243	.421
Connecticut, 1790–1920	—	−1.64	.080	infinite	.022	.025
Connecticut	.20±.02	.140±.11	.287±.031	7.33	.011	.013
New Jersey, 1790–1920	—	−.026	.224	infinite	.041	.046
New York, 1790–1920	—	.0394±.012	.244±.023	25.4	.239	.270
The Three States, 1790–1920	—	.0129	.221	77.3	.214	.241
The Three States	−2.2±1.8	−.040±.056	.105±.066	infinite	.161	.191
The Three States (*Relative*)	—	.0401	.278	24.9	5.4%	6.1%
The Three States (*Relative*)	−1.37	−.0113	.146	infinite	2.5%	3.0%
New York City and Environs. 1790–1920	—	.0366	.341	27.3	.129	.145
New York City and Environs	.079±.127	.0428±.0138	.363±.042	23.4	.127	.150
New York City and Environs (*Relative*)	—	.0027	.299	375	9.0%	10.2%
New York City and Environs (*Relative*)	.143	.0623	.413	16.2	4.6%	5.4%
New York City and Environs, 1850–1920	.713	.0640	.473	16.3	.148	.210
Area III thereof	—	−.411	.147	infinite	.0083	.0105
Area III thereof	.136	.265	.280	3.9	.0076	.0108
Area II thereof	.106	.278	.512	3.7	.019	.027
Area I thereof	.448	.0952	.498	11.0	.125	.176
United States, 1790–1910	—	.00510±.00027	.314±.0063	196	.446	.509
United States	−.29±.82	.00495±.00050	.308±.016	202	.443	.532
Chicago, 12 counties, 1840–1920	—	.181±.011	.587±.027	5.5	.038	.047
Chicago, 12 counties, 1840–1920	−.074±.068	.165±.019	.534±.051	6.0	.034	.045
Chicago, 15 counties, 1860–1920	—	.170	.553	5.9	.037	.049
Chicago, 15 counties, 1860–1920	.055	.179	.583	5.6	.036	.055

* This imaginary solution for Germany has not had a final least squares adjustment.

† σ is the standard deviation $\sqrt{S/k}$, σ' that adjusted for the number of constants used.

mented logistic becomes infinite in three out of fifteen cases if we exclude Germany (1816–1910) where we have to go over to the augmented critical case, and in four out of sixteen cases if we include it. The values of n, the survival rate in a sparse population, vary for the simple logistics from .029 to .587 or, if we exclude the logistics which become infinite, from .115 to .587 and thus offer little satisfaction to those in search of a natural survival rate. We should hesitate to interpret n in the case of the augmented logistic even if the range were not likewise exceedingly great. In the three cases in which we have standard errors for both the logistic and the augmented logistic we note that the errors for the latter are much larger than for the former, although the increase due merely to the introduction of an extra constant would not be great,[78] and we feel confident that were there more comparisons of this sort available we should find such a situation to be of frequent occurrence.

61. *Conclusions.* If by the statement that the logistic, whether augmented or not, is the law of population growth, one means only that the formula is well suited to fitting the census enumerations for the period of a century or so when such enumerations have actually been made, we can take no exception to it, for we have shown that those enumerations can be fitted even more closely than they have been fitted by others. But if the statement is to be considered as signifying that the formula affords a rational law to such an extent as to permit the extrapolation of the curve for forecasting purposes and the interpretation of the constants as constants of nature, we are forced to take serious exception to it, because we find that there are too many instances in which the curve becomes infinite in finite time or has a negative lower asymptote or both and because the constants are too often so poorly determined as to be practically undetermined; in all these cases we must at least withhold judgment until the populations have developed so far toward saturation that the fitting of the curve will give reasonably well determined indications of the saturation values. If however, we abandon the attempt to make any preassigned method of fitting give sensible results and avail ourselves of the very freedom furnished by the instability of the constants we may still get a tolerably good fit to the census enumerations with values of the constants that are not too unreasonable,[79] and although this method of using the curve leaves the determination of some of the constants personal to the individual worker, much as a graphical extrapolation of a curve is personal to the draughts-

man, we believe that the curve thereby becomes much more useful than otherwise to the student of population (or other types of) growth. Verhulst said in 1844: La loi de la population nous est inconnue, parce qu'on ignore la nature de la fonction qui sert de mesure aux obstacles, tant préventifs que destructifs, qui s'opposent à la multiplication indéfinie de l'espèce humaine. . . . Une longue série d'observations, non interrompues par de grandes catastrophes sociales ou des révolutions du globe, fera probablement decouvrir la fonction retardatrice dont il vient d'être fait mention. This remains true today; our series of observations has not yet been long enough; it is possible that the next thirty years will throw considerable light on the matter; by the end of that time we may be definitely rid of hypermalthusian growth and in many populations sufficiently near to saturation to determine whether the curve of growth is symmetric with respect to its half-way point and otherwise gives evidence of being satisfactorily close to the logistic. In the meantime it would seem to be of doubtful utility for demographic purposes to go to the trouble of fitting the curves by least squares.

V. APPENDIX. NOTES ON LEAST SQUARES.

62. *One Linear Unknown.* To clear up a number of matters relative to the method of least squares we submit the following remarks thereupon. For simplicity we may begin with the case of one linear unknown. The early discussion of the method was in terms of the normal law of errors which still is often made basic; we shall, however, adopt a strictly statistical procedure. It will be assumed that the n observations $E_i (i = 1, 2, \cdots, n)$ form a typical set or sample and a double index E_{ji} will be introduced in which j is constant for any one sample of n observations. Each observation E_{ji} for i fixed and j variable is supposed to be any one that could be made for that value of i; in other words there are n universes, one for each i, from which the elements E_{ji} are drawn. These n universes have their statistical characters, namely their means E_{mi}, their variances σ_i^2, and their higher moments μ_{3i}, μ_{4i}, $\cdots$ obtained conceptually by averaging E_{ji}, $(E_{ji} - E_{mi})^2$, $(E_{ji} - E_{mi})^3$, $\cdots$ over the index j. The problem is to determine the value of the unknown a_j which minimizes

$$S_j = \Sigma_i (a_j \varphi_i - E_{ji})^2, \quad i = 1, 2, \cdots, n \qquad (40)$$

and to determine the mean value a_m of a_j and the variance σ_a^2 of a_j

about a_m (possibly also the higher moments $\mu_{3a}, \cdots$ of a_j about a_m), where φ_i is a known quantity dependent on the index i.

We have at once, using / as a sign of division,

$$\begin{aligned} a_j &= \Sigma_i \varphi_i E_{ji}/\Sigma \varphi_i^2, \quad a_m = \Sigma_i \varphi_i E_{mi}/\Sigma \varphi_i^2 \\ a_j - a_m &= \Sigma_i \varphi_i \delta_{ji}/\Sigma \varphi_i^2, \quad (a_j - a_m)^2 = (\Sigma_i \varphi_i \delta_{ji})^2/(\Sigma \varphi_i^2)^2 \end{aligned} \tag{41}$$

where $\delta_{ji} = E_{ji} - E_{mi}$ is the variation of E_{ji} from its mean E_{mi}. The ith residual is the difference

$$v_{ji} = E_{ji} - a_j \varphi_i = E_{ji} - \varphi_i \Sigma \varphi_i E_{ji}/\Sigma \varphi_i^2$$

and is not the same as δ_{ji}. Indeed if a_m be the value of a when the means E_{mi} are fitted to the expression $a\varphi_i$ and if v_{mi} be the residual of this fit,

$$v_{ji} - v_{mi} = \delta_{ji} - \varphi_i \Sigma \varphi_i \delta_{ji}/\Sigma \varphi_i^2$$

To find the variance σ_a^2 it is necessary to average $(a_j - a_m)^2$ over all samples j. Then, squaring $\Sigma_i \varphi_i \delta_{ji}$ and averaging over j,

$$\sigma_a^2 = \Sigma_i \varphi_i^2 \sigma_i^2/(\Sigma \varphi_i^2)^2 + \Sigma_{ih} \varphi_i \varphi_h r_{ih} \sigma_i \sigma_h/(\Sigma \varphi_i^2)^2, \quad i \neq h \tag{42}$$

The expression involves the n variances σ_i^2 in the n universes and the $n(n - 1)/2$ correlation coefficients r_{ih} between δ_{ji} and δ_{jh}, i. e., between the variations in the different universes. If it may be assumed that the n observations E_{ji} for a given j are independent, the correlation coefficients r_{ih} vanish; sometimes this assumption is valid, sometimes it is not. If it may also be assumed that the variances σ_i^2 in the n universes are all equal, $\sigma_i^2 = \sigma^2$, then there is a major reduction to

$$\sigma_a^2 = \sigma^2/\Sigma \varphi_i^2 \tag{42'}$$

and sometimes this further assumption is valid, though often it is not.

63. *Higher Moments and Edgeworth's Theorem.* A word may be interpolated concerning the higher moments of a_j about a_m.

$$(a_j - a_m)^3 = (\Sigma_i \varphi_i \delta_{ji})^3/(\Sigma_i \varphi_i^2)^3 \tag{43}$$

$$(a_j - a_m)^3 = \frac{\Sigma_i \varphi_i^3 \delta_{ji}^3 + 3\Sigma_{ih} \varphi_i^2 \varphi_h \delta_{ji}^2 \delta_{jh} + \Sigma_{ihg} \varphi_i \varphi_h \varphi_g \delta_{ji} \delta_{jh} \delta_{jg}}{(\Sigma_i \varphi_i^2)^3}$$

where it is assumed that no two of the indices i, h, g are equal. The mean value will involve the moments μ_{3i} of the values δ_{ji} and also the mean values of such products as $\delta_{ji}^2\ \delta_{jh}$ and $\delta_{ji}\ \delta_{jh}\ \delta_{jg}$. If it may be assumed that the variations in the n different universes are independent[80] the higher products containing δ_{ji} or δ_{jh} to the first power must vanish because the mean of δ_{ji} over j is zero, and then the result is simply

$$\mu_{3a} = \Sigma_i \varphi_i^3 \mu_{3i}/(\Sigma_i \varphi_i^2)^3 \tag{43'}$$

In a similar manner

$$(a_j - a_m)^4 = [\Sigma_i \varphi_i^4 \delta_{ji}^4 + 3\Sigma_{ih} \varphi_i^2 \varphi_h^2 \delta_{ji}^2 \delta_{jh}^2 + \cdots]/(\Sigma_i \varphi_i^2)^4 \tag{44}$$

where the terms not written involve a δ_j to the first power and on the assumption of independence of the universes will vanish in the mean. Then

$$\mu_{4a} = [\Sigma_i \varphi_i^4 \mu_{4i} + 3\Sigma_{ih} \varphi_i^2 \varphi_h^2 \sigma^2_i \sigma_h^2]/(\Sigma_i \varphi_i^2)^4 \tag{44'}$$

and

$$\mu_{4a} - 3\sigma_a^4 = \Sigma_i \varphi_i^4 (\mu_{4i} - 3\sigma_i^4)/(\Sigma_i \varphi_i^2)^4$$

Finally if β_a and $\beta_{2a} - 3$ be defined as usual in the theory of moments,

$$\beta_a = \frac{\mu_{3a}}{\sigma_a^3} = \frac{\Sigma_i \varphi_i^3 \beta_i \sigma_i^3}{(\Sigma_i \varphi_i^2 \sigma_i^2)^{3/2}} \tag{43''}$$

$$\beta_{2a} - 3 = \frac{\mu_{4a} - 3\sigma_a^4}{\sigma_a^4} = \frac{\Sigma_i \varphi_i^4 (\beta_{2i} - 3)\sigma_i^4}{(\Sigma_i \varphi_i^2 \sigma_i^2)^2} \tag{44''}$$

If it can be assumed that the n universes have identical statistical properties with respect to moments these results reduce to

$$\beta_a = \beta \Sigma_i \varphi_i^3/(\Sigma_i \varphi_i^2)^{3/2}, \quad \beta_{2a} - 3 = (\beta_2 - 3)\Sigma_i \varphi_i^4/(\Sigma_i \varphi_i^2)^2 \tag{45}$$

where $\beta = \beta_i$ and $\beta_2 = \beta_{2i}$ are constant with respect to i.

Edgeworth emphasized the proposition that no matter what the universe to which the observations belonged, the mean of a sample of k observations tended, as k increased, to be distributed normally—indeed, this is sometimes referred to as Edgeworth's theorem. He further pointed out that our statistical quantities were often really mean values and that whenever this was so and the number in the sample was large we were certainly justified in assuming a practically normal distribution of the means. This theorem is a special case where $\varphi_i = 1$ of the results just obtained, at least insofar as concerns the determination of the frequency function to fourth moments. For in the case of means the values i are permutable and we have (a being now the mean of E)

$$\sigma_a^2 = \frac{\sigma^2}{n}, \quad \beta_a = \frac{\beta}{n^{1/2}}, \quad \beta_{2a} - 3 = \frac{\beta_2 - 3}{n}$$

The skewness as measured by β_a falls off inversely as the square root of the number in the sample; the kurtosis $\beta_{2a} - 3$, inversely as that number. It is, however, clear that the theorem Edgeworth[81] states for the mean must be true in essence for a great variety of least squares problems involving the fitting of an unknown a. Restricting the discussion to the assumption of independence of δ_{ji} and δ_{jh}, one may observe that $\Sigma_i\varphi_i^2\sigma_i^2$ may be written as n times the mean value of $\varphi_i^2\sigma_i^2$ and that so long as we are operating over a more or less fixed range for φ_i and σ_i, the value of σ^2_a will fall off as $1/n$, β_a as $1/n^{1/2}$ and of β_{2a} as $1/n$.

64. *Usual Expression for Variance.* In most least squares problems the statistical constants of the different universes corresponding to the different values of i are unknown; we have in fact merely the observations as given. From the solution a_j we may obtain the residuals v_{ji} and the value $S_j = \Sigma_i v_{ji}^2$ of the sum of the squares for the best fit. We have

$$\Sigma_i(v_{ji} - v_{mi})^2 = \Sigma_i\delta_{ji}^2 - (\Sigma\varphi_i\delta_{ji})^2/(\Sigma\varphi_i^2)$$

The mean value (over j) of the two sides is readily obtained on the assumption that δ_{ji} and δ_{jh} are uncorrelated.

$$\Sigma_i(v_{ji} - v_{mi})^2 = \Sigma_i v_{ji}^2 - 2\Sigma_i v_{ji}v_{mi} + \Sigma_i v_{mi}^2$$

Now the first term is S_j, the sum of the squares of the residuals in the given sample and the mean value of this will be some quantity S_m. The mean value of v_{ji} is v_{mi}; and $\Sigma_i v_{mi}^2$ is that value of S, say S_0 which would be obtained from fitting the means E_{mi} to $a\varphi_i$. Hence

$$S_m - S_0 = \Sigma_i\sigma_i^2 - \Sigma\varphi_i^2\sigma_i^2/(\Sigma\varphi_i^2) \tag{46}$$

If the further assumption be now made that $\sigma_i^2 = \sigma^2$ it is possible to eliminate σ^2 between this equation and that for σ_a^2 to obtain

$$\sigma_a^2 = \frac{S_m - S_0}{n - 1}\frac{1}{\Sigma_i\varphi_i^2} \tag{46′}$$

In a given case S_m is not known but if n is large the difference between S_m and the particular value S_j in the sample is probably small. Also S_0, the sum of the squares of the residuals obtained from fitting the universe means E_{mi} to $a\varphi_i$ is unknown.

The usual formula for the variance of the unknown is

$$\sigma_a^2 = \frac{S_j}{n - 1}\frac{1}{\Sigma_i\varphi_i^2} \tag{47}$$

We may now summarize the assumptions which will suffice to demonstrate that formula:

(i) The means E_{mi} are practically equal to $a\varphi_i$, that is, it is assumed that the "law" $a\varphi_i$ which is fitted to the observations E_{ji} is the law which fits the means so closely that the sum of the squares of the residuals in the case of the means is negligible compared with the sum of the squares of the residuals in the actual sample fitted.

(ii) The sample is large enough so that without important error the value of the sum of the squares of the residuals for the sample may be used for the mean value of that sum for all samples.

(iii) The values of σ_i^2, the variances of the possible observations for each value of i, are all equal.

(iv) The variations δ_{ji}, δ_{jh} of the observations from their means for two different values of i are not correlated.

It is clear that in any particular case of the application of least squares it would be difficult to assert that these four restrictions are in fact satisfied to all practical purposes; it is of course possible that their failures to be satisfied might result by accident in a balance of errors which would leave the final formula practically correct, but one cannot rely on such an accident.

65. *Several Linear Unknowns.* The analysis for k linear unknowns, the usual case, may be carried out almost letter for letter identically with that for one unknown provided the matrix or dyadic analysis be used.[83] The problem is to minimize

$$S_j = \Sigma_i(a_{j1}\varphi_{1i} + a_{j2}\varphi_{2i} + \cdots + a_{jk}\varphi_{ki} - E_{ji})^2 = \Sigma_i(\mathbf{a}_j \cdot \phi_i - E_{ji})^2 \quad (48)$$

where $\mathbf{a}_j$ is a "vector" unknown and ϕ is a known "vector" and the dot is the sign of scalar multiplication. Differentiate:

$$dS_j = d\mathbf{a}_j \cdot \nabla_{\mathbf{a}} S = 2d\mathbf{a}_j \cdot \Sigma_i \phi_i(\mathbf{a}_j \cdot \phi_i - E_{ji}) = 0 \quad (49)$$

for all $d\mathbf{a}_j$. Hence

$$\Sigma_i \phi_i(\mathbf{a}_j \cdot \phi_i - E_{ji}) = 0 \quad \text{or} \quad \Sigma_i(\mathbf{a}_j \cdot \phi_i \phi_i - \phi_i E_{ji}) = 0$$

or

$$\mathbf{a}_j \cdot \Sigma_i \phi_i \phi_i = \Sigma_i \phi_i E_{ji} \quad \text{or} \quad \mathbf{a}_j = (\Sigma_i \phi_i \phi_i)^{-1} \cdot \Sigma_i \phi_i E_{ji} \quad (50)$$

It may be observed that the dyadic $\Sigma_i \phi_i \phi_i$ is symmetrical or self-conjugate and that it may therefore be multiplied indifferently on either side by a vector such as $\Sigma_i E_{ji} \phi_i$. Again introducing $\delta_{ji} = E_{ji} - E_{mi}$ and using $\mathbf{a}_m$ for the unknowns obtained by fitting E_{mi} to $\mathbf{a}_m \cdot \phi_i$ we have

$$\mathbf{a}_j - \mathbf{a}_m = (\Sigma_i \phi_i \phi_i)^{-1} \cdot \Sigma_i \phi_i \delta_{ji} \tag{51}$$

Multiply $\mathbf{a}_j - \mathbf{a}_m$ by itself, thus forming a symmetrical dyad, we may write

$$(\mathbf{a}_j - \mathbf{a}_m)(\mathbf{a}_j - \mathbf{a}_m) = (\Sigma_i \phi_i \phi_i)^{-1} \cdot \Sigma_i \phi_i \delta_{ji} \Sigma_i \phi_i \delta_{ji} \cdot (\Sigma_i \phi_i \phi_i)^{-1} \tag{52}$$

Here we introduce assumption (iv) and take the average over j. The average of the dyadic will be a symmetrical dyadic which will be designated as μ_{2a}, the second moment dyadic.

$$\mu_{2a} = (\Sigma_i \phi_i \phi_i)^{-1} \cdot \Sigma_i \phi_i \phi_i \sigma_i^2 \cdot (\Sigma_i \phi_i \phi_i)^{-1} \tag{53}$$

Next introduce assumption (iii) and

$$\mu_{2a} = (\Sigma_i \phi_i \phi_i)^{-1} \sigma^2 \tag{53'}$$

where σ^2 is the variance of δ_{ji}, equal for all values of i.

The dyadic μ_{2a} gives the variances σ_i^2 of all k components of $\mathbf{a}$ and also the mean products π_{ih} of any two of the components. Indeed

$$\begin{aligned}(\Sigma_i \phi_i \phi_i)^{-1} \sigma^2 = {} & \sigma^2{}_1 \mathbf{u}_1 \mathbf{u}_1 + \pi_{12} \mathbf{u}_1 \mathbf{u}_2 + \cdots + \pi_{1k} \mathbf{u}_1 \mathbf{u}_k \\ & + \pi_{12} \mathbf{u}_2 \mathbf{u}_1 + \sigma_2^2 \mathbf{u}_2 \mathbf{u}_2 + \cdots\cdots\cdots \\ & + \cdots\cdots\cdots\cdots\cdots\cdots\cdots + \sigma_k^2 \mathbf{u}_k \mathbf{u}_k \end{aligned} \tag{54}$$

where $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ are units. On the other hand

$$\begin{aligned}\Sigma_i \phi_i \phi_i = {} & \Sigma_i \varphi_{1i}^2 \mathbf{u}_1 \mathbf{u}_2 + \Sigma_i \varphi_{1i} \varphi_{2i} \mathbf{u}_1 \mathbf{u}_2 + \cdots + \Sigma_i \varphi_{1i} \varphi_{ki} \mathbf{u}_1 \mathbf{u}_k \\ & + \Sigma_i \varphi_{1i} \varphi_{2i} \mathbf{u}_2 \mathbf{u}_1 + \Sigma_i \varphi_{2i}^2 \mathbf{u}_2 \mathbf{u}_2 + \cdots\cdot \\ & + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots + \Sigma \varphi_{ki}^2 \mathbf{u}_k \mathbf{u}_k \end{aligned} \tag{54'}$$

Owing to the relations between reciprocal dyadics or matrices the variances $\sigma_1^2, \cdots, \sigma_k^2$ of the k unknowns and the mean products $\pi_{12}, \cdots, \pi_{k-1,k}$ may be written as the quotients of the cofactors of the determinant of the matrix $\Sigma_i \phi_i \phi_i$ by that determinant (all multiplied by σ^2 which comes from the δ's).

Now using assumption (i) we may write

$$S_j = \Sigma_i \delta_{ji}^2 - \Sigma_i \delta_{ji} \phi_i \cdot (\Sigma_i \phi_i \phi_i)^{-1} \cdot \Sigma_i \phi_i \delta_{ji} \tag{55}$$

or in Gibbs's double multiplication

$$S_j = \Sigma_i \delta_{ji}^2 - (\Sigma_i \phi_i \phi_i)^{-1} : \Sigma_i \phi_i \delta_{ji} \Sigma_i \phi_i \delta_{ji} \tag{55'}$$

Using (iii) and (iv), we have for the mean over j

$$S_m = n\sigma^2 - (\Sigma_i \phi_i \phi_i)^{-1} : \Sigma_i \phi_i \phi_i \sigma^2 \tag{56}$$

As the double dot product of a dyadic and its reciprocal is equal to the number of dimensions, we have

$$S_m = (n - k)\sigma^2 \tag{56'}$$

Hence if S_m and S_j may be interchanged by assumption (ii),

$$\mu_{2a} = \frac{S_j}{n - k}(\Sigma_i\phi_i\phi_i)^{-1} \tag{57}$$

This contains the usual formulas for the variances of the unknowns and the less common formulas for the mean products of pairs of unknowns. The simplicity of the proof in matrix notation is noteworthy.

66. *Higher Moments.* If we write for brevity $\Phi = \Sigma_i\phi_i\phi_i$, the higher moments may be expressed in multiple product notation as

$$\begin{aligned}(\mathbf{a}_j - \mathbf{a}_m)&(\mathbf{a}_j - \mathbf{a}_m)(\mathbf{a}_j - \mathbf{a}_m)\\ &= (\Phi^{-1} . \Sigma_i\phi_i\delta_{ji})(\Phi^{-1} . \Sigma_i\phi_i\delta_{ji})(\Phi^{-1} . \Sigma_i\phi_i\delta_{ji}) \qquad (58)\\ &= \Phi^{-1}\Phi^{-1}\Phi^{-1} \vdots \Sigma_i\phi_i\delta_{ji}\Sigma_i\phi_i\delta_{ji}\Sigma_i\phi_i\delta_{ji}\end{aligned}$$

meaning that each of the three vectors $\Sigma_i\phi_i\delta_{ji}$ is multiplied into one[84] of the dyadics Φ^{-1}. The resulting third moment expression is a triad and its average will be a triadic which will express the third moments of each of the k unknowns and the mean products of the square of any unknown into any other and of the product of any three unknowns, all unknowns being measured from their means. Moreover if the independence of the δ's for different i's be assumed and the identity of the values μ_{3i} be further assumed,

$$\mu_{3a} = \Phi^{-1}\Phi^{-1}\Phi^{-1} \vdots \Sigma_i\phi_i\phi_i\phi_i\mu_3 \tag{58'}$$

This is the third moment triadic for the distribution of the unknowns about their means. It is not easy to write out the components, but no less tedious from this than from any other expression that might be obtained without the vector notation.[85]

Relative to Edgeworth's theorem we may note that if the value of n be increased and the range of ϕ_i be more or less fixed, $(\Sigma_i\phi_i\phi_i)^{-1}$ will vary inversely as n and hence μ_{2a} will vary inversely as n. As to μ_{3a}, each Φ^{-1} varies inversely as n whereas the term into which it is multiplied varies as n so that μ_{3a} varies as $1/n^2$. As μ_{2a} varies as $1/n$, the skewness of the n-fold distribution of the k unknowns would presumably vary as $1/n^{1/2}$.

It is possible to go on to higher moments. The fourth moment would involve a tetradic

$$\mu_{4a} = \Phi^{-1}\Phi^{-1}\Phi^{-1}\Phi^{-1} \vdots\, \Sigma_i\phi_i\delta_{ji}\Sigma_i\phi_i\delta_{ji}\Sigma_i\phi_i\delta_{ji}\Sigma_i\phi_i\delta_{ji} \tag{59}$$

where the mean must be taken over all samples j. Under the assumption of independence of the δ's and of identity of the constants of the different universes we have

$$\mu_{4a} = \Phi^{-1}\Phi^{-1}\Phi^{-1}\Phi^{-1} \vdots\, [\Sigma_i\phi_i\phi_i\phi_i\phi_i\mu_4 + \Sigma_{ih}(\phi_i\phi_i\phi_h\phi_h + \phi_i\phi_h\phi_h\phi_i + \phi_i\phi_h\phi_i\phi_h)\sigma^4] \tag{59'}$$

Now if we have a pair of dyads **ab** and **cd** any combination of them which is distributive would be called a product. Thus not only the tetrad **abcd** is such a product but the tetrads **acbd** and **acdb** formed by permuting the letters are such products and the tetradic

$$\mathbf{abcd} + \mathbf{acbd} + \mathbf{acdb}$$

may also be regarded as a type of product of **ab** and **cd** (for if any one of the four vectors **a**, **b**, **c**, **d** be written as the sum of two vectors as $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$ the product of **ab** and **cd** will be the sum of the products of $\mathbf{ab}_1$ and **cd** and of $\mathbf{ab}_2$ and **cd**). Let us denote the sum of the three terms as $\mathbf{ab} \times \mathbf{cd}$. Then any two dyadics may be multiplied in this fashion and in particular μ_{2a} may be multiplied by itself as $\mu_{2a} \times \mu_{2a}$. We have then

$$\mu_{4a} - \mu_{2a} \times \mu_{2a} = \Phi^{-1}\Phi^{-1}\Phi^{-1}\Phi^{-1} \vdots\, \Sigma_i\phi_i\phi_i\phi_i\phi_i(\mu_4 - 3\sigma^4) \tag{60}$$

The tetradic $\mu_{4a} - \mu_{2a} \times \mu_{2a}$ may be shown to vanish for the normal distribution of the k unknowns.[86] It will vanish if the δ's are distributed normally because then $\mu_4 = 3\sigma^4$. It will vary as $1/n^3$ and as the variances vary as $1/n$, the kurtosis coefficients would vary as $1/n$. Hence so far as the fourth moment we have the generalization under very moderate restrictions of Edgeworth's Theorem to any least squares solution: The distribution of the unknowns tends to normality (in k variables) when the size n of the sample increases indefinitely.

67. *The Case of k Unknowns with Weights.* The problem is to minimize

$$S_j = \Sigma_i w_i(\mathbf{a}_j \cdot \phi_i - E_{ji})^2, \quad i = 1, 2, \cdots, n \tag{61}$$

the results are

$$\mathbf{a}_j = (\Sigma_i w_i\phi_i\phi_i)^{-1} \cdot \Sigma_i w_i\phi_i E_{ji} \tag{62}$$

$$\mathbf{a}_j - \mathbf{a}_m = (\Sigma_i w_i\phi_i\phi_i)^{-1} \cdot \Sigma_i w_i\phi_i\delta_{ji} \tag{63}$$

$$\mu_{2a} = (\Sigma_i w_i\phi_i\phi_i)^{-1} \cdot \Sigma_i w_i^2\phi_i\phi_i\sigma_i^2 \cdot (\Sigma_i w_i\phi_i\phi_i)^{-1} \tag{64}$$

It is to be observed that $\Phi = \Sigma_i\phi_i\phi_i$ has become replaced by $\Sigma_i w_i\phi_i\phi_i$. It is readily shown that

$$S_m = \Sigma_i w_i\sigma_i^2 - \Sigma_i w_i^2\phi_i\phi_i\sigma_i^2 : (\Sigma_i w_i\phi_i\phi_i)^{-1} \qquad (65)$$

The reductions require assumptions (i) and (iv), and if we go on to assume $S_m = S_j$ we need (ii). What takes the place of (iii), the constancy of σ_i^2 for different values of i? Clearly what is needed for algebraic simplification is

(iii′) that $w_i\sigma_i^2$ shall be independent of i,

not that σ_i^2 shall be constant. That is, the weights w_i must be taken inversely proportional to the variances σ_i^2 of δ_{ji}. With this assumption the usual formula, as expressed in matrix notation, is

$$\mu_{2a} = \frac{S_j}{n-k}(\Sigma_i w_i\phi_i\phi_i)^{-1} \qquad (66)$$

Thus if we are to use the ordinary formula for the standard errors σ_a of the unknowns we must assume that the weights and the variances of the δ_{ji} have this specified adjustment.

68. *The Principle of Efficiency.* The matter may be put in a different light. R. A. Fisher has laid much emphasis on the concept of "efficiency" in statistics,[36] by which is meant the best use of the material, the use which gives the greatest precision to the results. Now any linear combination of the unknowns may be written as $\mathbf{l}\,.\,\mathbf{a}$ and as a change in $\mathbf{l}\,.\,\mathbf{a}$ may be written as

$$\triangle(\mathbf{l}\,.\,\mathbf{a}) = \mathbf{l}\,.\,\triangle\mathbf{a}, \quad [\triangle(\mathbf{l}\,.\,\mathbf{a})]^2 = \mathbf{l}\,.\,\triangle\mathbf{a}\triangle\mathbf{a}\,.\,\mathbf{l} \qquad (67)$$

We have by the familiar argument $\sigma^2(\mathbf{l}\,.\,\mathbf{a}) = \mathbf{l}\,.\,\mu_{2a}\,.\,\mathbf{l}$ where μ_{2a} is given by (66). The values of σ_i^2 being properties of the δ's in the k universes must be regarded as fixed even if unknown. The question may be asked, what system of weights is the most efficient, i. e., what system will make the variance $\sigma^2(\mathbf{l}\,.\,\mathbf{a})$ of a linear function of the unknowns as small as may be, and thus make the determination of the unknowns as precise as possible. We may differentiate (67) with respect to each of the k quantities w_i and show that the conditions resulting from the vanishing of the derivatives are equivalent to $w_i\sigma_i^2 = \text{const.}$, but a purely algebraic proof may be given. As a constant factor applied to the weights will not affect the value of $\mathbf{l}\,.\,\mu_{2a}\,.\,\mathbf{l}$ we may take $w_i\sigma_i^2 = 1$ for the case where $w_i\sigma_i^2$ is constant;

then $\mu_{2a} = (\Sigma_i w_i \phi_i \phi_i)^{-1}$. Now let $w_i = u_i \sigma_i^{-2}$. The differences of the variances in the two cases is

$$\mathbf{1} \,.\, [(\Sigma_i u_i \sigma_i^{-2} \phi_i \phi_i)^{-1} \,.\, \Sigma_i u_i^2 \sigma_i^{-2} \phi_i \phi_i \,.\, (\Sigma_i u_i \sigma_i^{-2} \phi_i \phi_i)^{-1} - (\Sigma_i \sigma_i^{-2} \phi_i \phi_i)^{-1}] \,.\, \mathbf{1}$$

This may be written as

$$\Sigma_i [\mathbf{1} \,.\, \{ (\Sigma_i u_i \sigma_i^{-2} \phi_i \phi_i)^{-1} \,.\, (u_i \sigma_i^{-1} \phi_i) - (\Sigma_i \sigma_i^{-2} \phi_i \phi_i)^{-1} \,.\, \sigma_i^{-1} \phi_i \}]^2$$

which is a sum of squares and cannot be negative, nor in general zero unless u_i be constant. Hence any assignment of weights other than $w_i \sigma_i^2 =$ const. will increase $\sigma^2(\mathbf{1} \,.\, \mathbf{a})$ and thus diminish the precision.[87]

69. *Some Comments.* Throughout these demonstrations it has been assumed that it is the mean values E_{mi} which fit the "law" exactly. This tacitly assumes that the means are the truly representative values. It is well known in statistical work that this assumption may not be true, that the median or the mode or the geometric mean may be preferable. In the same manner although we have used the variance as a measure of scatter, other measures may be preferred. It can only be remarked that the method of least squares does in fact use means and variances. This may be due largely to the fact that those quantities are the easiest to manipulate algebraically. If we had to reduce observations in which the geometric means were considered as better than arithmetic means we should presumably regard the observation as log E_{ji} rather than as E_{ji} and fit some expression to log E_{ji} adjusting the weights inversely as the squares of the variances of log E_{ji}, and consider this as the best approximation we could make to the conditions (i), (ii), (iii′), (iv). There is, however, one assumption that might perhaps be bettered, and that is (ii) the replacement of S_m by S_j. The quantity S_j averages at S_m, but a specific value of S_j owing to the skewness of the distribution of S_j may most probably be less than S_m.

The variance of S_j may readily be found. This will give an estimate of the error made by replacing S_j by S_m and is of more importance than to determine the most probable value of S_j which would depend on third moments, would be complicated, and we should still need some estimate of variability due to sampling.

$$S_j = \Sigma_i \delta_{ji}^2 - \Phi^{-1} : \Sigma_i \phi_i \delta_{ji} \Sigma_i \phi_i \delta_{ji}$$

$$
\begin{aligned}
S_j^2 = {} & \Sigma_i \delta_{ji}^4 + \Sigma_{ih} \delta_{ji}^2 \delta_{jh}^2 \\
& \qquad - 2\Phi^{-1} : [\Sigma_i \phi_i \phi_i \delta_{ji}^4 + \Sigma_{ih} \phi_h \phi_h \delta_{ji}^2 \delta_{jh}^2 + \cdots] \\
& + \Phi^{-1}\Phi^{-1} \vdots [\Sigma_i \phi_i \phi_i \phi_i \phi_i \delta_{ji}^4 + \Sigma_{ih} (\phi_i \phi_i \phi_h \phi_h \\
& \qquad + \phi_i \phi_h \phi_i \phi_h + \phi_i \phi_h \phi_h \phi_i) \delta_{ji}^2 \delta_{jh}^2 + \cdots]
\end{aligned} \tag{68}
$$

where the terms designated by $\cdots$ involve a δ linearly and vanish in the mean over j. Now

$$\Sigma_{ih} \phi_i \phi_i \phi_h \phi_h = \Sigma_i \phi_i \phi_i \Sigma_h \phi_h \phi_h - \Sigma_i \phi_i \phi_i \phi_i \phi_i$$

and

$$
\begin{aligned}
& \Phi^{-1}\Phi^{-1} \vdots (\Sigma_{ih} \phi_i \phi_i \phi_h \phi_h + \phi_i \phi_h \phi_i \phi_h + \phi_i \phi_h \phi_h \phi_i) \\
& \qquad = \Phi^{-1} \times \Phi^{-1} \vdots \Sigma_{ih} \phi_i \phi_i \phi_h \phi_h
\end{aligned}
$$

where the sign × again denotes the specially defined product which is symmetrical in the units. Hence

$$
\begin{aligned}
\overline{S_j^2} = {} & n\mu_4 + n(n-1)\sigma_4 - 2k\mu_4 - 2k(n-1)\sigma^4 \\
& + \Phi^{-1} \times \Phi^{-1} \vdots \Sigma_i \phi_i \phi_i \Sigma_i \phi_i \phi_i \sigma^4 \\
& - \Phi^{-1} \times \Phi^{-1} \vdots \Sigma_i \phi_i \phi_i \phi_i \phi_i \sigma^4 + \Phi^{-1}\Phi^{-1} \vdots \Sigma_i \phi_i \phi_i \phi_i \phi_i \mu_4
\end{aligned} \tag{68'}
$$

A simple way to calculate $\Phi^{-1} \times \Phi^{-1} \vdots \Phi\Phi$ is to consider that Φ is expressed in terms of.orthogonal units as

$$\Phi = c_1 \mathbf{u}_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 \mathbf{u}_2 + \cdots + c_k \mathbf{u}_k \mathbf{u}_k, \quad \Phi^{-1} = \frac{\mathbf{u}_1 \mathbf{u}_1}{c_1} + \frac{\mathbf{u}_2 \mathbf{u}_2}{c_2} + \cdots + \frac{\mathbf{u}_k \mathbf{u}_k}{c_k}$$

$$\Phi^{-1} \times \Phi^{-1} = 3\Sigma_i \frac{\mathbf{u}_1 \mathbf{u}_1 \mathbf{u}_1 \mathbf{u}_1}{c_i^2} + \Sigma_{ih} \frac{\mathbf{u}_i \mathbf{u}_i \mathbf{u}_h \mathbf{u}_h + \mathbf{u}_i \mathbf{u}_h \mathbf{u}_i \mathbf{u}_h + \mathbf{u}_i \mathbf{u}_h \mathbf{u}_h \mathbf{u}_i}{c_i c_h}$$

Multiplied by $\Phi\Phi$ the last two terms give nothing; the first term gives $3k$ and the second $k(k-1)$. Moreover as $\Sigma_i \phi_i \phi_i \phi_i \phi_i$ is completely symmetrical the result of applying $\Phi^{-1} \times \Phi^{-1}$ is the same as that of applying $3\Phi^{-1}\Phi^{-1}$. Hence

$$
\begin{aligned}
\overline{S_j^2} = {} & [n - 2k + \Phi^{-1}\Phi^{-1} \vdots \Sigma_i \phi_i \phi_i \phi_i \phi_i]\mu_4 \\
& + [n(n-1) - 2k(n-1) + k^2 + 2k - 3\Phi^{-1}\Phi^{-1} \vdots \Sigma_i \phi_i \phi_i \phi_i \phi_i]\sigma^4
\end{aligned} \tag{69}
$$

Subtracting $\overline{S_j^2} = (n-k)^2 \sigma^4$ gives

$$\sigma_S^2 = [n(\beta_2 - 1) - 2k(\beta_2 - 2) + (\beta_2 - 3)\Phi^{-1}\Phi^{-1} \vdots \Sigma_i \phi_i \phi_i \phi_i \phi_i]\sigma^4 \tag{70}$$

The last term varies as $1/n$ and vanishes when the distribution of the δ's is normal; it is presumably of small account in any case. Hence one may write approximately

$$S \pm \sigma_S = [n - k \pm \sqrt{(\beta_2 - 1)n - (\beta_2 - 2)k}\,]\sigma^2 \qquad (71)$$

which in the normal case reduces as is well known to

$$S \pm \sigma_S = (n - k)[1 \pm \sqrt{2}/\sqrt{n - k}\,]\sigma^2 \qquad (71')$$

If however the δ's were distributed upon a frequency function which gave $\beta_2 = 6$

$$S \pm \sigma_S = [n - k \pm \sqrt{5n - 8k}\,]\sigma^2$$

It may therefore be inferred that in most cases arising in fitting curves by least squares the determination of the variances of the unknowns is at best tolerably poor.

70. *The Non-linear Case.* Non-linearity may arise with respect to the unknowns or the observations or both. In fitting Ce^{nt} to E by minimizing $\Sigma(Ce^{nt} - E)^2$ the observations E enter linearly, so does the unknown C, but the unknown n enters non-linearly. If one takes logarithms and fits $nt + a$ where $a = \log C$ to $\log E$ by minimizing $\Sigma(nt + a - \log E)^2$ the two unknowns enter linearly but the observations enter non-linearly; and if it be desired to use weights so as better to approach the solution of the former case by minimizing $\Sigma E^2(nt + a - \log E)^2$ the quantity E enters in an additional manner. The assumptions which have been introduced will not apply identically to the several cases. Thus if the means E_{mi} of E_{ji} lie precisely on a curve Ce^{nt_i}, the means of $\log E_{ji}$ will not in general lie precisely on a line $nt_i + a$ because the mean of the logarithm is not the logarithm of the mean. Also the condition that $w_i\sigma^2_i$ be independent of i will not apply precisely to the two cases. A frequent non-linear case is that of the minimization of the sum of the squares of the relative residuals; another arises if one undertakes to minimize Pearson's χ^2.

The sum S_j to be minimized is always a sum over i of a function S_{ji} of the unknowns $a_1, a_2, \cdots, a_k$, symbolized as $\mathbf{a}$, and of E_{ji} with the characteristic that there exists a value $\mathbf{a}_0$ such that $S_{mi}(\mathbf{a}_0, E_{mi})$ vanishes, i. e., we assume the means E_{mi} all lie on the locus fitted to the observations. The function S_{ji}, except when it vanishes, is positive. An expression which seems to include all those which might arise in least squares is

$$S_{ji}(a_1, \cdots a_k, E_{ji}) = w_i(a_1, \cdots, a_k, E_{ji})[f_i(a_1, \cdots, a_k) - \varphi(E_{ji})]^2 \qquad (72)$$

where w_i and f_i contain also the index i. The function w_i is positive and $f_i(a_{10}, \cdots, a_{k0},) - E_{mi} = 0$ for every i. We may expand S_{ji}

about the values $\mathbf{a}_0$ and E_{mi} by Taylor's Theorem. The derivatives of S_{ji} for $\mathbf{a} = \mathbf{a}_0$ and $E_{ji} = E_{mi}$ are readily computed

$$\begin{aligned}
\left(\frac{\partial S_{ji}}{\partial a_h}\right)_0 &= \left(\frac{\partial w_i}{\partial a_h}\right)_0 [f_{i0} - \varphi(E_{mi})]^2 + 2w_{i0}[f_{i0} - \varphi(E_{mi})]\left(\frac{\partial f_i}{\partial a_h}\right)_0 = 0 \\
\left(\frac{\partial S_{ji}}{\partial E_{ji}}\right)_0 &= \left(\frac{\partial w_i}{\partial E_{ji}}\right)_0 [f_{i0} - \varphi(E_{mi})]^2 - 2w_{i0}[f_{i0} - \varphi(E_{mi})]\left(\frac{d\varphi}{dE_{ji}}\right)_0 = 0 \\
\left(\frac{\partial^2 S_{ji}}{\partial a_h^2}\right)_0 &= \left(\frac{\partial^2 w_i}{\partial a_h^2}\right)_0 [f_{i0} - \varphi(E_{mi})]^2 + 4\left(\frac{\partial w_i}{\partial a_h}\right)_0 [f_{i0} - \varphi(E_{mi})]\left(\frac{\partial f_i}{\partial a_h}\right)_0 \\
&\quad + 2w_{i0}\left(\frac{\partial f_i}{\partial a_h}\right)_0^2 + 2w_{i0}[f_{i0} - \varphi(E_{mi})]\left(\frac{\partial^2 f_i}{\partial a_h^2}\right)_0 = 2w_{i0}\left(\frac{\partial f_i}{\partial a_h}\right)_0^2 \qquad (73) \\
\left(\frac{\partial^2 S_{ji}}{\partial a_h \partial a_g}\right)_0 &= 2w_{i0}\left(\frac{\partial f_j}{\partial a_h}\right)_0 \left(\frac{\partial f_i}{\partial a_g}\right)_0 \\
\left(\frac{\partial^2 S_{ji}}{\partial a_h \partial E_{ji}}\right)_0 &= -2w_{i0}\left(\frac{\partial f_i}{\partial a_h}\right)_0 \left(\frac{d\varphi}{dE_{ji}}\right)_0 \\
\left(\frac{\partial^2 S_{ji}}{\partial E_{ji}^2}\right)_0 &= 2w_{i0}\left(\frac{d\varphi}{dE_{ji}}\right)_0^2
\end{aligned}$$

In vector notation if we designate differentiation by a by ∇ and by E_{ji} by D and differentiation by both by $\square$, we may write

$$\begin{aligned}
(\square\square S_{ji})_0 = {} & (\nabla\nabla S_{ji})_0 + (\nabla D S_{ji})_0 \mathbf{u}_{k+1} + \mathbf{u}_{k+1}(\nabla D S_{ji})_0 \\
& + (DDS_{ji})_0 \mathbf{u}_{k+1}\mathbf{u}_{k+1} = 2w_{i0}(\nabla f_i \nabla f_i)_0 \\
& - 2w_{i0}(D\varphi)_0[\nabla f_i \mathbf{u}_{k+1} + \mathbf{u}_{k+1}\nabla f_i]_0 + 2w_{i0}(D\varphi)^2 \mathbf{u}_{k+1}\mathbf{u}_{k+1}
\end{aligned}$$

where ∇ carries the units $\mathbf{u}_1, \cdots, \mathbf{u}_k$ assigned to $a_1, \cdots, a_k$ and $\mathbf{u}_{k+1}$ is the unit assigned to E_{ji}. It is observable that

$$(\nabla\nabla S_{ji})_0(DDS_{ji})_0 = (\nabla D S_{ji} \nabla D S_{ji})_0 \qquad (74)$$

and it will be assumed that this relation is satisfied by any function S_{ji} which enters into the specification of a least squares problem.

Now the sum to be minimized may be written with $\boldsymbol{a} = \mathbf{a} - \mathbf{a}_0$ as

$$S_j = \tfrac{1}{2}\boldsymbol{a}\boldsymbol{a} : \Sigma_i(\nabla\nabla S_{ji})_0 + \boldsymbol{a} \,.\, \Sigma_i \delta_{ji}(\nabla D S_{ji})_0 + \tfrac{1}{2}\Sigma_i \delta_{ji}^2 (D^2 S_{ji})_0 + \cdots$$

where the derivatives being taken for the values $\boldsymbol{a} = 0$ and $\delta_{ji} = 0$ are definite constant quantities, though unknown because the means E_{mi} and the value $\mathbf{a}_0$ are unknown. The expression is therefore only of theoretical value as a base from which to work. If the expansion be stopped with the quadratic terms the results are as in the linear case.

$$\boldsymbol{a}_j = -\,(\Sigma_i \nabla\nabla S_{ji})_0^{-1} \cdot \Sigma_i \delta_{ji} (\nabla D S_{ji})_0 \tag{75}$$

As δ_{ji} occurs linearly, the mean of $\boldsymbol{a}_j$ is zero or $\mathbf{a}_j = \mathbf{a}_0$. For the variance

$$\boldsymbol{a}_j \boldsymbol{a}_j = (\Sigma_i \nabla\nabla S_{ji})_0^{-1} (\Sigma_i \nabla\nabla S_{ji})_0^{-1} : \Sigma_i \delta_{ji}^2 (\nabla D S_{ji})_0 \Sigma_i (\nabla D S_{ji})_0$$

$$\overline{\boldsymbol{a}_j \boldsymbol{a}_j} = \mu_{2a} \doteq (\Sigma_i \nabla\nabla S_{ji})_0^{-1} (\Sigma_i \nabla\nabla S_{ji})_0^{-1} : \Sigma_i (\nabla D S_{ji})_0 (\nabla D S_{ji})_0 \sigma_i^2 \tag{76}$$

There is, in this form, no indication of cancellation whatever the value of σ_i^2. We may, however, use (74). Then

$$\mu_{2a} = (\Sigma_i \nabla\nabla S_{ji})_0^{-1} (\Sigma_i \nabla\nabla S_{ji})_0^{-1} : \Sigma_i (\nabla\nabla S_{ji})_0 (D^2 S_{ji})_0 \sigma_i^2 \tag{77}$$

Now if it be assumed that $(D^2 S_{ji})_0 \sigma_i^2 = w\sigma^2$ is constant

$$\mu_{2a} = (\Sigma_i \nabla\nabla S_{ji})_0^{-1} w\sigma^2 \tag{77'}$$

Similarly with the use of (74) one may show that the mean of S_j is

$$S_m = \tfrac{1}{2}(n-k) w\sigma^2 \quad \text{and} \quad \mu_{2a} = \frac{S_m}{n-k} (\Sigma_i \tfrac{1}{2} \nabla\nabla S_{ji})_0^{-1} \tag{78}$$

The solution is as in the former case with $(\Sigma_i \frac{1}{2} \nabla\nabla S_{ji})_0$ taking the place of $\Sigma_i \phi_i \phi_i$ or $\Sigma_i w_i \phi_i \phi_i$. The quantities S_m and $\Sigma_i (\nabla\nabla S_{ji})_0$ being both unknown can only be replaced by values presumably near to them, viz., the actual sum S_j and the second derivative $\nabla\nabla S_j$ taken at the values of the unknowns which minimize S_j. But there is a difference here which may be noted; in the linear case $\nabla\nabla S_j$ is constant, i. e., independent of the sample, whereas in the non-linear case it depends on the sample.

For the general form of function (72) assumed for least squares

(iii″) $(D^2 S_{ji})_0 = 2 w_{i0} (D\varphi)_0^2$ and $w_{i0} (D\varphi)_0^2 \sigma_i^2$ must be constant.

Thus the values of the variances in the n universes do not vary inversely with the values of the weights unless $D\varphi$ is constant. For example if one were fitting $nt + a$ to log E without weights one should expect that σ_i^2 varied inversely as $(1/E)^2$ or directly as E^2; but if one takes the weights as E^2, then σ_i^2 should be constant. These results are as might be expected. If one were minimizing

$$\Sigma \left(\frac{1}{a_j} - \frac{1}{E_{ji}} \right)^2, \; a_j \text{ being the harmonic mean of } E_{ji},$$

the variances σ_i^2 must be supposed to vary as E^4; if in fact the variances were the same in the different universes one would have to use weights as E^4 if he would apply the ordinary formulas for the variance of the unknowns.

71. *A Second Approximation.* The next step is to remove the restriction of the expansion to terms of the second order. As the formulas become involved it will be desirable to limit the discussion to the case of one variable. Further let us suppress the S_{ji} and the subscript 0 and write $(\nabla\nabla S_{ji})_0$ as simply ∇_i^2, $(\nabla DS_{ji})_0$ as ∇D_i, etc., keeping the index i to indicate that the values are different for different indices i though taken for $a = a_0$ and $E_{ji} = E_{mi}$ and thus independent of the sample. (The symbol ∇, retained to denote differentiation by a, has now no vectorial significance.) Then, including terms of the third order,

$$S_j = \tfrac{1}{2}\alpha^2\Sigma_i\nabla_i^2 + \alpha\Sigma_i\delta_{ji}\nabla D_i + \tfrac{1}{2}\Sigma_i\delta_{ji}^2D_i^2 + \tfrac{1}{6}\alpha^3\Sigma_i\nabla_i^3$$
$$+ \tfrac{1}{2}\alpha^2\Sigma_i\delta_{ji}\nabla^2D_i + \tfrac{1}{2}\alpha\Sigma_i\delta_{ji}^2\nabla D_i^2 + \tfrac{1}{6}\Sigma_i\delta_{ji}^3D_i^3$$

Differentiate and set the derivative equal to zero:

$$0 = \alpha_j\Sigma_i\nabla_i^2 + \Sigma_i\delta_{ji}\nabla D_i + \tfrac{1}{2}\alpha_j^2\Sigma_i\nabla_i^3 + \alpha_j\Sigma_i\delta_{ji}\nabla^2D_i + \tfrac{1}{2}\Sigma_i\delta_{ji}^2\nabla D_i^2$$

For the solution put $dS_j = 0$. The first two terms give for α_j the value (75) which may be called α_{j1}. If we set $\alpha_j = \alpha_{j1} + \varepsilon_j$ we need retain ε only in the first term. Hence

$$\alpha_j = -\frac{\Sigma_i\delta_{ji}\nabla D_i}{\Sigma_i\nabla_i^2} - \frac{1}{2}\,\frac{(\Sigma_i\delta_{ji}\nabla D_i)^2\Sigma_i\nabla_i^3}{(\Sigma_i\nabla_i^2)^3}$$
$$+ \frac{\Sigma_i\delta_{ji}\nabla D_i\Sigma_i\delta_{ji}\nabla^2D_i}{(\Sigma_i\nabla_i^2)^2} - \frac{1}{2}\,\frac{\Sigma_i\delta_{ji}^2\nabla D_i^2}{\Sigma_i\nabla_i^2}$$

$$\bar{\alpha}_j = -\frac{1}{2}\,\frac{\Sigma_i(\nabla D_i)^2\sigma_i^2\Sigma_i\nabla_i^3}{(\Sigma_i\nabla_i^2)^3} + \frac{\Sigma_i\nabla D_i\nabla^2D_i\sigma_i^2}{(\Sigma_i\nabla_i^2)^2} - \frac{1}{2}\,\frac{\Sigma_i\nabla D_i^2\sigma_i^2}{\Sigma_i\nabla_i^2}$$

The mean value of α_j is no longer zero, but involves the squares of σ_i. The mean of α_{j1}^2 is known, that of α_j^2 may be had by adding that of $2\alpha_{j1}\varepsilon_j$

$$\alpha_{j1}\varepsilon_j = \frac{1}{2}\,\frac{(\Sigma_i\delta_{ji}\nabla D_i)^3\Sigma_i\nabla_i^3}{(\Sigma_i\nabla_i^2)^4} - \frac{(\Sigma_i\delta_{ji}\nabla D_i)^2\Sigma_i\delta_{ji}\nabla^2D_i}{(\Sigma_i\nabla_i^2)^3}$$
$$+ \frac{1}{2}\,\frac{\Sigma_i\delta_{ji}\nabla D_i\Sigma_i\delta^2_{ji}\nabla D_i^2}{(\Sigma_i\nabla_i^2)^2}$$

$$\overline{\alpha_j^2} = \frac{\Sigma_i(\nabla D_i)^2\sigma_i^2}{\Sigma_i\nabla_i^2} + \frac{\Sigma_i(\nabla D_i)^3\mu_{3i}\Sigma_i\nabla_i^3}{(\Sigma_i\nabla_i^2)^4} - 2\,\frac{\Sigma_i(\nabla D_i)^2(\nabla^2 D_i)\mu_{3i}}{(\Sigma_i\nabla_i^2)^3} + \frac{\Sigma_i\nabla D_i\nabla D_i^2\mu_{3i}}{(\Sigma_i\nabla_i^2)^2}$$

A certain amount of change in this relation may be made by (74) but without any evident simplification. It is clear that $\overline{\alpha_j^2}$ involves the third derivatives of the function S and the third moments μ_{3i} of the δ's. To obtain σ_a^2 we should substract $\overline{\alpha_j^2}$ which would involve σ_i^4. Now although $\mu_{3i} = \beta_i\sigma_i^3$ may be small because β_i may be, so that terms in σ_i^4 would not be negligible compared with those in $\beta_i\sigma_i^3$, it is clear that the next approximation will bring in additional terms in $\mu_{4i} = \beta_{2i}\sigma_i^4$ and in σ_i^4 so that it is unnecessary to include terms in σ_i^4 until that approximation is reached. If, however, we are to have convergence which is rapid it is clear that terms like

$$\sigma^3\,\frac{\Sigma_i(\nabla D_i)^3\Sigma_i\nabla_i^3}{(\Sigma_i\nabla_i^2)^4},\qquad \sigma^3\,\frac{\Sigma_i(\nabla D_i)^2(\nabla^2 D_i)}{(\Sigma_i\nabla_i^2)^3},\qquad \sigma^3\,\frac{\Sigma_i\nabla D_i\nabla D_i^2}{(\Sigma_i\nabla_i^2)^2}$$

must be small compared with $\sigma^2\Sigma_i(\nabla D_i)^2/(\Sigma_i\nabla_i^2)^2$, and so on to higher orders. The third derivatives in terms of the expression (72) are

$$\nabla_i^3 = 6\left(\frac{dw_i}{da}\right)_0\left(\frac{df_i}{da}\right)_0^2 + 6w_{i0}\left(\frac{df_i}{da}\right)_0\left(\frac{d^2f_i}{da^2}\right)_0,\ \text{etc.},$$

that is, they are not simple expressions. That the correction terms are extremely complicated even in the simplest case is therefore apparent; if one goes on to take terms of the fourth order and more unknowns than one, the complications become very formidable and it is extremely doubtful whether there would be any practical advantages in pursuing the matter further.

The result is that we may infer that the questions discussed in Arts. 27–29 relative to the expansion of the variation before squaring and to the exclusion or inclusion of certain presumably small terms in the true expansion of S and to the exclusion or inclusion of such terms in the Hessian of S which is used to express the value of the variance of the unknowns cannot be answered without comparing in detail expressions which are so complicated that comparison in general appears impracticable. It may be assumed that for any practical case the values of the variances might equally well be taken from either expression because in all cases where the expressions differed appreciably it would be extremely doubtful whether either of them could be considered adequate without an elaborate discussion of terms of higher order in the expansions for those variances. This corresponds to our actual findings in all cases we have examined,—there is little difference in the immediate vicinity of the minimum between the inclusion or exclusion of the extra terms. The artificial example which we concocted[47] to show that there might be such great differences that the solution could be found by one set of equations, but not by the other, while informative for that purpose, does not upset the statements just made because we were fitting $y = mx + e^{\beta}$ or $y = mx + c^2$ to the points $(0, -1)$ and $(1, 1)$ with the solution as $y = x$. Now if the problem be set up as a sampling problem we should assume either that (0, 1) and (1, 1) were merely representative points from universes $(0, -1 + \varepsilon)$, $(1, 1 + \eta)$, in which case the means $y = -1$ and $y = 1$ do not lie on a line of the type considered, or that the points $(0, -1)$ and $(1, 1)$ were sample variations from some means (0, a), (1, b), in which case if a be negative the means do not lie on the type line whereas if a be positive the variations to -1 leads across a discontinuity in representation by reals and is scarcely a permissible variation.

72. *Germany and Chicago.* We may finally discuss the cases of Germany and Chicago. Monk and Jeter pointed out that for Chicago the constants of the augmented logistic turned out imaginary for some choices of the four points, but real for the other choices. We have shown that for Germany the constants appeared to be imaginary and have raised the question as to whether the populations could be regarded as fluctuations from a (real) augmented logistic. Now there is this difference between the two cases: A least squares procedure applied to the case of Chicago leads to a minimum with real values of the constants whereas applied to the case of Germany it leads out to the critical case and a much better fit of the imaginary augmented logistic can be found. We should therefore consider that the logistic was truly real for the case of Chicago and that the fact that imaginary values were obtained for the constants when certain points were used to determine them did not preclude us from considering the populations as fluctuations from a real logistic, but rather indicated that sort of anomaly which is not uncommon in statistical work, namely, the accumulation of errors due to statistical fluctuation.[88] In the case of Germany, however, it would appear that the augmented logistic is truly imaginary and that if for

a set of four equally spaced censuses we should happen to get real values for the constants those values would have to be interpreted as due to accumulation of errors due to statistical fluctuations. In other words it seems reasonable to assume that if we had an infinity of hypothetical Chicagos the censuses for them at each date could naturally be assumed to have means which might lie on a real augmented logistic from which the actual censuses for one actual Chicago would vary by fluctuations that would be in the nature of sampling fluctuations; but that if we had an infinity of hypothetical Germanys it would be an imaginary augmented logistic that would have to be assumed as the locus of the line of means. At all events we hope we have made it clear that this sort of hypothetical sampling set up is at the basis of the method of least squares and must be kept in our mental background in thinking about the method. One further and final word: Granted the hypothesis about the Chicagos as a sampling problem, one could raise the question as to whether some sample Chicago might not turn up with a set of censuses for which the best logistic would be imaginary. There seems to be no reason why such might not be the case; but if it were, the result of working with that solution would be wholly misleading so far as revealing the behavior of the mean of the universe or the future behavior of that particular sample. That is the present situation with respect to Germany. If we believe that populations grow on augmented logistics, except for variations that can be regarded as sampling fluctuations from means which lie on such curves, Germany, 1816–1910, is simply an exceptional sample which happens to fit very closely on an imaginary augmented logistic.

73. *A Mistake of Edgeworth's.* In a brochure Mathematical Representation of Statistics, dated 1900, F. Y. Edgeworth reprinted four papers[89] with an introduction. On pages 4–5 he discussed the fitting of the frequency function

$$y = \frac{N}{\pi}\,\frac{c}{c^2 + x^2} \quad \text{or} \quad y = \frac{1}{A + Bx^2}$$

to data for the heights of 25,878 American recruits. The details of the discussion are given on pages 23–25 in Note 3. Edgeworth knew, of course, that the fit would be bad at best; he knew that the standard deviation of this frequency function is infinite and the mean therefore indeterminate, indeed he probably knew that the means of samples of n from a universe with this frequency distribution are distributed on the same curve as the individual elements; he knew that means of samples of heights of recruits would show stability much greater than the heights of the individuals. It may therefore be presumed that he fitted the function merely for illustrative purposes. What he did was to write $n_r = 1/(A + Bx_r^2)$ where x_r is the distance from the center of the strips in the histogram to the mean (or median) of the whole distribution, and n_r is the number of observations falling in that strip of the histogram. Next he cleared of fractions to obtain $A + Bx_r^2 - 1/n_r = 0$ as an equation of condition,—thus getting the equations in linear form. Then he paused briefly to discuss the weights of these equations which he took on analogy to be n_r and therefore minimized

$$\Sigma n_r(A + Bx_r^2 - 1/n_r)^2 \qquad (79)$$

finding, as he states, with some surprise a negative value for A; and this solution he discussed, coming to the conclusion that the method of Least Squares showed that the proposed representation was inadmissible. He reinforced the argument for the demonstration of inadmissibility by showing that if one wrote $A + Bx_r^2 = \log n_r$, as would be natural when fitting the normal law $y = e^{A+Bx_r^2}$ and minimized

$$\Sigma n_r(A + Bx_r^2 - \log n_r)^2$$

he obtained sensible values for A and B.

The trouble is with the weighting. In the second case the variances of the universes for each value of r are by (iii″) inversely proportional to

$$w_r(D\mathfrak{p})^2 = n_r(1/n_r)^2 = 1/n_r$$

and therefore proportional to n_r. This is a reasonable assumption, because in fitting a frequency function where the actual numbers in the different strips of the histogram are supposed to result by sampling fluctuations, the variances should be approximately proportional to the numbers themselves (being Np_rq_r with q_r near 1 and $Np_r = n_r$). In the first case, however, the variances would have to be inversely proportional to

$$w_r(D\mathfrak{p})^2 = n_r(1/n_r^2)^2 = 1/n_r^3$$

and therefore proportional to n_r^3 which is not sensible. The weights are therefore not n_r as in fitting by minimizing (79); it is readily seen that the expression to be minimized should be

$$\Sigma n_r^3(A + Bx_r^2 - 1/n_r)^2 \qquad (79')$$

which will make the variances again proportional to n_r, and thus the weights of the equations $A + Bx_r^2 - 1/n_r = 0$ are n_r^3. If these weights be used we are really minimizing

$$\Sigma n_r(A + Bx_r^2)^2\left(n_r - \frac{1}{A + Bx_r^2}\right)^2 = \Sigma n_r(A + Bx_r^2)\,\frac{[n_r - (A + Bx_r)^{-1}]^2}{(A + Bx_r)^{-1}}$$

which is Pearson's χ^2 except for the multiplier $n_r(A + Bx_r^2)$ that must be nearly equal to 1 if the fit is at all good. With these weights (n_r^3 instead of n_r as used by Edgeworth) the value of A does not come out negative, it is positive—we find[90]

$$y = \frac{1}{.000236 + .00003089x^2}$$

The fit is assuredly not good, as was evident from the start, but fitting by least squares is not inadmissible, there are no paradoxical results. It is difficult to imagine how Edgeworth who was so able a mathematician and who was especially distinguished for the excellence of his judgment of situations, whether mathematical or factual, could have been misled by his erroneous solution into giving in support of it such explanations as he gave instead of seeing in it indications of its own erroneous character even to the discovery of the particular error which had been made.

REFERENCES.

[1] Read in preliminary form before the Seminar in Zoology, University of California, Berkeley, April 3, 1929; read in part before joint session of Sections A and K of the Amer. Assoc. Adv. Sci., New Orleans, December 31, 1931. Presented to the Academy, April 12, 1933.

[2a] Pearl, R. and Reed, L. J. Proc. Nat. Acad. Sci., 6, 275–288, 1920.

[2b] Knibbs, G. H. Appendix A, vol. I, Census of Commonwealth of Australia, 1–33.

[2c] DuPasquier, L. G. Vierteljahrschr. Naturforsch. Ges., 63, 1918, pp. 236–249.

[2d] Hain in Statistics of the Austrian Empire (1852). This reference is quoted from Edgeworth, J. Roy. Statist. Soc., 88, 1925, p. 59, because of his comments on it; we have not had access to it.

[3] It is curious to call the exponential law Malthusian when Malthus' chief thesis was that it could not be followed, but the usage seems established.

[4] The constant n has "dimensions" of the reciprocal of time and its magnitude varies inversely to that of the unit of time, i. e., nt is a pure number.

[5] Verhulst, P. F. Correspondence Mathematique et Physique par Quetelet,

vol. X, 1838, pp. 113–121. Nouv. Mem. Acad. Roy. Sci. Belles-Let., Bruxelles, 18, 1845, pp. 1–38; 20, 1847, pp. 1–32.

[6] We may also, as Verhulst points out, integrate if $\phi(P) = naP^{i+1}$ as $P^{-i} = a + be^{-nit}$. This is a four constant equation which reduces to the logistic when $i = 1$. For the general form the population converges to $\sqrt[i]{1/a}$ and the inflection point of the plot of P on time is when $P = \sqrt[i]{[a(i + 1)]^{-1}}$. The ratio of the population at the inflection to the asymptote is $(i + 1)^{-1/i}$ which is .5 when $i = 1$ and .577 when $i = 2$ and .630 when $i = 3$.

[7] Hotelling, H., J. Amer. Statist. Assoc., 22, 1927, pp. 283–314, works directly with the differential equation, without integration, and there are some advantages in so doing. He also has some sound remarks about probable errors of extrapolations. This paper should be read in connection with that of Schultz.[32] The reason that Schultz and Hotelling give such different probable (or standard) errors of a forecast is due to their approaching the problem from such different viewpoints; indeed the viewpoints may be considered independent so that the errors should be superposed. Schultz considers the standard errors of the parameters of an empirical formula such as the logistic when the formula is fitted to the observations by least squares, Hotelling considers the deviations arising by a gradual accumulation of independent variations in the growth rates dP/dt.

[8] Many references will be found in Pearl, R. Medical Biometry and Statistics, 2nd Ed., 1930, pp. 427–428. We shall cite specifically those we use later.

[9] Yule, G. U. J. Roy. Statist. Soc., London, 88, 1925, pp. 1–62 (with discussion by Edgeworth and others). See also Stevenson, T. H. C. *Ibid.* pp. 63–90 (with discussion by Yule, Bowley, and others).

[10] East, E. M. Mankind at the Crossroads, Chap. IV.

[11] Abstract U. S. Census of 1920, p. 584. Unfortunately for our purposes the system of land classification seems to have been so different for the 1930 census that we are unable to get a figure for that year. The table and graphs show that the figure for 1870 was out of line; we are particularly in need of the figure for 1930 in order to judge as to whether that for 1920 was similarly out of line.

[12] For discussion of growth paper see Wilson, E. B., Proc. Nat. Acad. Sci., 11, 1925, pp. 451–456.

[13] For a critical discussion reference may be made to East.[10]

[14] Beloch, J. Zeitschr. Socialwiss., 2, 18, (1899), pp. 505–514, 600–621; *Ibid.* 3, 18, 1900, pp. 405–423, 765–786.

[15] One might even surmise that instead of approaching the asymptote steadily a population would be far more likely to overshoot the asymptote and settle down to it by oscillating about it, or at least to over-shoot it and approach it from the upper side.

[16] Knibbs, G. H. J. Amer. Statist. Assoc., 21, 1926, pp. 381–398, *Ibid.*, 22, 1927, pp. 49–59.

[17] The values of n were found graphically. The figures for population were taken from Pearl,[8] p. 421 and Reed and Pearl.[34]

[18] Wilson, E. B. and Luyten, W. J. Proc. Nat. Acad. Sci., 11, 1925, pp. 137–143.

[19] Wilson, E. B. Science 61, 1925, pp. 87–89 showed this for Canada, and numerous other cases will be noted in this paper. They are often easy to recognize from the semi-log plot of the population by the upward concavity of the curve (Wilson and Luyten, footnote 7).

[20] Taken from Pearl, R. and Reed, L. J., Predicted Growth of Population of New York and Its Environs, Plan of New York and Its Environs, 1923, 42 pp. (p. 27).

[21] What this means graphically is that if the populations are plotted on semi-log paper there is no *obvious* concavity of the set of points as a whole either up or down (Art. 10).

[22] Pearl, R. and Reed, L. J., Science, 72, 1930, pp. 399–401. They did not claim a correction to the 1920 figure making it 106.1 instead of 105.7 due to the fact that the census was taken as of Jan. 1 instead of as April 15. They had a right to the benefit of this correction which would have improved their forecast.

[23] Firenczi, I. International Migrations, vol. I, 1929, pp. 374–500.

[24a] Cannan, E. Economic J., 5, 1895, pp. 505–515.

[24b] Bowley, A. L. Economic J., 34, 1924, pp. 188–192.

[24c] Dublin, L. I. Forum, 86, 1931, pp. 270–276.

[25] For Least Squares we may refer to Whittaker and Robinson, Calculus of Observations, Chap. IX. It should not be necessary to state the definition of least squares, which has been accepted for more than a century, but Pearl and Reed[2a] after fitting the logistic to three censuses precisely and computing the residuals and the square root of the sum of their squares wrote: "It must not be forgotten, however, that the root mean square error is reduced in the present case by virtue of the fact that in three out of the thirteen ordinates theory and observation are made, by the procrustean method of fitting, to coincide exactly,"—and the context seems to show that they meant *reduced* and not increased. It would seem therefore that they thought that with three zero residuals the sum of squares would be less than in a least squares fit. To us this seems wrong; we believe that if three deviations are made zero the squares of the others will add up to more than that for a least squares fit,—and we shall proceed on the assumption that Least Squares minimizes the sum of the squares.

[26] The index i ranges over the values of 0 to $k - 1$ if there are k censuses available. We shall drop this index in formulas subsequent to (12) and leave it to be understood by the reader.

[27] The $\frac{1}{2}$ is inserted before $\partial S/\partial a$ and the other derivatives to make the normal equations in the linear case come out as usually written. Thus if the linear equations are

$a_i x + b_i y + \cdots = n_i$ so that the errors are $a_i x + b_i y + \cdots - n_i$,

where $x, y, \cdots$ not $a, b, \cdots$ are the unknowns and n is the quantity observed, the derivatives of

$$S(x, y, \cdots) = \Sigma(a_i x + b_i y + \cdots - n_i)^2$$

are

$$\partial S/\partial x = 2\Sigma(a_i x + b_i y + \cdots - n_i)a_i = 2(\Sigma a_i^2 x + \Sigma a_i b_i y + \cdots - \Sigma a_i n_i)$$

$$\partial S/\partial y = 2\Sigma(a_i x + b_i y + \cdots - n_i)b_i = 2(\Sigma a_i b_i x + \Sigma b_i^2 y + \cdots - \Sigma b_i n_i)$$

etc., and the standard normal equations are $\frac{1}{2}\partial S/\partial x = 0$, $\frac{1}{2}\partial S/\partial y = 0$, etc.

[28] These "weights of the residuals" in the three equations must not be confused with the "weights of the observations" (Art. 22) in a least squares solution or with the "weights of the unknowns" (Whittaker and Robinson,[25] Art. 122).

[29] The conditions (11) have been interpreted neglecting mention of the constants $1/b$ and $1/K$ which multiply the equations. Some modification of the statement might be necessary if b or K vanished or were infinite.

[30] The method of moments is not a method of least squares for the logistic.

[31] Pearl, R. Studies in Human Biology, 1924, pp. 578–581. Our statement as to the conditions which are equivalent to his method is based on the procedure of the computation form. It seems difficult to state precisely what the conditions would be on the basis of the algebraic theory which leads up to that form because the author seems to have confused E and P and thus to have been led to perform operations which are inadmissable in least squares.

[32] Schultz, H., J. Amer. Statis. Assoc., 25, 1930, pp. 139–185. Schultz states (p. 164) that it is difficult to give meaning to the "residuals" Pearl and Reed are minimizing. We hope that the discussion in Art. 21 and the example in Art. 42, though not supplying any strict meaning to those residuals, will throw some light on the difficulty.

[33] To illustrate the difference which would arise by putting $E = P$ in the weights of the residuals we may consider the first of conditions (14). The difference

$$\Sigma E(E - P) - \Sigma P(E - P) = \Sigma(E - P)^2 = S$$

Thus if we replace $\Sigma P(E - P)$ by $\Sigma E(E - P)$ and then make the latter vanish, $\Sigma P(E - P)$ will not vanish but will be negative and equal to $-S$. In case the fit is very good S will be small and the conditions will be practically equivalent.

[34] It should not be necessary to define weights but Reed and Pearl writing in the Journal of the Royal Statistical Society of London, 90, 1927, pp. 729–746, state definitely (p. 742) "The use of absolute residuals weights each observation in proportion to its magnitude. This is in accord with the customary usage in the fitting of curves." To us this seems incorrect. We believe it would have seemed incorrect to two great predecessors of Pearl and Reed at the Johns Hopkins University—C. S. Peirce and Simon Newcomb,

who were expert in least squares and in the reduction of observations. The definitions which we give would require us to mimimize $\Sigma E(E - P)^2$ if we were to weight each observation in proportion to its magnitude.

[35] Wilson and Luyten,[18] while not actually fitting by least squares procedures, did try to render small the sum of the squares of the relative residuals (rather than that of the absolute residuals). They did not, however, advocate this method as preferable. The chief reason for their choice of relative residuals was that they were working graphically on semi-log paper with method (8) without making allowances for change of scale in the different parts of the graph. They state in their footnote 15 that they believe the Pearl-Reed values (for Greater New York) are very near to those obtained by basing the least squares criterion on actual instead of percentage differences. We shall find that they erred in this belief: The solution by actual differences gives the limiting population 23.4 whereas Pearl and Reed found 34.9. Wilson and Luyten's approximate relative difference solution assumed 16.7 whereas the least squares solution gives 16.2 (Art. 55).

[36] Pearson actually fits his frequency functions by the method of moments and then applies χ^2 as a test of goodness of fit. For a discussion of this matter see Fisher, R. A., Phil. Trans. Roy. Soc., London, Series A, 222, pp. 309–368.

[37] Of course if the fit is really close the different systems of weighting all lead to essentially the same values of the empirical constants. Whenever the fitted curves lead to distinctly different results with different systems of weighting in the least squares solution it is obvious that inferences from those results are largely dependent upon the appropriateness of different systems of weighting.

[38] Generally done graphically on semi-log paper or on ordinary paper using a logarithmic scale.

[39] Obviously the unweighted logarithmic fit is very nearly equivalent to fitting by relative residuals. It is not absolutely the equivalent of such a fit, as sometimes stated, because $E - P$ are not really infinitesimals and to be assimilated to differentials, they are finite.

[40] We drop here and hereafter, the notation $(\ )_0$ leaving it to be understood.

[41] As the ordinary conditions for a minimum are $\partial^2 S/\partial C^2 > 0$, $\partial^2 S/\partial n^2 > 0$ and $4H = \partial^2 S/\partial C^2 \cdot \partial^2 S/\partial n^2 - (\partial^2 S/\partial C \partial n)^2 > 0$, the determinant of the two equations in δC and δn cannot vanish at the minimum, nor (by virtue of continuity) for some finite though possibly small region around the minimum, and hence if we are near enough to that position the solution of the equations is possible. (It is possible to have a minimum for which the stated conditions vanish as in the case $S = c^4 + n^4$, but this requires special and very complicated considerations; fortunately it is not likely to arise in practical least squares solutions.)

[42] Whittaker and Robinson,[25] p. 214. Schultz[32] follows this same procedure.

[43] Except such an artificial case as is discussed in footnote 47.

[44] The conditions for the minimum being $\partial S/\partial C = \partial S/\partial n = 0$ our problem

is to solve those equations. It is to be noted that the terms at the extreme left of equations (22), (24), (25), are all identical, being in fact the half derivatives of S, and only the coefficients of δC and δn are affected by using the modification of the approximation equations. If then we can get from either set of equations, or from any others, values of δC and δn which make the derivatives vanish, we have solved our problem, and the simpler the equations we can use, having regard to the rapidity of the convergence of these derivatives toward 0, the better.

[45] This statement is perhaps too strong. We have never found in practice a case in which the standard errors computed from (26) with and without the correction terms dependent on $E - P$ have differed appreciably. We are not sure just what might be considered the best theory of least squares, including the standard errors of the unknowns, when the unknowns enter the equation non-linearly. We are not convinced of the appropriateness of a least squares solution in such cases or even in some cases in which the constants do enter linearly. As, however, there is a tendency to express conditions for a particular state by minimizing something, the probability is that the problem of fitting will be expressed as the minimization of some function, and it would appear as though the various estimates of goodness of fit and of the precision of the constants as determined should be in terms of the same function. The discussion offered in the Appendix seems to indicate that when the constants enter non-linearly, the problem of determining the standard deviation of the unknowns involves serial expansions of so high a degree of complexity as to make any precise determination of these standard deviations wholly impractical and further that the use or rejection of the extra terms would ordinarily not be a matter of importance.

[46] Whittaker and Robinson,[25] p. 246. If a_{ij} are the coefficients of the normal equations and D their determinant and A_{ij} their cofactors, the expressions for the standard errors of the unknowns involve the ratios $A_{ii} : D$. We may speak not only of the standard errors of the unknowns but of their correlation or of their mean products, which obviously must involve $A_{ij} : D$ where $i \neq j$. In case we solve for a and σ_a we may get $K = 1/a$ and $\sigma_K = \sigma_a/a^2$; and if we solve for B and C we have $K = B/C$ immediately but we cannot get σ_K from B, C, σ_B, σ_C—we must have also the mean product Π_{BC} or the correlation coefficient r_{BC}. We shall have occasion to speak qualitatively of the correlation between the variables but shall not need to develop the quantitative side of the matter as might readily be done following Whittaker and Robinson, Art. 171, or A. L. Bowley, Elements of Statistics, p. 405–407. These authors treat correlation as a probability problem, but G. U. Yule, Introduction to the Theory of Statistics, does not thus restrict it, and Whittaker and Robinson give the treatment of least squares without such restrictions. We have really to do with properties of quadratic forms or symmetrical systems of linear equations, and in particular with the "reciprocal properties" of such forms or systems. The details of the discussion may be left to the Appendix.

[47] If we try to fit $y = mx + b$ to some points and choose to write $y = mx + e^{\beta}$ we may get the wrong solution as may be seen by taking $x = 0$, $y = -1$ and $x = 1$, $y = 1$ as the (two) points. The line $y = mx + b$ is $y = 2x - 1$, and $S = 0$ because the fit is perfect. With $y = mx + e^{\beta}$ we must have a minimum, either actually attained or approached as a limit, for $S = (mx + e^{\beta} - y)^2$ because this quantity is necessarily positive or zero and is continuous. The solution is $\beta = -\infty$, $m = 1$, $y = x$, $S = 1$. For any value of m other than 1 and for any value of β other than $-\infty$ the sum is greater than 1. The minimum is in a sense a false minimum since $\beta = -\infty$ is not truly a value for β, but the line $y = mx + e^{\beta}$ approaches the definite position $y = x$ as m approaches 1 and as β becomes negatively infinite, and S approaches the limit 1. If however we attempt to find this minimum by using the proper approximation equations we shall find that $\delta\beta$ is near -1 when β is large and negative and m is near 1,—in other words we cannot find the minimum without an infinite number of steps.

If one objects to the value $\beta = -\infty$, the principle may be illustrated by taking $y = mx + c^2$ as the form to fit. Then

$$\tfrac{1}{2}\partial S/\partial c = 4c^3 + 2mc, \quad \tfrac{1}{2}\partial S/\partial m = m + c^2 - 1$$

The solution is $c = 0$ and $m = 1$ and the line is $y = x$. Also

$$\tfrac{1}{2}\partial^2 S/\partial c^2 = 12c^2 + 2m, \quad \tfrac{1}{2}\partial^2 S/\partial c\partial m = 2c, \quad \tfrac{1}{2}\partial^2 S/\partial m^2 = 1$$

The minimum is a true minimum because for $c = 0$, $m = 1$ the first and last second derivatives are positive and the determinant H is also positive. The approximation equations for finding the minimum are

$$4c^3 + 2mc + (12c^2 + 2m)\delta c + 2c\delta m = 0, \quad m + c^2 - 1 + 2c\delta c + \delta m = 0$$

and δc and δm converge properly toward 0 with c and m converging toward 0 and 1 respectively. The formulas for the standard deviations give definite results. If however we expand before squaring and reject δc^2 we have

$$\begin{aligned} S &= (c^2 + 2c\delta c + 1)^2 + (m + \delta m + c^2 + 2c\delta c - 1)^2 \\ &= 2c^4 + m^2 - 2m + 2 + 2mc^2 + (8c^3 + 4cm)\delta c \\ &\quad + (2m + 2c^2 - 2)\delta m + 8c^2\delta c^2 + 4c\delta c\delta m + \delta m^2 \end{aligned}$$

The "normal" equations are

$$4c^3 + 2cm + 8c^2\delta c + 2c\delta m = 0, \quad m + c^2 - 1 + 2c\delta c + \delta m = 0$$

and $\delta c = -c/2 - 1/2c$, $\delta m = -m + 2$; there is no convergence to the minimum at $c = 0$ and $m = 1$. These illustrations show that the minimum in a least squares problem may depend on the form in which the constants enter into the equation to be fitted. When the constants enter linearly, as is usual, the expression to be minimized is a quadratic form in the constants and has a definite minimum because it is always positive or zero and becomes indefinitely large as the values of the constants retreat indefinitely from the values which give the minimum. When the constants do not enter linearly the function to be minimized though always positive or zero is not quadratic, there may be no minimum (except at infinity), and the empirical expression may not be fittable (with real values of the constants) to a number of points equal to the number of constants.

[48] Such a situation arises frequently in fitting logistics, the value of S may depart very little from its minimum and yet the errors in the unknowns may be considerable.

[49] If we are interested solely in the closeness of fit over the observed range of data, we can estimate the convergence by the behavior of S and if we do not care about a least squares fit we may be content with a value of S considerably removed from the minimum; but if we desire to extrapolate the curve we need to know the values of the constants, particularly if we desire to estimate the asymptotic population. Not only is the elongation of the ellipsoid for the logistic often extreme, but we shall see that the shape of the range of values for the constants over which S hovers near its minimum may be far from elliptical (Art. 53).

[50] Population figures taken from Yule.[9] Time in decades measured from the initial year, here and elsewhere.

[51] We give in this table and in subsequent tables the figures we got without claiming, and indeed under the distinct understanding that we disclaim, validity for the last few places. Thus in our second least squares procedure the corrections to a and n were only about 1.5% of the next previous corrections, whereas the correction to b was 12%. The second values of δa, δb, δn may presumably be taken to be on the whole decidedly larger than the errors in the final result if further approximations were made, but it cannot be assumed that each of the three is decidedly larger as it might be that one or even two of them had happened exceptionally to be very small. In respect to the value of S it should be especially remarked that although the enumerated populations E have been taken to a limited number of places (to tens of thousands in the present case following Yule, but generally to thousands) more places have generally been carried in the computed populations P in order to give better possibilities of estimating the change in S.

[52] It must not be overlooked that in calculating the determinant D which goes in the denominator of the formula for the standard errors[46] we lose a considerable number of places and hence the values of the standard errors are not particularly good. These errors were calculated from the coefficients of the simplified equations

[53] Population figures taken from Pearl[31] (p. 607).

[54] Time measured from 1861 in our figures and from 1800 in Pearl's. We have fitted to the six censuses 1861–1910; Pearl states that he fitted from 1855 on and we should interpret this to mean that he used the data for 1855, but the table gives no evidence to support such an interpretation except that if we include 1855 the sum of the differences for Pearl's solution is only −.0011 and thus comes near vanishing as it should when an additive constant is

available; however, in the tabulations he does not include the calculated value of his fit for the year 1855 and as it is seen that the error of the fit for that year is considerably larger than for any of the years tabulated we may infer that probably he did not fit to that year.

[55] To remind the reader that not all the places written are good we have in these four cases underlined (italicized) the places which we believe to be good. In making another least squares approximation we should start with all the places given as the values of the constants are highly correlated and a good many places are lost in the work and as there is very little saving in machine calculation by reducing the number of places. The values of σ in the last column are $\sqrt{S/k}$, where k is the number of censuses and thus give the standard deviation of the residuals. (The value $\sigma' = \sqrt{S/(k-3)}$ would of course have to be used to estimate the "error to be feared.") The 1920 populations of the four regions were respectively, 1.4, 3.2, 10.4, 14.9 so the percentage errors based on the terminal populations are respectively 1.6, 1.3, 2.3, 1.4; they would be higher if based on the mean, median or mid population as during the period 1790–1920 the populations have increased by the respective factors 6, 17, 30, 20.

[56] How is it that Pearl and Reed will offer as a least squares fit, or even as a satisfactory fit, one which can be bettered by a graphic Malthusian, one which is hypomalthusian whereas the series of populations plotted on semi-log paper is distinctly concave up, one which though containing three constants cuts the curve of observations (minor fluctuations excepted) in only two instead of in three or more points, in fine, one which satisfies none of the qualitative inspection criteria that would be applied to see whether or not the fit was at all possible as a least squares solution let alone the quantitative criteria (11) which might not be known to them? These qualitative criteria give the same indications for any part of the period 1790–1920 ending with 1920.

The situation with respect to New Jersey and the other regions is not quite so bad. The following is a summary of the Reed and Pearl asymptotic populations obtained (p. 735) in "fitting each of these populations with separate logistic curves by using the theory of least squares, with absolute residuals," and the values we find (in millions).

	K(R. and P.)	K(Correct)	Comment
Connecticut	8.771	− .610	Curve infinite in finite time
New Jersey	15.053	−38.920	Curve infinite in finite time
New York	26.596	25.370	Nearly right
Three States	49.510	77.330	Error is −36%

The other constants needed for their summation theory are also naturally wide of the mark, some of them being indeed properly imaginary because of the form in which they took their logistic (Art. 30); that is, the values one should find in the cases of Connecticut and New Jersey for their a, b, K (which correspond to our log b, n, $1/a$) from the proper fitting by least squares of a logistic in any form which is universally applicable would make their a imaginary because our b is negative. But if the logistic is taken in their form and a least squares procedure is set up for that form, no imaginary value of their a can be found, and presumably no real value except $+\infty$ or $-\infty$ (see footnote 47 and Arts. 47–49). We should expect therefore that, with infinite repetition the least squares procedure would converge toward the Malthusian Ce^{nt}, and in fact the best Malthusian fitted to the whole period has $C = .146$, $n = .169$ and $S = .0314$ (for Connecticut). However, as the curve is concave up and the indicated type is hyper-Malthusian one might try fitting a critical case (7). Indeed a rough fitting of this type gives $S = .027$ instead of .0478 and thus the critical case fits the data better than their logistic and better than any Malthusian can be made to fit. It is perhaps too much to expect that biologists, even those who use considerable mathematics, should be so expert mathematically as to be right in handling problems of fitting equations involving constants non-linearly by least squares, but as one of the authors of this paper in the Journal of the Royal Statistical Society was trained as a mathematician, it is surprising to find their definitions departing from accepted usage and their fittings quite obviously impossible.

[57] This can be illustrated by the simple numerical case. Using the Reed and Pearl summation formulas, corrected for typographical errors, we find $K_s = 45.571$, $m_s = e^{2.9078}$, $n_s = 1.1367$. This curve does not pass through the values $E + E' = 2, 7, 17$ but takes for $t = 0, 1, 2$ the values $P = 2.359$, 6.626, 15.791, so that the errors $E - P$ are $-.36$, $+.37$, 1.21. If the comparison were made for subsequent times differences would likewise be shown between the sum of the two component curves and the curve obtained by the summation theory, but they might not be considered serious and the theory might well be considered applicable in this fictitious case. We have queried fitting logistics by fitting their derivatives, and we see no special merit in the particular summation theory derived therefrom as compared to some other, possibly simpler theory. Thus if we wish to have the curves add at $t = \infty$ we are forced to set $K'' = K + K'$ and we might naturally take the a'' and b'' (of Reed and Pearl) as simply the weighted means of a and b (weight K) and a' and b' (weight K'). In this fictitious example the result seems quite as satisfactory. But no summation theory can be satisfactory in any case in which one component logistic is hypermalthusian and the logistic for the total region in hypomalthusian. Wilson and Luyten were entirely familiar with the fact that this must be the situation with respect to Connecticut, New Jersey, New York and the three states combined, and considered using this particular illustration of non-additivity; the reason they preferred to give merely a fictitious illustration was simply to avoid the difficulties attendant on fitting long series of actual figures. If one is willing so to force the fit of the individual components as to make them all hypomalthusian no matter what their real type one may possibly get a satisfactory addition theory, but one cannot at the same time talk of fitting the components by least squares.

[58] In commenting on Wilson and Luyten's figures Reed and Pearl say "their equation does not fit the observations, which are the significant realities in the situation." It is to be borne in mind that Wilson and Luyten were not fitting by least squares but merely trying graphically to make the relative residuals small. The success they had as compared with the least squares solution may be seen from the table, their value for S is .066 compared with the least squares value .041, an error of 60% and their value for K was 22 compared with 25, an error of 12%. If these be compared with errors of Reed and Pearl in solutions claimed to be by least squares, it is doubtful if they will be considered bad. As to the significant realities of the situation, what are they? That every one who works in population theory must fit by least squares with absolute residuals irrespective of how he may wish to define his problem? Nobody has yet shown that this is the case and nobody has stated it to be the case except Reed and Pearl whose figure by that method is 36% in error and whose statements about least squares differ from accepted usages. As a matter of information it should be stated that the manuscript of the paper by Wilson and Luyten was sent to Pearl before being sent to press, and on its return was somewhat modified and sent to him in that form. Presumably it was shown to Reed. If Pearl had likewise sent the copy of the manuscript of his reply (with Reed) to Wilson, the Journal of the Royal Statistical Society of London might have been spared the printing of egregious errors of theory and practice. What service the editor of that journal thought he was rendering science by printing an article obviously in part polemic and obviously wrong in this part is not easily imagined. For the standards which should be expected of those who use mathematics see E. B. Wilson, Presidential Address, American Statistical Association, "Mathematics and Statistics," J. Amer. Statist. Assoc., March, 1930.

[59] The comparisons should presumably be all by absolute or all by relative residuals. The figure for the United States by relative residuals we have not computed, but the United States is fitted so well by the logistic that we believe it is safe to assume that the fit by relative residuals would show an asymptotic population not very different from 200 and decidedly less than 375.5.

[60] If we note that $\delta(1/P) = -\delta P/P^2$ and if we may assimilate $E - P$ to a differential, we could consider the problem of fitting by absolute residuals to be the minimization of $S = \Sigma(1/P - 1/E)^2E^4$ and by relative residuals to be the minimization of $S = \Sigma(1/P - 1/E)^2E^2$. These expressions for S are linear in two of our variables, a and b and the computation form for minimizing S is considerably simpler than that we have used (Art. 31). What we have said about the logarithmic fit (Art. 25) applies,—the quantities may not strictly be assimilated to infinitesimals and the fit on this assumption is in some cases not particularly close to the true fit by absolute residuals. For the sake of getting a first approximation to the fit by absolute residuals we do not believe that this method would be notably superior to that based on (31) which is

simpler. The conditions for relative residuals are as a matter of fact identical with (16b) and it is this form we have used in fitting logistics by relative residuals; on the other hand in fitting augmented logistics (Arts. 54 and 55) we based our computation form on (16a) because it was simpler than the one based on (16b).

[61] Figures for populations taken from Monk and Jeter, J. Amer. Statist. Assoc., 23, 1928, pp. 361–385; we shall have occasion to refer to results taken from other parts of this paper.

[62] In such cases it is necessary to go back to general exploratory methods. With experience one learns to get valuable indications of the direction in which the least squares solution lies even from the failure of the least squares procedure to converge. We have often found, particularly in the case of the augmented logistic, that the constants of the trial graphic solution must be decidedly nearer to those of the least squares solution than their standard errors if the least squares procedure is to converge.

[63] At least as early as 1922 (Proc. Nat. Acad. Sci., 8, pp. 365–368, 1922) they were using the augmented form. They returned to the simple form in 1927. They seem often to be considering the possibility of using (Pearl, Biology of Population Growth, see p. 17)

$$P = d + \frac{k}{1 + me^{a_1x + a_2x^2 + a_3x^3}}$$

But in his Medical Biometry and Statistics, 2nd. Ed., 1930, Pearl works exclusively with the augmented logistic (Chap. XVII) which he calls the logistic—which it is geometrically as a curve but which it is not as a "law."

[64] The statement in this form assumes what is the natural assumption of any cycle theory namely that one cycle runs its course to or near to its appropriate upper asymptote d and the subsequent cycle begins from or from near the lower asymptote d. The condition of non-realization of the upper or lower asymptote complicates the interpretation of the cycle theory and makes the existence of the asymptotes or of cycles problematical. Possibly as Hotelling[7] suggests the real reason for the additional constant is to get greater fittability.

[65] In respect to the interpretation of the constants it is necessary to note that the augmented logistic may be written identically[61]

$$d + \frac{K}{1 + me^{-nt}} = d + K + \frac{-K}{1 + m^{-1}e^{nt}} = d' + \frac{K'}{1 + m'e^{-n't}}$$

so that $d' = d + K$, $K' = -K$, $m' = 1/m$, $n' = -n$ and $r' = 1/r$, where $r = \log n$. If, in practice, we never found anything but hypomalthusian augmented logistics the two forms could be kept separate and the first would be used with all the parameters (except possibly d) positive; it is clear, however, that the hypercritical case becomes confused—indeed we can always have $n > 0$ and abolish the hypercritical case if we choose.

[66] Equations are given in Pearl's Medical Biometry and Statistics, 2nd Ed.,

p. 424. These seem to be based on (24), and are preferable, at least in most cases, to the longer equations based on (25). They are, however, based on a form of the logistic which has a discontinuity (infinite values) of two of the unknowns as the curve passes through the Malthusian configuration.

[67] We have identically, if $1/A = C + e^{-nt}$,

$$S = \Sigma\left[d + \frac{B}{C + e^{-nt}} - E\right]^2 = \Sigma(E - E_m)^2 + k\left[d - E_m + \frac{\Sigma A\,\Sigma A(E - E_m)}{k\Sigma A^2 - (\Sigma A)^2}\right]^2$$

$$+ 2\Sigma A \cdot \left[d - E_m + \frac{\Sigma A\,\Sigma A(E - E_m)}{k\Sigma A^2 - (\Sigma A)^2}\right]\left[B - \frac{k\Sigma A(E - E_m)}{k\Sigma A^2 - (\Sigma A)^2}\right]$$

$$+ \Sigma A^2 \cdot \left[B - \frac{k\Sigma A(E - E_m)}{k\Sigma A^2 - (\Sigma A)^2}\right]^2 - k\frac{[\Sigma A(E - E_m)]^2}{k\Sigma A^2 - (\Sigma A)^2}$$

Since the quadratic form consisting of the 2nd, 3rd, and 4th terms is definitely positive (provided A varies at all with t, as it must if $n \neq 0$) we can minimize S only by putting these terms equal to zero which gives (37) and (36). If then we find C and n (which enter into A) so as to minimize (36) and then take B and d from (37) we have the solution of (35). *We do not get the solution* that way but by a least squares procedure on all four variables. However, if we wish to take S larger than its minimum and discuss the locus S = const. we can use these formulas. We find the values of C and n which give (36) the desired value greater than its minimum and then by virtue of the identity we can find from (37) the values of B and d which do not change this value of S (See Art. 53).

[68] The solution is of course biologically inappropriate in that the population goes to infinity in finite time, and this will be true in any critical case; but if we do not try to use the equation for long range forecasting it may be considered quite satisfactory biologically in that it shows that conditions in Germany have been such as to give no indication of approaching saturation insofar as that could be indicated by fitting an augmented logistic (and we may recall that the simple logistic was hypermalthusian). What we really desire to emphasize by expressing doubt as to the appropriateness of the solution is the possible if not probable mathematical inappropriateness of taking as the *mathematical solution of a problem in curve fitting a real solution obtained* by a procedure of successive approximations when the actual solution is imaginary. Such questions arise because of the non-linear form in which the constants enter our empirical expressions. We have tried to illustrate these difficulties by reference to the imperfect form in which the discussion of least squares is given in the books, by showing that the least squares solution in simple fictitious examples may depend upon the form of the empirical expressions, and by showing that this sort of consideration is forced on us in the discussion of the (augmented) logistic theory of the population. Unless a student is enough of a mathematician to perceive such difficulties let alone being enough of a mathematician to get results right in cases where such

difficulties do not arise, he can have little claim to real competence in this line of work.

[69] The least squares fit to the data from 1861 to 1910 is $d = 19.671$, $a = -.024557$, $b = .078826$, $n = .10709$, $S = .35357$. The lower asymptote is about 20 and the curve becomes infinite in finite time. For Pearl's solution $S = 1.0568$ which is about three times as large as for ours. If one is to maintain that the best way to fit the augmented logistic is by least squares with absolute residuals he must prefer our solution to Pearl's, and to interpret it as showing that the period 1861–1910 indicates no saturation as yet in sight for the German population. The least squares fit to the data from 1816–1855 is

$$d = 14.903, \quad a = .039909, \quad b = .060255, \quad n = .54480, \quad S = .059754$$

The lower asymptote is not much less than that of the period 1861–1910, the upper asymptote is $d + 1/a = 39.96$. For Pearl's fit $S = .3858$ or more than six times that for the least squares fit. If we are to interpret these figures on the cycle theory we have to state that from 1816 to 1855 Germany was growing on one cycle based on a lower asymptote of 15 million and approaching an upper asymptote of 40 to which it had approximated with the figure 36 by the latter date. From 1861 on it was growing on a cycle based on a lower asymptote not of 40 (the upper asymptote of the previous cycle), not even on anything approaching the population 36 actually attained, but on one of 19.7 (certainly not significantly greater than 15) and was growing with less than no saturation in its second cycle. Such results would seem to leave Pearl's cycle theory of Germany with much to be desired—unless one should insist that his values of the constants obtained one knows not how and giving a relatively poor fit have a greater scientific validity than the least squares constants.

[70] As seven places are lost in computing the determinant D in the denominator we cannot, even with 10 place work, claim any precision in these standard errors.

[71] Yule,[9] while warning against the danger of distant extrapolation, points out that with Pearl's equation the population in 1751 would have been almost exactly 5, as contrasted with 4.2 for one of Yule's logistics and with 6.3 as estimated for that time. If we use our least squares fit for extrapolation we find for 1751 the population 3.4 which is worse, not better, than the figure 4.2 or 4.3 given by a variety of good logistics, just as the forecast of the population of England and Wales based on the asymptote 383 is (presumably) worse than that on the asymptote 107. Thus at both ends the augmented logistic fitted by least squares with absolute residuals (the method given by Pearl[8] in what is perhaps his latest, 1930, pronouncement on the subject) is worse than the logistic fit, and much worse than the particular augmented logistic which Pearl has given without statement of the method by which it was obtained and which is comparatively a poor fit to the censuses available (1801–1911). What does this mean for the scientific methodology of the subject of this paper? And why should Yale extrapolate a four-constant curve which fitted ($S = .6$)

worse than his three-constant fits? The sorts of values of the constants which are compatible with a fit as bad as $S = .6$ are treated in Art. 53.

[72] For the Three States we found graphics as follows:

$$P = d + Ce^{nt}, \quad d = -.8, \quad C = 1.66, \quad n = .172, \quad S = .47$$

$$P = d + (\alpha - \beta t)^{-1}, \quad d = -.7, \quad \alpha = .131, \quad \beta = .00656, \quad S = .39$$

The value of S for the logistic (Art. 39) is .64. We are unable to build up much confidence in the reliability of the logistics as forecasters from such a welter of contradiction (or instability) as is revealed here or in the case of England and Wales.

[73] We do not understand why Reed and Pearl should characterize this as small; it is larger than any they give; possibly they obtained a value of d very different from ours as is the case with many of their other values.

[74] Wilson and Luyten might not have written their article[18] on New York if the Pearl-Reed values had been right, large as they were, and it is certain that they could not have written it if Pearl and Reed had obtained the right values. The main fact that must strike the student of populations is that the 1920 population of 106 for the U. S. A. and a forecasted population of less than 200, forms too great a contrast with the 1920 population of 9 for Greater New York and a forecasted population of 35. There is presumably little validity in any forecast but figures around half of those obtained by Pearl and Reed seem more reasonable and it may be some merit (in this special case) that their method really gives the smaller figures.

[75] Monk and Jeter[61] seem to favor estimates of public service corporations with higher asymptotic values than those given by logistics (simple or augmented). They are entitled to their opinion; the discussion they gave of the Chicago region was scientific. When they conclude from that discussion that the logistics give poor forecasts we agree with them and the 1930 census confirms them. But we see no reason to believe that there is anything in the application of the logistics themselves which would indicate any greater degree of inapplicability than in other cases. Not only are the standard errors of the constants of normal size but the actual fit has a standard deviation σ' (error to be feared) of around .05 in the table of Art. 60 (the 1920 population being about 3.8) whereas New York City and Environs has $\sigma' = .21$ for the comparable period of time 1850–1920 (the 1920 population being 9) and $\sigma' = .15$ for the whole period 1790–1920. Based on the terminal populations the errors to be feared are 1.3% for Chicago and 2.3% or 1.7% for New York City according as we use the comparable or the whole period of time. In developing a scientific method we are not at liberty to make a special explanation or a special adjustment for each special case, for thus we should lose all hope of the method being scientific and lapse back upon an art. Our chief interest here is to examine the logistic (simple or augmented) as a method and its breakdown in the case of Chicago as indicated by the enumerations of 1930 is to our way of thinking an indication that the method is not sound.

We have not overlooked the comments of Pearl and Reed, J. Amer. Statist. Assoc., 24, 1929, pp. 66–67, which, in view of the results of Art. 55, perhaps need not be taken very seriously.

[76] It must be borne in mind that fitting by absolute residuals tends to make the absolute residuals independent of the population and to make the relative residuals larger for the smaller populations. So fitting by relative residuals tends to make the absolute residuals less for the smaller than for the larger populations. These tendencies may not be in evidence in a particular case. It is, of course, not necessary that we should adopt a system of weights which would on the whole make the corresponding errors trendless (for if we had real evidence as to variability we should use it for an a priori determination of the proper system of weighting), but from an a posteriori point of view that is about all we can do.

[77] To advocate a complicated mathematical technique, to fail to follow it, and yet to get better forecasts that could have been had by following it—that is not science but Magic! We do not wish to underestimate the importance of magic in human affairs. To some our final conclusions (Art. 61) may seem from a practical point of view to be an advocacy of magical practice in the use of the logistic in studies of population or other growth, but we should beg to differ!

[78] Only the ratio $\sqrt{k-3 : k-4}$ is involved due to the increase of the number of constants. The large increase in the standard errors of the unknowns must therefore be due chiefly to the fact that $A_{ii} : D$ is much larger for the augmented logistic than for the logistic.

[79] For Scotland, Pearl (Human Biology, p. 617) gives

$d = .178, \quad a = .1240, \quad b = .5697, \quad n = .1636, \quad S = .076$

We may offer a series of graphical fits with S only about two-thirds as great:

$d = .213, \quad a = .130, \quad b = .565, \quad n = .1629, \quad S = .050$

$d = .0, \quad a = .120, \quad b = .490, \quad n = .1495, \quad S = .051$

$d = -.30, \quad a = .110, \quad b = .408, \quad n = .1361, \quad S = .049$

$d = -.50, \quad a = .100, \quad b = .372, \quad n = .1253, \quad S = .051$

$d = -.80, \quad a = .090, \quad b = .322, \quad n = .1130, \quad S = .050$

Just where the least squares solution would be found we have not determined, but the table shows the instability of the constants. With about equal precision of fitting the lower asymptote may be taken between the limits +.21 and −.80, the upper asymptote between 7.9 and 10.3 and the rate of increase between .163 and .113 provided the constants which are highly correlated are properly adjusted to one another.

[80] The quantities δ_{ji} and δ_{jh} are uncorrelated if $r_{ih} = 0$ but the products $\delta_{ji}\delta_{jh}^2$, $\delta_{ji}\delta_{jh}\delta_{jg}$ need not therefore vanish when averaged over j; we used the term "independent" to indicate that as many of these higher products as interest us shall also vanish.

[81] From early days Edgeworth was pointing out by illustrations that as n increased the mean of a sample of n tended to lie on a normal curve. See the Introductory Description to his collection of papers[89] with references to a date as early as 1888. The theorem was given a satisfactory mathematical formulation only by Isserlis, J. Roy. Statist. Soc., London, 81, 1918, pp. 75–81, which was the stimulus for a further contribution by Edgeworth, Ibid., pp. 624–632.

[82] If, for instance, the values $i = 1, \cdots, n$ correspond to n values of t which ranges from t_0 to t_1 with a frequency $P(t)$ and if the particular set of n values be a fair sample we should have approximately

$$\Sigma_i \varphi_i^2 \sigma_i^2 = n \int_{t_0}^{t_1} \varphi^2(t) \sigma^2(t) P(t) dt, \quad \Sigma_i \varphi_i^3 \mu_{3i} = n \int_{t_0}^{t_1} \varphi^3(t) \mu_3(t) P(t) dt,$$

etc. It is by no means necessary that the range be fixed. For example, if we fit $y = ax$ to a set of values of y assigned at $x = 1, 2, \cdots, n$ we have

$$\Sigma_i \varphi_i^2 \sigma_i^2 = \Sigma_i x_i^2 \sigma_i^2, \quad \Sigma_i \varphi_i^3 \mu_{3i} = \Sigma_i x_i^3 \mu_{3i}, \text{ etc.}$$

Now if $\sigma_i^2 = \sigma^2$ and $\mu_{3i} = \beta\sigma^3$ are independent of i, these sums vary with n chiefly as n^3 and as n^4, etc., and we have

$$\beta_a = C_1 \frac{n^4}{n^{9/2}}, \quad \beta_{2a} - 3 = C_2 \frac{n^5}{n^6}$$

where C_1 and C_2 are constants except for powers of $1/n$. Thus β_a and $\beta_{2a} - 3$ still vary chiefly as $1/n^{1/2}$ and $1/n$ although the range increases with n. However, the value of σ_a^2 will in this case vary as $1/n^3$ and as the range increases σ_a will drop off much faster than $1/n^{1/2}$. If the standard deviations σ_i^2 varied as i^2 and μ_{3i} as i^3, etc., we should have

$$\sigma_a^2 = C_0 \frac{n^5}{n^6}, \quad \beta_a = C_1 \frac{n^7}{n^{15/2}}, \quad \beta_{2a} - 3 = C_2 \frac{n^9}{n^{10}}$$

and the variation is as in the simplest case. (But if σ_i^2 varies we should do better to reduce the observations with weights; see infra.)

[83] See for example, E. B. Wilson, Gibbs' Vector Analysis, Chap. V for the case $k = 3$. If the vectors be written in terms of k units as

$$a_j = a_{j1}\mathbf{u}_1 + a_{j2}\mathbf{u}_2 + \cdots + a_{jk}\mathbf{u}_k, \quad \phi_i = \varphi_{1i}\mathbf{u}_1 + \varphi_{2i}\mathbf{u}_2 + \cdots + \varphi_{ki}\mathbf{u}_k$$

and the law of scalar multiplication $\mathbf{u}_h \cdot \mathbf{u}_h = 1$, $\mathbf{u}_h \cdot \mathbf{u}_g = 0$, $h \neq g$, we have

$$\mathbf{a}_j \cdot \phi_i = a_{j1}\varphi_{1i} + a_{j2}\varphi_{2i} + \cdots + a_{jk}\varphi_{ki}$$

which is a scalar linear in the k unknowns $a_{j1}, a_{j2}, \cdots, a_{jk}$ and dependent on i through the k values of ϕ which involve i with $i = 1, 2, \cdots, n$ $n > k$. These laws of multiplication are equivalent to those of matrix analysis, though the notations may differ in different presentations of the subjects as in M. Bocher's Introduction to Higher Algebra or A. N. Whitehead's Universal Algebra, Vol. I (all that was published of that treatise).

[84] If **ab**, **cd** are dyads the double multiplication **ab : cd** means the scalar (**a** . **c**)(**b** . **d**). The notation **abc** : $\Phi\Psi\Omega$, however, when **a**, **b**, **c**, are vectors and Φ, Ψ, Ω are dyadics means the triadic (**a** . Φ)(**b** . Ψ)(**c** . Ω).

[85] The meaning will perhaps be clearer if we write the expression out. There will be k^3 moments μ_{prt} of the third order each being the mean of a product $\delta a_p\ \delta a_r\ \delta a_t$ of the variations of three specified components of a from their means, but some of the components may be repeated since p, r, t are not restricted to be different. If now $\Phi^{-1} = \Sigma C_{pq}\mathbf{u}_p\mathbf{u}_q$

$$\Phi^{-1}\Phi^{-1}\Phi^{-1} = \Sigma C_{pq}C_{rs}C_{tu}\mathbf{u}_p\mathbf{u}_q\mathbf{u}_r\mathbf{u}_s\mathbf{u}_t\mathbf{u}_u$$

the quantities C_{pq} are the mean products Π_{pq} of the second order divided by σ^2. Also

$$\Sigma_i\boldsymbol{\phi}_i\boldsymbol{\phi}_i\boldsymbol{\phi}_i = \Sigma_{qsu}\varphi_{qi}\varphi_{si}\varphi_{ui}\mathbf{u}_q\mathbf{u}_s\mathbf{u}_u$$

then

$$\mu_{3a} = \mu_3\Sigma_{prt}\mathbf{u}_p\mathbf{u}_r\mathbf{u}_t[\Sigma_{qsu}C_{pq}C_{rs}C_{tu}\Sigma_i\varphi_{qi}\varphi_{si}\varphi_{ui}]$$

or the third moment

$$\mu_{prt} = \mu_3\Sigma_{qsu}C_{pq}C_{rs}C_{tu}\Sigma_i\varphi_{qi}\varphi_{si}\varphi_{ui}$$

It does not appear that in general there will be any reduction in this expression because the sum over the φ's is cubic in the φ's whereas the C's are the quotients of cofactors of a determinant by the determinant itself, each term of which is a sum quadratic in the φ's, viz., $C_{pq} = \Sigma_i\varphi_{pi}\varphi_{qi}$. The values of μ_{prt} are the same if the values of p, r, t are unchanged except for permutations. There are k moments of the type μ_{ppp} and $k(k-1)$ of the type μ_{ppr} and $k(k-1)(k-2)/6$ of the type μ_{prt} with the letters p, r, t having distinct values which makes in all $k(k+1)(k+2)/6$ different moments to be computed.

[86] By definition μ_{4a} is made up of tetrads of the type

$$\overline{\delta a_p\delta a_r\delta a_t\delta a_v}\mathbf{u}_p\mathbf{u}_r\mathbf{u}_t\mathbf{u}_v,$$

where the bar denotes that the expression is to be averaged for all samples. There are k fourth moments obtained by averaging $(\delta a_p)^4$, there are $k(k-1)$ arising from $(\delta a_p)^3\delta a_r$, there are $\frac{1}{2}k(k-1)$ from $(\delta a_p)^2(\delta a_r),^2$ and $\frac{1}{2}k(k-1)(k-2)$ from $(\delta a_p)^2\delta a_r\delta a_t$, finally $k(k-1)(k-2)(k-3)/24$ of the type $\delta a_p\delta a_r\delta a_t\delta a_v$ with p, r, t, v all different. This makes $k(k+1)(k+2)(k+3)/24$ in all. The values are, for the normal distribution,

$$\mu_{1111} = 3\sigma_1^4, \quad \mu_{1112} = 3r_{12}\sigma_1^3\sigma_2, \quad \mu_{1122} = (1+2r_{12}^2)\sigma_1^2\sigma_2^2,$$

$$\mu_{1123} = (r_{23}+2r_{12}r_{13})\sigma_1^2\sigma_2\sigma_3,$$

$$\mu_{1234} = (r_{12}r_{34}+r_{13}r_{24}+r_{14}r_{23})\sigma_1\sigma_2\sigma_3\sigma_4,$$

where the symbols σ_i refer to the standard deviation of a_i and r_{ij} to the correlation of a_i and a_j. [The only value hard to obtain is that of μ_{1234} and that becomes simple if it is observed that the mean of $x_1x_2x_3x_4$ is one third of the mean of the sum of the three equal quantities $(x_1x_2)(x_3x_4)$, $(x_1x_3)(x_2x_4)$, $(x_1x_4)(x_2x_3)$ which may be found by the usual process of differentiating under the sign in the generalized probability integral.] Although there are only $k(k+1)(k+2)(k+3)/24$ different fourth moments, there are of course k^4 different tetrads $\mathbf{u}_p\mathbf{u}_r\mathbf{u}_t\mathbf{u}_v$ in the tetradic sum μ_{4a}. The value of the fourth moment μ_{prtv} is the same for any given set of p, r, t, v, but there will be a number of tetrads with the units $\mathbf{u}_p$, $\mathbf{u}_r$, $\mathbf{u}_t$, $\mathbf{u}_v$ arranged in a variety of orders. Thus there must be for

μ_{pppp}, k tetrads; μ_{pppv}, $4 \times k(k-1)$ tetrads;

μ_{pptt}, $6 \times \frac{1}{2}k(k-1)$ tetrads; μ_{pptv}, $12 \times \frac{1}{2}k(k-1)(k-2)$ tetrads;

μ_{prtv}, $24 \times k(k-1)(k-2)(k-3)/24$ tetrads;

and adding we find that there are k^4 tetrads accounted for.

Now $\mu_{2a} = \Sigma \delta a_p \delta a_r \mathbf{u}_p\mathbf{u}_r = \Sigma \mu_{pr}\mathbf{u}_p\mathbf{u}_r$ with $\mu_{pr} = \mu_{rp}$. The product $\mu_{2a}\mu_{2a}$ has the following types of terms where only the indices of the units $\mathbf{u}$ are given

$\mu_{11}^2(1111)$, k terms; $\mu_{11}\mu_{22}(1122 + 2211)$, $\frac{1}{2}k(k-1)$ terms;

$\mu_{12}^2(1212 + 1221 + 2112 + 2121)$, $\frac{1}{2}k(k-1)$ terms;

$\mu_{11}\mu_{12}(1112 + 1121 + 1211 + 2111)$, $k(k-1)$ terms;

$\mu_{11}\mu_{23}(1123 + 2311 + 1132 + 3211)$, $\frac{1}{2}k(k-1)(k-2)$ terms;

$\mu_{12}\mu_{13}(1213 + 1231 + 2113 + 2131$
$+ 1312 + 1321 + 3112 + 3121)$, $\frac{1}{2}k(k-1)(k-2)$ terms;

$\mu_{12}\mu_{34}(1234 + 2134 + 1243 + 2143$
$+ 3412 + 3421 + 4312 + 4321)$, $\frac{1}{8}k(k-1)(k-2)(k-3)$ terms.

There are k^4 such individual tetrads in the product $\mu_{2a}\mu_{2a}$. With the interchanges of units corresponding to **abcd** + **acbd** + **adcb** we find that the product defined as $\mu_{2a} \times \mu_{2a}$, which must contain $3k^4$ tetrads, has these types

$3\mu_{11}^2(1111)$, k terms; $3\mu_{11}\mu_{12}(1112 + 1121 + 1211 + 2111)$, $k(k-1)$ terms;

$(2\mu_{12}^2 + \mu_{11}\mu_{22})(1122 + 2211 + 1221 + 2112 + 2121 + 1212)$, $\frac{1}{2}k(k-1)$ terms;

$(2\mu_{12}\mu_{13} + \mu_{11}\mu_{23})(1213 + 2113 + 1312 + 1231 + 2131 + 1321 + 3113 +$
$3121 + 1123 + 2311 + 1132 + 3211)$, $\frac{1}{2}k(k-1)(k-2)$ terms;

$\mu_{12}\mu_{34}$(1234 and all permutations thereof), $\frac{1}{8}k(k-1)(k-2)(k-3)$ terms.

It is clear that $\mu_{2a} \times \mu_{2a}$ has complete permutability of the units in each of the term types. Now as μ_{11}^2 is σ_1^4 and $\mu_{11}\mu_{12}$ is $r_{12}\sigma_1^3\sigma_2$, etc., it is clear that μ_{4a} and $\mu_{2a} \times \mu_{2a}$ are identical, as stated, when the distribution is normal.

(The only identification which is not perfectly obvious is the last involving μ_{1234}. We have $k(k-1)(k-2)(k-3)/8$ coefficients of the form $\mu_{12}\mu_{34}$ each multiplied by 24 tetrads; we have to collect according to the tetrads. Consider $\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3\mathbf{u}_4$. This has $\mu_{12}\mu_{34} = r_{12}\sigma_1\sigma_2 r_{34}\sigma_3\sigma_4$ as written. The same tetrad will occur with coefficients $\mu_{13}\mu_{24}$ and $\mu_{14}\mu_{23}$ and only these.)

[87] The theorem holds under assumption iv, namely, that $r_{ih} = 0$. If the

variations δ in the different universes are correlated the relation of the weights to the values of σ would be much more complicated and would involve the r's. Even in the simplest case, $n = 2$, $k = 1$, the minimization of

$$\sigma_a^2 = [w_1^2\varphi_1^2\sigma_1^2 + w_2^2\varphi_2^2\sigma_2^2 + 2rw_1w_2\varphi_1\varphi_2\sigma_1\sigma_2]/(w_1\varphi_1^2 + w_2\varphi_2^2)^2$$

leads to

$$\frac{w_2\sigma_2^2}{w_1\sigma_1^2} = \frac{1 - r_{12}\varphi_1\sigma_2/\varphi_2\sigma_1}{1 - r_{12}\varphi_2\sigma_1/\varphi_1\sigma_2} \quad \text{or} \quad \frac{w_2\sigma_2\varphi_2}{w_1\sigma_1\varphi_1} = \frac{\varphi_2/\sigma_2 - r_{12}\varphi_1/\sigma_1}{\varphi_1/\sigma_1 - r_{12}\varphi_2/\sigma_2}$$

[88] Thus a tolerable smooth curve defined by statistical data may appear very rough if one tries to estimate its smoothness from third or fourth differences in the data. Such arithmetical roughness is presumably just as imaginary or unreal so far as concerns the phenomenon defined by the curve as the imaginary constants for Chicago.

[89] Journal of the Royal Statistical Society, London, December 1898, March, June, September 1899.

[90] When fitting n_r to $(A + Bx_r^2)^{-1}$, the constant $N = 25878$, which is the total number of items in the sample, is not introduced explicitly and is not reproduced in the result, for if we take the expression for y and integrate from $-\infty$ to $+\infty$ we find the number 36800 which is nearly 50% in excess of N. As the theory of χ^2 depends on the distribution of a specified number N of items, we can hardly apply χ^2 to compare the actual distribution in the histogram with the distribution computed from the frequency function here fitted.

R. A. Fisher,[36] in discussing efficiency, points out (p. 331) that one should minimize the likelihood $L = S \log f$ and that the standard deviations of the unknowns can be determined (p. 332) from the second derivatives of L. It is interesting to apply that method of fitting to the case of

$$y = \frac{N}{\pi}\frac{c}{c^2 + (x - m)^2}, \quad \text{then } \frac{N}{\pi}\frac{1.591}{(1.591)^2 + (x - 66.634)^2}$$

with $\sigma_c = .0125$ and $\sigma_m = .0162$. It is noteworthy that as the mean of the observations is 66.701 and median is 66.651, the value of m at 66.634 differs from the mean by four times σ_m. Further, if the standard deviation of the mean of the observations is estimated by the usual method we find $\sigma_{\text{mean}} = .0159$ which is essentially the same as σ_m although if the distribution of the universe were actually on the curve used, the sample should not have a determinate mean and σ_{mean} should presumably be much larger. We may also fit the same type of curve directly by least squares though the case is non-linear, and what we actually did was to minimize (over the range for which there were observations)

$$\Sigma \frac{(P - E)^2}{P} \text{ to find } y = \frac{N}{\pi}\frac{1.9786}{(1.9786)^2 + (x - 66.630)^2}$$

with $\sigma_c = .193$ and $\sigma_m = .202$ when determined by the formulas usually applied for least squares. These values are much larger due to the multiplier

$S/(n-2)$ in σ^2 where S is the value of χ^2 which for the fit is about 5000 though $n = 27$. Now, is the least squares method worth anything for this case? We incline to think that it is not. The effort to minimize χ^2 assumes tacitly that the sample varies from the universe by fluctuations which are roughly proportional to the square root of the numbers in the different intervals and if the application of the method is to be considered justified a posteriori, the observed fluctuations $P - E$ must be approximately equal to $\sqrt{P}$ and $\chi^2 = S$ must be approximately equal to n, whereas we find the fit so bad that χ^2 actually turns out to be 5000. Under these conditions the sample cannot arise as a chance variation from a universe of the type fitted and the actual value of χ^2 for the sample cannot be put equal to the mean value of χ^2 for all samples which could arise by chance fluctuation.

Finally it may be observed that we could fit the curve by taking into account the region beyond that over which the sample is distributed, and probably should do so, because one of the facts of the observations is that although $N = 25878$ no recruit was found with a height greater than 78″ or less than 51″ and furthermore the distribution tailed off to these figures in such a way as to imply that the termination of the distribution was natural and not forced by regulations. The results of including the range out to $\pm\infty$ make, of course, a large difference (between 40% and 50%) in the value of χ^2 because of the inappropriateness of the curve type selected. The result of minimizing χ^2 on this hypothesis is

$$y = \frac{N}{\pi}\frac{1.8694}{(1.8694)^2 + (x - 66.616)^2}, \quad \sigma_c = .248, \quad \sigma_m = .249$$

The value of the constant c is materially different, that of m is not much different, those of σ_c and σ_m are larger, but chiefly because χ^2 is larger, an increase from around 5000 to around 7400 amounting to one of about 25% in σ, which by the formula varies as the square root of S.

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THE APPLICATION OF METHODS OF GEOMETRICAL INVERSION TO THE SOLUTION OF CERTAIN PROBLEMS IN ELECTRICAL RESONANCE.

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THE APPLICATION OF METHODS OF GEOMETRICAL INVERSION TO THE SOLUTION OF CERTAIN PROBLEMS IN ELECTRICAL RESONANCE.

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Summary.

The type of circuit considered is that containing a resonating element of very low damping constant, consisting of resistance, inductance, and capacitance in series, in parallel with which is a second capacitance. Although the method was developed with particular reference to the piezo-electric resonator, it is equally applicable to any type of alternating current network containing a sharply tuned resonating element.

The distinguishing feature of the present method lies in the use of a single circular locus for both admittances and impedances. The process of performing an inversion, instead of requiring the construction of a new diagram, now demands nothing more than a change in the value of the unit vector. The solution of problems is thus greatly simplified, and the effect of varying the frequency or any other parameter is easily followed.

The frequency-calibration of the circular locus is described, and the properties of the quadrantal points and especially of the points of maximum and minimum admittance are discussed. The effects upon the performance of the circuit of the addition of fixed impedances in series or parallel with the resonating element are considered. As illustrations of the method the theory of the determination of the constants of the resonating element from observations of the maximum and minimum impedance of the entire circuit is given, and also the problem of finding the length of air-gap of a piezo resonator for maximum voltage across the gap.

Symbols.

$R\ L\ C\ C_1$	Constants of resonating element, Fig. 1.
Z_2	Impedance in series with resonating element, having components R_2 and X_2. Z_2 may be a pure capacitance C_2.

C_3	Parallel capacitance, with reactance X_3 and susceptance b_3
Z Y X g b	Impedance, admittance, reactance, conductance, and susceptance of the RLC branch.
Z_1' and components R_1', X_1'	Impedance of $RLCC_1$ mesh.
C_1'	Equivalent series capacitance of $RLCC_1$.
Y_1' and components g_1', b_1'	Admittance of $RLCC_1$ mesh.
R_1'' X_1'' C_1''	Equivalent parallel parameters of resonating element $RLCC_1$.
R' L' C' C_1'	Constants of network equivalent to $RLCC_1C_2$.
Z_m Z_n Y_m Y_n	Maximum and minimum values of impedance and admittance of $RLCC_1$
Z_2' and components R_2' and X_2'	Resultant values for $RLCC_1Z_2$ network.
Y_2' and components g_2', b_2'	Resultant values for $RLCC_1Z_2$ network.
R_2'' X_2'' C_2''	Equivalent parallel values for $RLCC_1Z_2$ network.
ρ	Radius of circle.
a a' a''	Distance from point A on diagrams to origin of vectors F or F′.
s	Scale-value for impedance diagram for RLC branch.
s_y	Scale-value for admittances, RLC or $RLCC_1$ mesh, also with C_3 in parallel.
s_z	Scale-value for impedances, $RLCC_1$ mesh, also with Z_2 or C_2 in series.
s_y'	Scale-value for admittances, $RLCC_1C_2$ network.
s_y''	Scale-value for admittances, $RLCC_1Z_2$ network.
σ	Scale-value for frequencies.
f_o ω_o	Frequency and angular velocity corresponding to maximum admittance of RLC branch.
f_{ym} f_{yn}	Frequencies corresponding to Y_m and Y_n respectively.
f_{q1} f_{q2}	Frequencies at upper and lower quadrantal points respectively on admittance circle.
$2n_q$	Difference between the frequencies f_{q2} and f_{q1}.
α	Damping constant $= R/2L$.

Other special symbols are defined in the text.

INTRODUCTION.

The method of geometrical inversion as applied to circular loci in alternating current problems is well known.[1] The particular development about to be described was occasioned by the study of the piezo-electric resonator, but it is of sufficient generality to be applicable to any network containing a resonating element that possesses a damping constant small in comparison with that of any other portion of the circuit. It offers the advantage of rapid and simple solution of problems. The method is also applicable to mechanical or acoustical resonating systems.

In the past it has been customary, when performing inversions of vectors or of their loci, to construct a separate diagram for each inversion. The advantage in the present method lies in carrying out all operations on a single diagram. This involves a departure from convention, in that all vectors representing inductive quantities are drawn upward (in the positive direction), and capacitive downward, whether the quantities are impedances or admittances. This feature is, however, in conformity with the procedure commonly observed in the theory of geometrical inversion. The use of the same circular locus for both impedances and admittances requires also that the unit vector, or scale-value, be introduced explicitly in each equation. Problems involving damped free oscillations and products of complex vectors are not considered.

DEVELOPMENT OF THE METHOD.

1. We are concerned chiefly with the type of circuit shown in Fig. 1. We proceed to find a general graphical solution for the resultant impedance and admittance, with their components, and also for the selectivity of the network, as functions of the frequency. The *RLC* branch, with its parallel capacitance C_1, is the resonating element.

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[1] For the mathematical principles of inversion see, for example, Graustein, Introduction to Higher Algebra, N. Y. 1930, or Ziwet and Hopkins, Analytic Geometry and Principles of Higher Algebra, N. Y. 1922. For applications to alternating currents, see Lee, Graphical Analysis of Alternating Current Circuits, Baltimore 1928. Applications to vibrating systems: Kennelly, Electrical Vibration Instruments, N. Y. 1923. Applications to piezo-electric resonator circuits: Vigoureux, Quartz Resonators and Oscillators, London 1931; D. W. Dye, Proc. Phys. Soc. Lond. 38, pp. 399–457, 1926; Y. Watanabe, Elek. Nachr. Techn. 5, pp. 45–64, 1926, or Proc. Inst. Radio Eng. 18, pp. 695–717 and 862–893, 1930.

In the case of the piezo resonator, R, L, C and C_1 are the equivalent electrical constants introduced by Van Dyke and Dye.[2] Z_2 may represent the air-gap between crystal and electrodes, and C_3 the capacitance of mounting and leads, or, as proved below, $RLCC_1$ may include these capacitances, while Z_2 and C_3 represent the constants of other parts of the circuit.

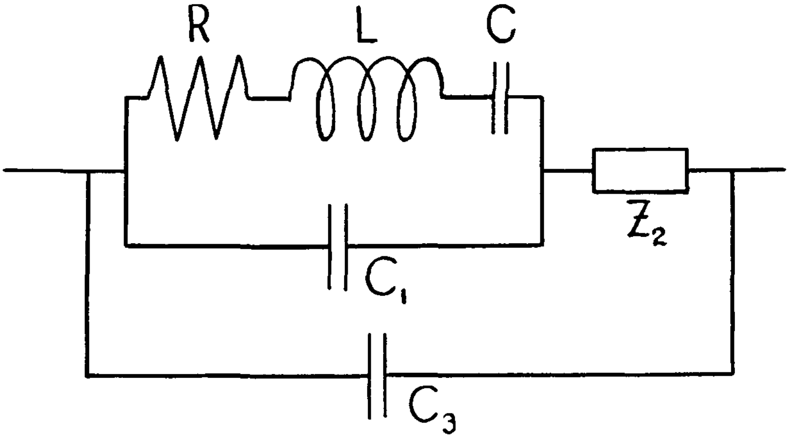

FIGURE 1. Electrical Network Including the Resonating Element $RLCC_1$.

2. *The Resonating Element.* This is the $RLCC_1$ portion of Fig. 1. As starting point we take the impedance of the RLC branch. In Fig. 2, let the resistance vector be represented by any convenient horizontal distance AB. The unit vector, or scale-value, is then $R/\text{AB} = s$ (ohms per division). By "division" is meant unit distance on the diagram. We now write

$$\begin{aligned} R &= s\cdot\text{AB} \\ X &= \omega L - \frac{1}{\omega C} = s\cdot\text{BS} \qquad (1) \\ Z &= s\cdot\text{AS} \end{aligned}$$

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[2] K. S. Van Dyke, Phys. Rev. 25, p. 895, 1925 and Proc. Inst. Radio Eng. 16, pp. 742–764, 1928; D. W. Dye, Proc. Phys. Soc. Lond. 38, pp. 399–457, 1926.

As the frequency varies, the locus of the impedance vector is of course the line BS. B is the point of resonant frequency f_o, at which the reactance $X = 0$, that is $\omega_o^2LC - 1 = 0$.

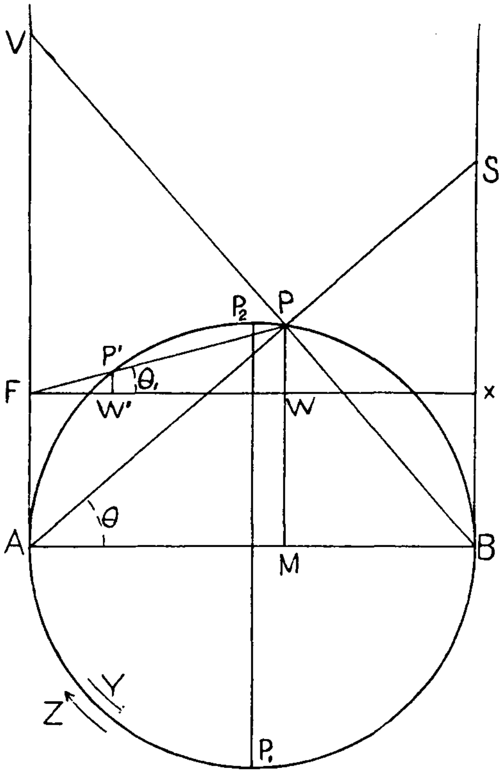

FIGURE 2. Circle Diagram, showing Admittance and Impedance of Resonating Element.

The circular locus of the admittance Y for RLC is found by inverting AS with respect to the point A as center. According to the usual practice, we should assign the value unity to the radius of the circle of inversion, write Z = AS, Y = 1/AS, and plot a circular locus for Y to some convenient scale, the arguments of the vectors having signs opposite to those for Z. Instead of this, we take AB as radius of the circle of inversion, with center at A. It is unnecessary to draw this circle in the figure, for it is evident from the principles of inversion theory that under these circumstances the straight line BS inverts into a circle having AB as diameter. The inverse of any point S is P, where AS cuts the circle, while the point B inverts into itself. Hence with suitable choice of scale-value we may use the circle APB as the admittance locus for *RCL*. AP now represents the admittance $Y = 1/Z$, by the equation $Y = s_y \cdot \mathrm{AP}$. Here s_y is the admittance scale-value in ohms per division. Since $\mathrm{AP} \cdot \mathrm{AS} = 1/(\mathrm{AB})^2$, it follows that $ss_y = 1/(\mathrm{AB})^2 = 1/4\rho^2$, where ρ is the radius of the circle APB. Thus s_y is determined in terms of s and ρ. Or, in terms of R and ρ, we have by (1), $s_y = 1/2\rho R$. Hence

$$\frac{1}{R} = s_y \cdot 2\rho \tag{2}$$

For the admittance, conductance and susceptance of *RLC* at any frequency f we have, according to Fig. 2,

$$\begin{aligned} Y &= \frac{1}{Z} = s_y \cdot \mathrm{AP} \\ g &= \frac{R}{Z^2} = s_y \cdot \mathrm{AM} \\ b &= \frac{X}{Z^2} = s_y \cdot \mathrm{MP} \end{aligned} \tag{3}$$

At frequencies above the resonant value f_o, X and b are positive and, according to the method here adopted, P lies above the horizontal axis. The frequency-scale runs counter-clockwise around the circle, having values zero and infinity at A.

In order to represent the *admittance of the entire resonating element* $RLCC_1$, we determine the point F, Fig. 2, such that $s_y \cdot a$ is numerically equal to ωC_1, where a denotes the distance AF. Since the damping constant of the resonator is assumed to be small, a may be considered as constant over the resonant range, which, as will appear, includes almost the entire circle.

The point F is to be taken as the origin of vectors for $RLCC_1$. FA is then negative, and FP is the vector sum of FA and AP. In other words,

$$s_y \cdot a = -\omega C_1$$

$$s_y \cdot \mathrm{AP} = Y = g + jb \tag{4}$$

$$s_y \cdot \mathrm{FP} = Y_1' = g_1' + jb_1'$$

Y_1' is the admittance of $RLCC_1$, having components $g_1' = g = s_y \cdot \mathrm{AM}$ and $b_1' = b - \omega C_1 = s_y \cdot \mathrm{WP}$. The phase angles for RLC and $RLCC_1$ are Θ and Θ_1 respectively. A positive phase angle (*e. g.* for point P in Fig. 2) indicates an inductive admittance and a lagging current. Over certain frequency ranges it is evident that Θ and Θ_1 have opposite signs. We may call the circle in Fig. 2 either the "admittance circle" or the "impedance circle." When it serves as an admittance circle, the frequencies run counter-clockwise, as indicated by the arrow Y in Fig. 2. On the impedance circle the direction is clockwise (arrow Z).

Impedance circle for the element $RLCC_1$. We perform a second inversion, this time taking F, Fig. 2, as center and a = FA as radius of the circle of inversion. The circle inverts into itself, and the point inverse to P is P′. This does not mean that FP′ is the reciprocal of FP, for a differs in general from unity. It does mean, however, as is easily seen from geometrical considerations, that $\mathrm{FP}' = a^2/\mathrm{FP}$, whence if $s_y \cdot \mathrm{FP} = Y_1'$ we may write $Z_1' = 1/Y_1' = 1/s_y \cdot \mathrm{FP} = \mathrm{FP}'/s_y \cdot a^2 = s_z \cdot \mathrm{FP}'$, where s_z is the impedance scale-value, given by the equation

$$s_z = \frac{1}{a^2 s_y} = \frac{1}{2\rho R \omega^2 C_1^2} \text{ (ohms per div.)} \tag{5}$$

Conversely, if FP′ represents the given admittance of $RLCC_1$, FP represents the impedance at the same frequency.

From Eqs. (4) and (5) it is evident that on the impedance diagram a represents the reactance of C_1.

Draw the horizontal axis Fx in Fig. 2, and the vertical line W′P′. We have the following relations for the impedance circle, analogous to Equations (4):

$$s_z \cdot a = -\frac{1}{\omega C_1} \tag{6}$$

$$s_z \cdot \mathrm{FP}' = Z_1' = \frac{1}{Y_1'} = R_1' + jX_1' \tag{7}$$

In Eq. (7) P′ denotes any point on the impedance circle, the frequency being that characteristic of the inverse point P on the admittance circle. R_1' and X_1' are the equivalent series resistance and reactance of $RLCC_1$. Their values, which are easily derived, are now given:[3]

$$s_z \cdot \mathrm{FW}' = R_1' = \frac{R}{(1 - \omega C_1 X)^2 + \omega^2 C_1^2 R^2} \tag{8}$$

$$s_z \cdot \mathrm{W'P'} = X_1' = \frac{X - \omega C_1 (R^2 + X^2)}{(1 - \omega C_1 X)^2 + \omega^2 C_1^2 R^2} \tag{9}$$

It is sometimes convenient to regard the resonating element $RLCC_1$ as equivalent to a resistance R_1' and capacitance C_1' in series. The equivalent series capacitance, which for all points lying above the Fx axis assumes negative values, and which, like R_1', is a function of the frequency, is found from the equation $C_1' = -1/\omega X_1'$.

As is easily verified, the diameter of the $RLCC_1$ impedance circle, in ohms, is given by the maximum value of R_1':

$$s_z \cdot 2\rho = \frac{1}{\omega^2 C_1^2 R} \tag{10}$$

The frequency for maximum R_1' comes at point B on the impedance circle.

The element $RLCC_1$ may also be represented as a resistance R_1'' in parallel with a capacitance C_1''. These values are of course dependent upon the frequency. We find $R_1'' = 1/g_1' = 1/(s_y \cdot \mathrm{FW})$; $X_1'' = 1/b_1' = 1/(s_y \cdot \mathrm{WP}) = -1/\omega C_1''$. As is seen from Fig. 2, the minimum value of R_1'' occurs at point B on the admittance circle. C_1'' has maximum positive and negative values at the two quadrantal points P_1 and P_2 on the admittance circle, Fig. 2, that is, the points for which $\Theta = \pm 45°$.

It is interesting to note that at any frequency the impedance Z_1' of $RLCC_1$ is a mean proportional between the equivalent series and parallel resistances R_1' and R_1''. This is easily proved analytically or graphically.

3. *Calibration of the circle for frequency.* This process is most easily carried out for the admittance circle. To find the frequency corresponding to any point P′ on the impedance circle, one has only to find the inverse point P, Fig. 2, which locates the admittance corresponding to the given impedance, and then to find the frequency for P by the method about to be described.

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[3] These values correspond to Eqs. (a) and (b) in Dye, l.c., p. 403.

The distribution of frequencies around the circle depends only on R, L and C and not at all on C_1. From Eqs. (1) and Fig. 2 we have

$$\tan\Theta = \frac{X}{R} = \frac{\omega^2 LC - 1}{\omega RC} = \frac{4\pi L}{R}(f - f_o) = \frac{2\pi n}{\alpha} = \frac{\mathrm{BS}}{2\rho} = \frac{2\rho}{\mathrm{AV}} \tag{11}$$

approximately, since the resonance range is assumed small in comparison with the frequency. In (11), $\alpha = R/2L$ is the damping constant, and n is an abbreviation for the frequency-difference $f - f_o$. We may now define the frequency scale-value σ in cycles per division of the BS scale as follows:[4]

$$\sigma = \frac{n}{\mathrm{BS}} = \frac{\alpha}{4\pi\rho} \tag{12}$$

In order to find the frequency for point P we extend the line AP to meet the vertical axis through B at S. We then have for the frequency-difference $n = f - f_o$ between the points P and B,

$$n = \sigma\cdot\mathrm{BS} \tag{13}$$

For points below the diameter AB, BS is negative and $f < f_o$.

When P is to the left of the center of the circle, the distance BS becomes inconveniently great. In that case we use the distance AV, obtained by producing BP to meet the vertical axis through A at V. Then by Eq. (11) we have

$$n = \frac{\alpha\rho}{\pi}\cdot\frac{1}{\mathrm{AV}} \tag{14}$$

A method for the experimental determination of σ will be discussed in Sec. 12.

Since by Eq. (13) the frequency-scale on BS is linear, it is more convenient to use this equation whenever possible, rather than (14).

From (12) and (14) it is evident that the smaller the damping constant α the more wide-open is the frequency-scale (see Sec. 15).

4. *Critical Points on the Circular Locus.* Regarding Fig. 3 as the admittance diagram for $RLCC_1$, we have as before $\mathrm{AF} = a = \omega C_1$ and $\mathrm{AB} = 2\rho = 1/s_y R$. *At point B the frequency* f_o is that at which the reactance of RLC vanishes. For $RLCC_1$, with F as origin, B is the point at which, from Sec. 2, the equivalent parallel resistance R_1'' is a maximum. The frequency at B is still f_o. On the admittance

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[4] Cf. Mallett, Telegraphy and Telephony, London 1929, p. 135.

circle the point P_7, inverse to B, gives the frequency for *maximum equivalent series resistance* R_1'; for upon inverting to an impedance diagram, P_7 inverts into B, and by Eq. (8) we then have $R_1' = s_z \cdot AB$. The frequency at B on the impedance circle is of course the same as at P_7 on the admittance circle.

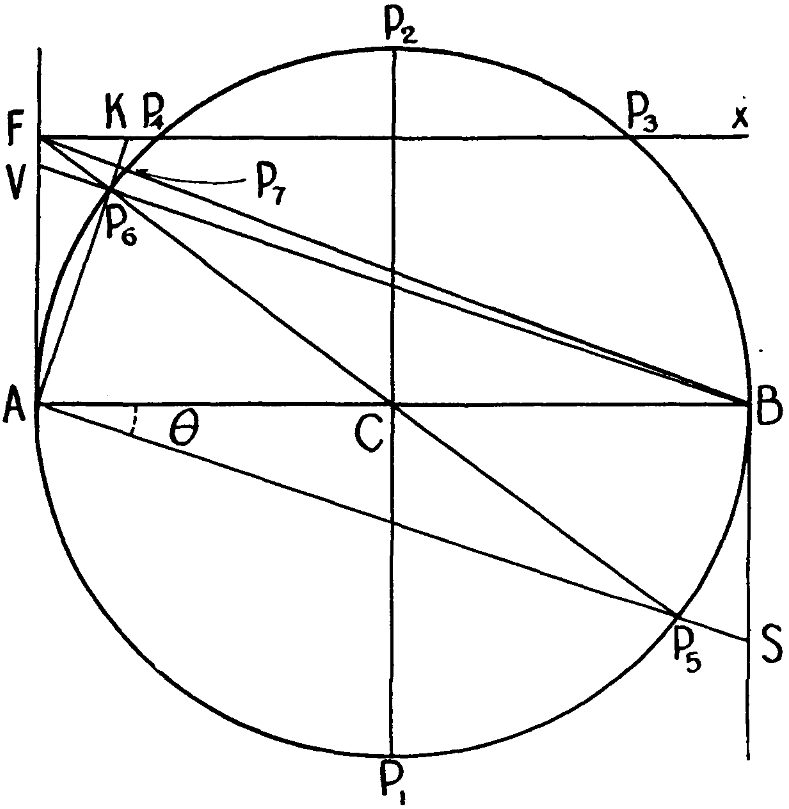

FIGURE 3. Critical Points on Circle Diagram for Resonating Element.

Consider next the mutually inverse points P_3 and P_4, at each of which the $RLCC_1$ resonator acts as a *pure resistance*. At P_3 the admittance, now a pure conductance, is large. This point has sometimes been called the point for "series resonance," since if C_1 is small,

P_3 is close to B, that is, the frequency is nearly that for series resonance of the *RLC* branch. On the other hand at P_4 we have the condition of "parallel resonance," in which C_1 plays a determining part. Upon inverting, we find that the high series resistance at parallel resonance is given by $s_z \cdot FP_3$. If a resonator is to show marked reduction in current at parallel resonance, C_1 and hence AF should be small. The distance AF is exaggerated in Fig. 3 beyond the value generally found for the piezo-electric resonator, in order to make the various points stand out more clearly. The values of R_1' and R_1'' become identical at P_3; the same is also true at P_4.

The quadrantal points are P_1 and P_2. On the admittance diagram, these are the points for maximum positive and negative values, respectively, of the equivalent parallel capacitance C_1'', as explained in Sec. 2. On the impedance circle, they are the points for greatest positive and negative equivalent series capacitance C_1'. Their frequencies are found from the points inverse to them, as explained at the beginning of Sec. 3.

The quadrantal points on the admittance circle are of greater usefulness than those on the impedance circle. The angular velocities at the admittance quadrantal points are well known to be

$$\omega_q = \sqrt{\omega_o^2 + \alpha^2} \pm \alpha \tag{15}$$

When α is small this may with sufficient accuracy be written $\omega_q = \omega_o \pm \alpha$. The following relations may also be mentioned, calling f_{q1} and f_{q2} the quadrantal frequencies on the admittance circle:

$$f_{q1} - f_{q2} = f_o^2 \tag{16}$$

$$\alpha = \pi(f_{q2} - f_{q1}) = \pi n_q = 2\pi(f_{q2} - f_o) = 2\pi(f_o - f_{q1}) \tag{17}$$

approximately, where n_q is the quadrantal frequency-range.

Many writers have found the points of *maximum and minimum impedance* of great importance. The relations between them, the corresponding frequencies, and the damping constant, are clearly brought to light by the present method. In Fig. 3, these points are P_5 and P_6. Using subscripts *m* and *n* to denote maximum and minimum values respectively, we have $Y_m = s_y \cdot FP_5$, $Y_n = s_y \cdot FP_6$, $Z_m = s_z \cdot FP_5$, $Z_n = s_z \cdot FP_6$. From the principle of inversion it is easily proved[5] that

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[5] Kobzarew, Jour. Appl. Phys., Moscow, Vol. 6, p. 31, 1929.

$$Y_m Y_n = (\omega C_1)^2 \tag{18}$$

$$Y_m - Y_n = \frac{1}{R} \tag{19}$$

Thus when Y_m and Y_n are measured, their product gives C_1 and their difference gives R.

When R is small in comparison with $\sqrt{L/C}$, the frequency at maximum admittance is given[6] by

$$\omega^2 = \frac{C + C_1 + C_2}{LC\,(C_1 + C_2)} \tag{20}$$

5. *Constants of Resonating element in Terms of Y_m and Y_n.* We must first write the equations relating maximum and minimum admittance to the corresponding frequencies f_m and f_n. Referring to Fig. 3 and Eqs. (12) and (14), we find for the differences n_m and n_n between these frequencies and f_o,

$$f_o - f_m = n_m = \frac{\alpha}{4\pi\rho}\cdot \mathrm{BS} \quad \text{and} \quad f_n - f_o = n_n = \frac{\alpha\rho}{\pi}\cdot\frac{1}{\mathrm{AV}} \tag{21}$$

It is to be observed that BS = AV, and $\mathrm{BP}_6 = \mathrm{AP}_5$. From similar triangles and the above equations we then have

$$\frac{n_n}{n_m} = \frac{4\rho^2}{\mathrm{AV}\cdot\mathrm{BS}} = \frac{\mathrm{FP}_5}{\mathrm{FP}_6} = \frac{Y_m}{Y_n} \tag{22}$$

$$f_o - f_m = n_m = \frac{Y_n}{Y_m + Y_n}(f_n - f_m) \tag{23}$$

From (23) f_o may be precisely determined from observed values of f_m and f_n.

In terms of these same admittances and frequencies the damping constant may by Eq. (11) be expressed as follows (see Fig. 3):

$$\tan\Theta = \frac{\mathrm{BS}}{2\rho} = \frac{\mathrm{AV}}{2\rho} = \frac{\mathrm{FK}}{\mathrm{AF}} = \frac{\mathrm{FP}_6}{\mathrm{AF}} = \frac{Y_n}{\omega C_1} = \frac{2\pi n_m}{\alpha}$$

On substituting the value of n_m from (22) and of ωC_1 from (18), we have

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[6] Boella, l'Elettrotecnica, Vol. 17, p. 672, 1930, or Proc. Inst. Radio Eng. Vol. 19, p. 1252, 1931.

$$\alpha = \frac{2\pi\sqrt{Y_m Y_n}}{Y_m + Y_n}(f_n - f_m) \tag{24}$$

In terms of frequency determinations alone, the damping constant may be derived from Eqs. (21), remembering that BS = AV:

$$\alpha^2 = 4\pi^2 n_m n_n \tag{25}$$

A comparison of Eq. (25) with (17) shows that the quadrantal frequency-range is a mean proportional between the frequency-differences n_m and n_n.

An expression for the self-inductance L can now be derived, by substituting from Eqs. (19) and (25) in the equation $\alpha = R/2L$:

$$L = \frac{1}{4\pi(Y_m - Y_n)\sqrt{n_m n_n}} = \frac{R}{4\pi\sqrt{Y_m Y_n}} \cdot \frac{Y_m + Y_n}{f_n - f_m} \tag{26}$$

When the frequency scale-value σ has been determined from observations, for example by the method outlined in Sec. 12 below, and when R and the diameter 2ρ are known, we may, by combining the equation for the damping constant, $\alpha = R/2L$, with Eq. (12), write the following very simple equation for L:

$$L = \frac{R}{8\pi\rho\sigma} \tag{26a}$$

From Eq. (17), L may be expressed as $L = R/n_q$. This equation is not likely to be as useful as (26) in the experimental determination of L. In general, observations of the two extreme values of admittance and of the corresponding frequencies are made more readily than for the quadrantal points, and as we have seen they are quite as useful in deriving the values of the electrical constants.

6. *Capacitance C_2 in Series with Resonating Element.* In the present section we will assume that the impedance Z_2 in Fig. 1 is a capacitance C_2. In the case of the piezo-electric resonator, C_2 may represent the air-gap between crystal and electrodes.

Fig. 4 represents the admittance diagram for $RLCC_1$ if we take as origin the point F such that $s_y a = \omega C_1$ where a = AF. We shall prove that when C_2 is included in the circuit, the $RLCC_1C_2$ combination is equivalent to a *new resonating element* of the same form as

$RLCC_1$ having values $R'L'C'C_1'$ expressed in terms of R, L, C, C_1 and C_2, and independent of frequency.[7]

In Fig 4 let f_o be the frequency at B, and f that at any other point P. On inverting to an impedance diagram, the admittance FP becomes the impedance FP_1, of value $Z_1' = s_z \cdot FP_1$. This impedance is to be added vectorially to that of C_2. We lay off FF′ such that

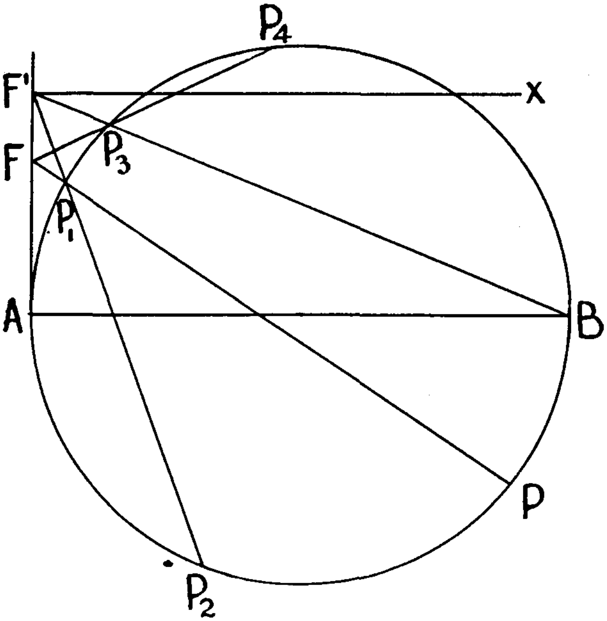

FIGURE 4. Graphical Construction for Capacitance C_2 in Series with Resonating Element.

$s_z \cdot F'F = -1/\omega C_2$. F′ becomes the new origin, the resistance axis being F′x, and $F'P_1$ is the vector sum of F′F and FP_1, so that for the impedance of $RLCC_1C_2$ we have $Z_2' = s_z \cdot F'P_1$. Let the distance AF′ be denoted by a'. From Eq. (6) we see that

$$-s_z a' = \frac{1}{\omega C_1} + \frac{1}{\omega C_2} \qquad (27)$$

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[7] See Watanabe, Proc. Inst. Radio Eng., Vol. 18, pp. 705–707, 1930.

Fig. 4 may be inverted back into an admittance diagram by performing inversions with F′ as center and a' as radius of the circle of inversion. Thus P$_1$ inverts into P$_2$, and the circle is the admittance locus for $RLCC_1C_2$ with F′ as origin of vectors. Besides the shift in origin, the introduction of C_2 has brought about two further changes: we have a new admittance scale-value s_y' given by

$$s_y' = \frac{1}{a'^2 s_z} \tag{28}$$

and the frequency-distribution around the circle is altered.

If the series reactance were inductive, the distance FF′ would of course be laid off in the opposite direction, and ωL_2 would replace $1/\omega C_2$.

The admittance of $RLCC_1C_2$, at the frequency corresponding to P on the Y circle for $RLCC_1$, is $Y_2' = s_y' \cdot \mathrm{F'P_2}$. This is the vector sum of two admittances, represented by F′A, which is practically constant over the resonance range, and AP$_2$, which has a circular locus. Since this is exactly the condition that obtained in the case of the diagram for $RLCC_1$, it is clear that the new diagram represents a series chain, which we will call $R'L'C'$, in parallel with a fixed capacity C_1' which, from Eq. (27), has the value

$$C_1' = \frac{C_1C_2}{C_1 + C_2} \tag{29}$$

The value of R' is found from Eqs. (2) and (28):

$$R' = \frac{1}{s_y' \cdot 2\rho} = R\left(1 + \frac{C_1}{C_2}\right)^2 \tag{30}$$

When the frequency-scale for the Y-circle for $RLCC_1C_2$ has been determined, L' and C' may be derived as in Sec. 5. The values are[8]:

$$L' = L\left(1 + \frac{C_1}{C_2}\right)^2 \tag{31}$$

$$C' = C\frac{C_2^2}{(C_1 + C_2)\ (C + C_1 + C_2)} \tag{32}$$

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[8] Watanabe, l.c., p. 707. For a general treatment of the equivalent constants of networks, see T. E. Shea, Transmission Networks and Wave Filters, New York, 1929.

7. *Frequency-scale for $R'L'C'C_1'$.* This is easily derived from the frequency-scale for $RLCC_1$. The scale-value σ, as defined by Eq. (12), remains unaltered, since the damping constant $\alpha = R/2L = R'/2L'$ is unchanged. All frequencies are, however, shifted clockwise around the circle, so that with C_2 in, the resonant frequency f_o' at B is higher than the value f_o for $RLCC_1$ alone. The relation between f_o and f_o' is easily found graphically, by the following procedure.

In Fig. 4, $s_y' \cdot \mathrm{F'B} = Y_2'$ at frequency f_o'. On performing inversions we have $s_z \cdot \mathrm{F'P_3} = Z_2'$, $s_z \cdot \mathrm{FP_3} = Z_1'$, and $s_y \cdot \mathrm{FP_4} = Y_1'$. Hence $\mathrm{P_4}$ marks the point of frequency f_o' on the Y-circle for $RLCC_1$. On this latter circle the frequency at B is f_o; the frequency-difference between f_o and f_o' can be found graphically according to Sec. 3. Analytically, from (31) and (32), the difference is given by the equation

$$\omega_o' - \omega_o = \frac{1}{2\omega_o L(C_1 + C_2)} \qquad (33)$$

8. *Capacitance C_3 in Parallel with Resonating Element.* The element may be either $RLCC_1$ or the equivalent $R'L'C'C_1'$ with C_2 in series. In either case the admittance locus is the circle in Fig. 2, with $s_y \cdot \mathrm{AF} = \omega C_1$ (or $\omega C_1'$). The addition of C_3 (which may be a condenser or the capacitance of lead wires) in parallel simply necessitates moving the origin upward exactly as in the case of a series capacitance C_2, Fig. 4, only here the new origin F′ is determined by the equation $s_y \cdot \mathrm{FF'} = \omega C_3$, or $s_y \cdot a'' = \omega(C_1 + C_3)$. This does not alter the frequencies around the Y-circle, but all admittance values are of course changed, and the critical points $\mathrm{P_4}$, $\mathrm{P_5}$, $\mathrm{P_6}$ and $\mathrm{P_7}$, Fig. 3, come at lower frequencies and $\mathrm{P_3}$ at a higher frequency.

If it is desired to invert to an impedance diagram with C_3 in, the process is as before, except that a new scale-value s_z'' must be employed, given by the equation $s_y s_z'' = 1/a''^2$.

A similar construction holds for an inductance L_3 in parallel with $RLCC_1$.

9. *Introduction of Series Impedance Z_2 and Parallel Capacitance C_3.* Let the point P, Fig. 5, on the Y-circle for $RLCC_1$ be given, at any frequency f, and let it be required to find the vector representing the impedance of $RLCC_1Z_2$ at this frequency. The impedance of $RLCC_1$ is $Z_1' = s_z \cdot \mathrm{FP'}$. Draw the line OF of such length that $s_z \cdot \mathrm{OF} = Z_2 = R_2 + jX_2$, and with a slope making $s_z \cdot \mathrm{OM} = R_2$, $s_z \cdot \mathrm{MF} = X_2$. As represented in Fig. 5, X_2 is capacitive. If it were inductive, M would lie below F.

The vector sum of Z_2 and Z_1' is Z_2', given by the equation $Z_2' = s_2 \cdot \mathrm{OP}'$. Its components are $R_2' = s_2 \cdot \mathrm{OW}'$ and $X_2' = s_2 \cdot W'\mathrm{P}'$.

If now a circuit element, as C_3 Fig. 1, is to be placed in parallel with $RLCC_1Z_2$, it becomes necessary to invert to an admittance diagram. If O is taken as the center of inversion the circle remains the same, and P′ inverts into P″. In performing this operation we must use as the radius of the circle of inversion the distance ON from O to the point of tangency on the circle. ON may be expressed in terms of ρ, C, C_1, and Z_2, and if we designate it by a_1'' we have for the new

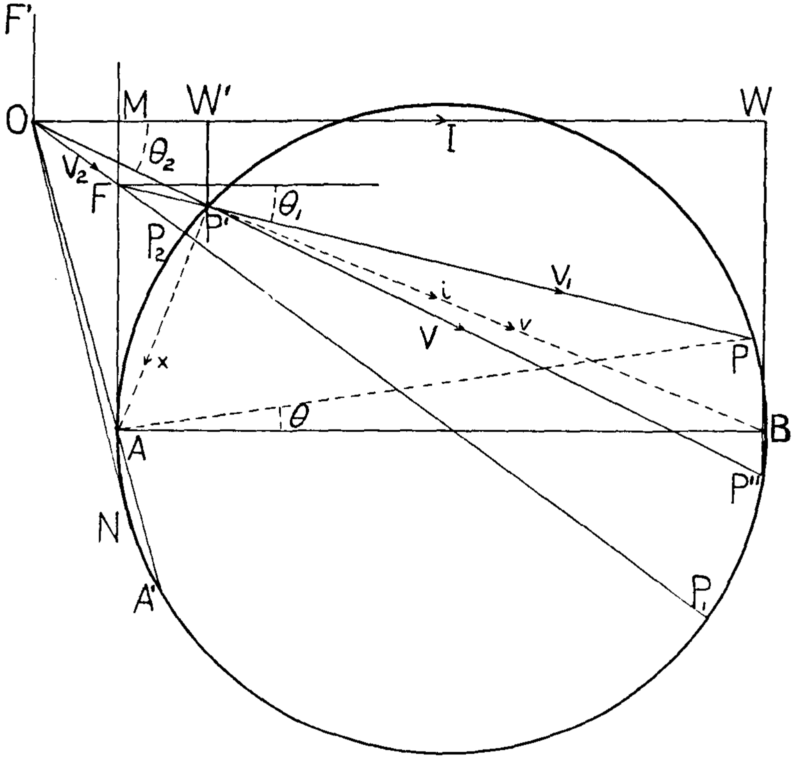

FIGURE 5. Graphical Construction for Impedance Z_2 in Series with Resonating Element.

scale-value on the admittance diagram for $RLCC_1Z_2$ $s_y'' = 1/(a_1''^2 s_y)$. The admittance of $RLCC_1Z_2$ at frequency f is then $Y_2' = s_y'' \cdot \mathrm{OP}''$.

As in Sec. 2, we find for the equivalent parallel resistance and capacitance of $RLCC_1C_2$, $R_2'' = 1/(s_y'' \cdot \mathrm{OW})$ and $X_2'' = -1/(\omega C_2'') = 1/(s_y'' \cdot \mathrm{WP}'')$.

Owing to the presence of R_2 in Z_2 it is not in general possible to replace $RLCC_1Z_2$ by an equivalent $R'L'C'C_1'$ in which the four parameters are independent of frequency.

With regard to the distribution of frequencies around the Y_2' circle, it is to be noted first that upon the addition of Z_2, A inverts into A′, which becomes the point for zero and infinite frequency. At points P_1 and P_2, on the extension of OF, frequencies are unchanged. All of the frequencies that on the $RLCC_1$ admittance circle lay on the lower (low frequency) portion between A and P are now compressed in the arc A′P″. All points on the admittance circle with Z_2 out (origin at F) are, upon the introduction of Z_2 (origin at O), shifted in both directions away from P_2 toward P_1. The frequencies are discussed further in Sec. 12.

The capacitance C_3 may now be taken into account by drawing a vertical reactance-axis through O and laying off a distance OF′ upward, such that $s_y'' \cdot \mathrm{OF}' = \omega C_3$. F′ is then the origin of admittance vectors for the network.

10. *Relative Phases of Current and Voltage Components.* Consider Fig. 5 as the impedance diagram for Z_2 in series with the resonating element. The arrows in this figure indicate relative phases, without regard to magnitudes of voltages and currents. Let the total current be denoted by I, its vectorial direction being horizontal as shown. Since by Sec. 9 the total impedance is $Z_2' = s_z \cdot \mathrm{OP}'$, it follows that the voltage V impressed upon the entire network lags behind I by the angle Θ_2. Similar reasoning shows that the voltage V_1 across the resonating element lags behind I by the the angle Θ_1, and that the voltage V_2 across Z_2 is parallel to OF. The direction of the current-vector for i, the current in the RLC branch, is obtained from P, the operating point on the admittance locus for $RLCC_1$. For we have by Equations (3) $Y = s_y \cdot \mathrm{AP}$, and since P is on the high-frequency side of the circle, i lags behind V_1 by the angle Θ. Now noting that the angles PP′B and PAB are equal, we see that PP′B also equals Θ, whence the direction of i relatively to I is given by the line P′B.

In the case of the piezo-electric resonator the velocity v of mechanical vibration is in phase with i, while the mechanical strain x lags behind v by 90° and is represented by a vector parallel to P′A.

11. The case may arise in which it is desired to find the values of R, L, C, and C_1 from observations of the impedance of $RLCC_1Z_2$ at various frequencies. Z_2 may, for example, be the resistance of a thermo-element, and we will here assume it to be a pure resistance, R_2. There are several alternatives.

a. If only the maximum and minimum values Z_{2m}' and Z_{2n}' are observed, R and C_1 can be derived. We have

$$\frac{1}{R} = s_y'' \cdot 2\rho = \frac{1}{Z_{2n}'} - \frac{1}{Z_{2m}'} \tag{34}$$

The value of C_1 is found from Eq. (18), in which Y_m and Y_n are the maximum and minimum admittances of $RLCC_1$. When R_2 is present, Y_m and Y_n must be derived from the observed Z_{2m}' and Z_{2n}'. This may be done graphically or analytically. For the graphical method, we construct an impedance diagram, Fig. 6. Starting with O as origin, the distance OF $= R_2/s_z$ is first laid off, and then FD $= (Z_{2m}' - Z_{2n}')/2s_z$, which is the radius of the impedance circle. To locate the center C, a perpendicular to FD is erected at D. The point C must then lie on this perpendicular, at such a distance from O that $s_z \cdot$ OC $= (Z_{2m}' + Z_{2n}')/2$. The point A is found by drawing AC parallel to FD. C_1 may now be derived from the equation FA $= 1/(s_z \cdot \omega C_1)$. Upon drawing a line from O through C we have $s_z \cdot$ OP$_1 = Z_{2n}'$, $s_z \cdot$ OP $= Z_{2m}'$. The values of Z_m and Z_n for $RLCC_1$ are obtained by drawing a line, not shown in Fig. 6, from F through C. The distances from F to the points where this line cuts the circle, multiplied by s_z, give Z_m and Z_n. We may then compute C_1 according to Eq. (18): $(\omega C_1)^2 = Y_m Y_n = 1/Z_m Z_n$.

The algebraic expressions for Z_m and Z_n are readily deduced from geometrical considerations. For Z_n we find

$$Z_n = \frac{1}{2}\left\{\sqrt{(Z_{2m}' + Z_{2n}')^2 + 4R_2^2 - 4R_2(2R_2 + Z_{2m}' - Z_{2n}')} - (Z_{2m}' - Z_{2n}')\right\} \tag{35}$$

Z_m may be derived from the equation $Z_{2m}' - Z_{2n}' = Z_m - Z_n$.

b. If in addition to Z_{2m}' and Z_{2n}' we know also the corresponding frequencies f_{2m} and f_{2n}, we can determine L and C. In order to avoid cumbersome expressions, it is usually advisable to derive the difference $f_n - f_m$ between the frequencies corresponding to Y_n and Y_m for $RLCC_1$, graphically according to Sec. 12. L is then found by Eq. (26),

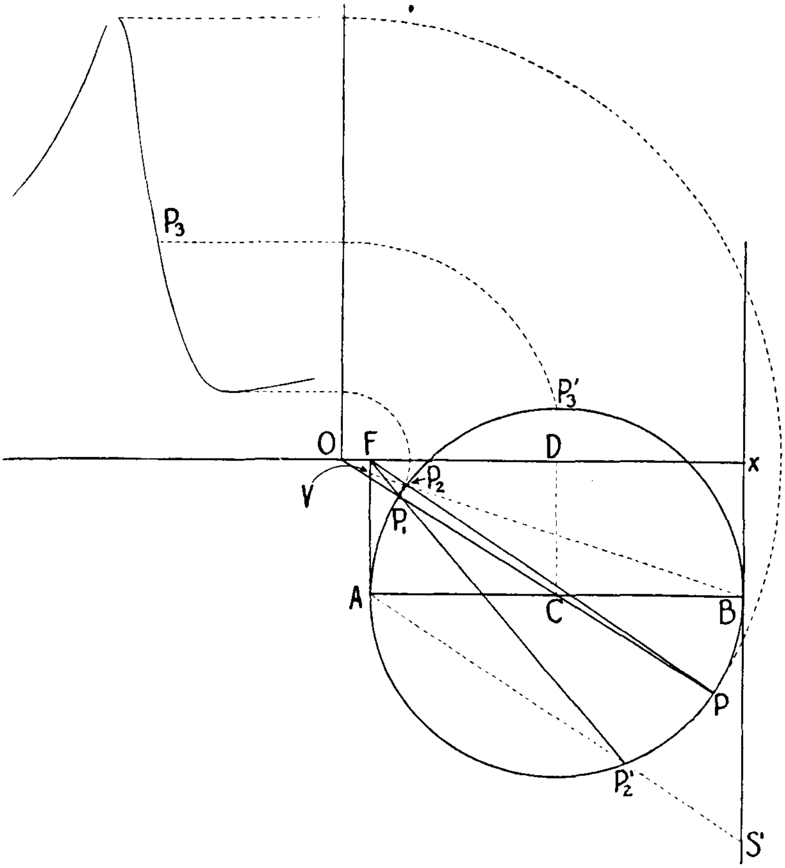

FIGURE 6. Resonance Curve with Associated Impedance Circle.

using values of Y_m and Y_n obtained graphically or derived from Eq. (35). From L and the resonant frequency, C is determined.

c. Greater precision is secured if impedances are observed at other frequencies in the neighborhood of resonance. R and C_1 must still be found from the maximum and minimum values alone, but the other observations may be utilized for fixing the frequency-scale around the circle, as described in Sec. 12. If the observed impedances are plotted in the form of a resonance curve, with frequencies as abscissas, as shown in Fig. 6, then after the impedance circle has been drawn in the manner indicated above, points on the circle corresponding to various points on the curve may be located.[9] Thus in Fig. 6, P_3' on the circle is found from P_3 on the resonance curve. By the reverse process, the frequency f_o at point B may be found from the resonance curve. Also, by the use of the circular locus the phase-angle for any impedance may be found.

d. If in addition to the above mentioned data the phase angles corresponding to the various impedances are known, each observation may be used independently to fix one of the points on the impedance locus. The circle that best fits the points is then drawn. Each point thus established may be made to play its part in fixing the frequency scale, as described in the next section.

12. *Determination of Frequency Scale for the Circular Locus, from Observations with Z_2 in Series with the Resonating Element.* If only Z_{2m}', Z_{2n}', and the corresponding frequencies f_{2m} and f_{2n} are observed, we construct the impedance circle as in Sec. 11, and locate P and P_1 Fig. 6. By inverting these points with respect to F we obtain P_2 and P_2' such that $s_y \cdot FP_2 = Y_n$ and $s_y \cdot FP_2' = Y_m$ for $RLCC_1$. If P_2 and P_2' are not too close to A we draw lines AP_2 and AP_2', and produce them to meet the axis Bx at points S and S′ (only S′ is shown in Fig. 6). From Eq. (12) we then have as the frequency scale-value $\sigma = (f_{2m} - f_{2n})/SS'$ from which by interpolation the value of f_o at B can be found, as well as the frequency at any other point on the circle.

If one of the points, as P_2, is rather close to A, it is better, as was shown in Section 3, to draw the line BP_2V. From AV the corresponding value of BS is found from Eq. (11): $BS = 4\rho^2/AV$. The value of σ is then $(f_{2m} - f_{2n})SS'$ as indicated above.

When the locations and frequencies of a number of points on the circle are experimentally determined, as stated at the end of Section 11, we make a similar inversion for each point with respect to F, thus obtaining a number of values of AV or of BS, with accompanying frequencies. From the values of AV the corresponding BS values are first computed by the equation BS $= 4\rho^2/AV$, after which the average value of σ is found either graphically or algebraically. When σ has thus been determined, α may be found from Eq. (12) and L from the equation $\alpha = R/2L$.

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[9] Cf. Dye, l. c., p. 433, Mallett, l. c., p. 134; Mallett and Blumlein, Jour. Inst. El. Eng., Vol. 63, p. 397, 1925.

If a line is drawn from F Fig. 6 through the center of the circle, it cuts the circle at two points which mark the maximum and minimum impedance for $RLCC_1$, as stated in Sec. 11a. By the method just described the two corresponding frequencies may be found.

When the method outlined above is followed, it is unnecessary to draw the resonance curve of Fig. 6.

13. *Effect of* C_1, C_2, *and* Z_2 *on Selectivity.* When anything beyond R, L, and C is taken into account, we can no longer use the damping constant $\alpha = R/2L$, for in all such cases the effective series R and L vary with frequency. Instead of this, we shall use as measure of selectivity the *change in phase accompanying a small change in frequency*, $\delta\Theta/\delta\omega$.[10]

As an illustration let us select two points close together, as P and Q, Fig. 7, on the admittance circle for $RLCC_1$. Then if the origin were at A (for RLC alone), the difference in phase-angle corresponding to (or causing) a frequency-difference PQ would be $\delta\Theta$. With C_1 in, the corresponding angular difference is $\delta\Theta_1$. If an impedance Z_2 is in series with $RLCC_1$ we first find P′ and Q′ inverse to PQ with respect to F; then using as origin the point O such that $s_z \cdot \mathrm{OF} = Z_2$, as in Fig. 5, we draw OP′ and OQ′, giving the angular difference $\delta\Theta_2$. Thus, with Z_2 in, a very small change in phase will shift the frequency from P to Q.

If the series impedance is a capacitance C_2 instead of Z_2, the origin O will be on the extension of the line AF, as in Fig. 4, but the statements regarding selectivity remain otherwise unaltered.

For a given resonator and for a given operating frequency there is an optimum position of the point F, Fig. 7, such that the selectivity $\delta\Theta/\delta\omega$ is a maximum. In Fig. 7, F is shown approximately in the optimum position for an operating frequency at P. This position may be attained by assigning the proper value to the parallel capacitance C_1 or by the use of a series capacitance C_2 of suitable value, as indicated in Eq. (27). The insertion of a series resistance generally lowers the selectivity; nevertheless, if the distance AF is greater than the optimum value it may happen that a series resistance, by bringing the origin O slightly to the left of F, actually improves the selectivity somewhat. Still, such improvement can never be as great as that brought about by proper choice of C_1 or C_2.

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[10] Mallett, l. c., p. 135, takes $\delta\omega/\delta \tan \Theta$ as measure of selectivity. It seems preferable to use the angle instead of its tangent for cases in which the origin is not at the point A, even though this makes the selectivity depend upon the frequency, because the change in phase accompanying a small change in frequency is an important criterion of resonator performance.

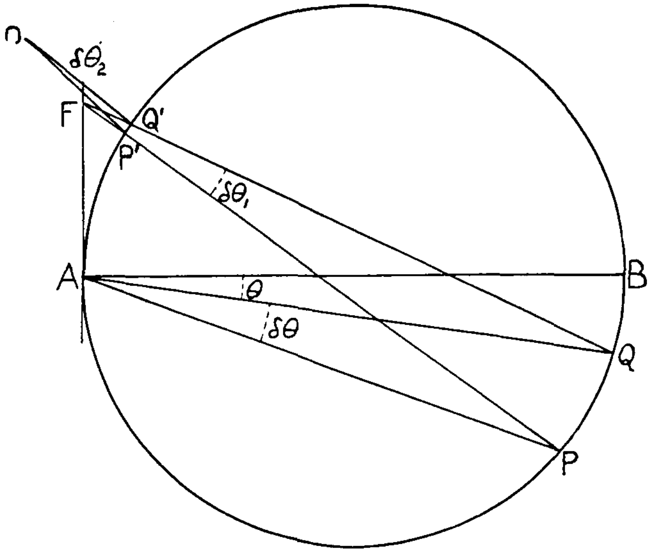

FIGURE 7. Effect of Additional Circuit Elements on Selectivity.

Whether the crystal is to be used as a resonator or an oscillator it is in general desirable to have the operating region in the neighborhood of the point B, where the frequency scale is wide open (See Sec. 15). For this region it is always an advantage in the interest of selectivity to have the origin near the point A.

A complete definition of selectivity should include the rate of change with respect to frequency of the *modulus* of the resonator impedance as well as that of its phase angle, since it is the rate of change in *vector* impedance that determines the selective properties of the resonator in any given network. The dependence of the modulus upon frequency may be obtained graphically from Fig. 7. For frequencies in the neighborhood of P, and with A as origin (*RLC* branch alone), this variation may be expressed, for example, as

$$\frac{(\mathrm{AQ} - \mathrm{AP})/\mathrm{AP}}{\delta\omega}.$$

When O is the origin (Z_2 in series with $RLCC_1$), the expression is

$$\frac{(\mathrm{OQ}' - \mathrm{OP}')/\mathrm{OP}'}{\delta\omega}.$$

This quantity, like $\delta\Theta/\delta\omega$, is in general greater the nearer the origin of vectors is to the point A.

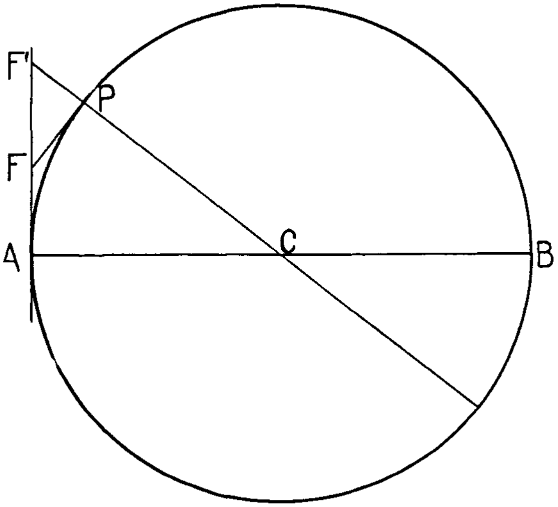

FIGURE 8. Graphical Construction for Determination of Condition for Maximum Voltage across an Air-gap.

14. *Condition for Maximum Voltage Across the Air-gap of a Piezo-electric Resonator.* As an illustration of the use of the impedance circle, let it be required to find the length of air-gap of a piezo-electric resonator necessary to build up maximum voltage across the gap with a given supply voltage. While the analytical solution is rather cumbersome, the solution is found graphically with great ease. The

problem was discussed by Dye[11] in relation to the luminous resonator of Giebe and Scheibe.

We seek the values of C_2 and f that make the ratio of X_2, the reactance of C_2, to Z_2' the total impedance of $RLCC_1C_2$, a maximum. In Fig. 8 is represented an impedance circle, the origin being at F for $RLCC_1$ and at F′ for $RLCC_1C_2$. We are dealing with a type of series resonance, in which the operating point P is on the high-frequency side of the circle. We have $s_z \cdot \mathrm{FA} = 1/\omega C_1$, $s_z \cdot \mathrm{F'F} = 1/\omega C_2 = X_2$, $s_z \cdot \mathrm{F'P} = Z_2'$. The problem is reduced to finding that position of F′ and also of P which makes F′F/F′P = sin FPF′/sin F′FP a maximum. It is not difficult to prove that this condition is satisfied when FP is tangent to the circle and the angle FPF′ is a right angle. The location of P determines the frequency, while the location of F′ determines C_2 and hence the length of air-gap. For a pronounced effect, C_1 should be as small as possible. It is to be noted that since F′P passes through the center C, the frequency is that of minimum impedance for the particular value of C_2, and that at this frequency the reactance of C_1 is numerically equal to the impedance of $RLCC_1$, because FA = FP. The point P is here unique in that it marks the same frequency on the impedance circle as on the admittance circle for $RLCC_1$.

15. *Frequency Calibration of the Admittance Circle as Affected by the Damping Constant.* From Eqs. (11) and (12), Sec. 3, it is evident that if the damping constant α decreases in a given ratio from α to α_1, the distance BS, Fig. 9, corresponding to any small frequency-difference $n = f_o - f$, where f_o is the frequency at B, must increase approximately in the same ratio, say from BS to $\mathrm{BS_1}$. Similarly, the distance SS′, corresponding to a small change δn in frequency from f to $f - \delta n$, becomes enlarged to S_1S_1' in the same ratio.[12] On the admittance circle for RLC, the arcs corresponding to δn are PP′ and $\mathrm{P_1P_1'}$ for large and small α respectively. We shall now prove that the arc corresponding to the difference between the same two frequencies f and $f - \delta n$, which obviously cannot increase in length everywhere around the circle in consequence of a diminution in α, increases until the quadrantal point $\mathrm{Q_1}$ is reached, and thereafter decreases toward zero. The same is true of the lower quadrantal point $\mathrm{Q_2}$ for frequencies below f_o. This property of the quadrantal points does not appear to have been mentioned hitherto, and it imposes a special interpreta-

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[11] D. W. Dye, Proc. Phys. Soc. London, Vol. 38, p. 454, 1926.

[12] In Fig. 9, since P is on the high-frequency side of the circle, the values of n and δn are negative.

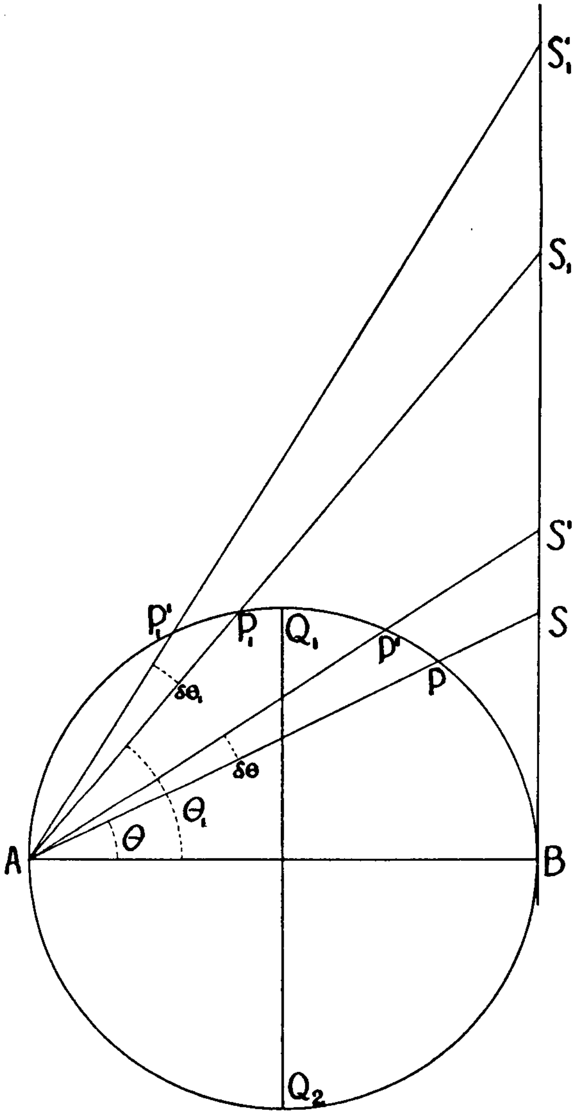
S_1'
S_1
S'
S
P_1'
P_1
Q_1
P'
P
$\delta\theta_1$
$\delta\theta$
θ_1
θ
A
B
Q_2

FIGURE. 9. Effect of a Change in Damping Constant upon Distribution of Frequencies around the Admittance Circle.

tion upon the statement that as the damping constant diminishes the frequency scale becomes "more wide-open."

It is convenient to use the small phase-angles $\delta\Theta$ and $\delta\Theta_1$ as the measures of the arcs, rather than the arcs themselves. Taking Θ as the phase-angle for damping constant α, it is easily shown from Eqs. (11) and (12) that

$$\frac{\delta\Theta}{\delta n} = \frac{2\pi}{\alpha}\cos^2\Theta = \frac{2\pi\alpha}{\alpha^2 + 4\pi^2 n^2} \qquad (36)$$

It is evident from (36) that so long as n, and hence Θ, is small, $\delta\Theta/\delta n$ increases as α decreases, while for large values of n the opposite is true. An increase in $\delta\Theta/\delta n$ means an increase in the arc corresponding to δn, as α decreases.

We will call $\delta\Theta/\delta n = \psi$, and find the value of n for which ψ is independent of α. This value is found from Eq. (36) by setting the derivative of ψ with respect to α equal to zero. The result is the equation $\alpha = \pm 2\pi n$. Now it is well known (see Eq. (17) Sec. 4) that this relation between α and n is characteristic of the two quadrantal points. Hence the values of n for which $\delta\psi/\delta\alpha = 0$ are those at the quadrantal points.

The conclusion may be summed up in the statement that for one and the same frequency-interval any decrease in the damping constant increases the corresponding arc on the circle, and thereby the phase-interval $\delta\Theta$, so long as the arc is on the resonance side of the quadrantal points, and decreases it when the arc is on the other side.

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THE LOWER PERMIAN INSECTS OF KANSAS. PART 6.

DELOPTERIDAE, PROTELYTROPTERA, PLECTOPTERA AND A NEW COLLECTION OF PROTODONATA, ODONATA, MEGASECOPTERA, HOMOPTERA, AND PSOCOPTERA.

BY FRANK M. CARPENTER.

THE LOWER PERMIAN INSECTS OF KANSAS. PART 6.

Delopteridae, Protelytroptera, Plectoptera and a New Collection of Protodonata, Odonata, Megasecoptera, Homoptera, and Psocoptera.*

By Frank M. Carpenter.

During the summer and fall of 1932, since the previous part of this series of papers was sent to the press, the writer was enabled by a grant from the Milton Fund of Harvard University to make an extensive trip to the central and western parts of the United States for the purpose of collecting fossil insects. One of the formations investigated was the Lower Permian limestone at Elmo, Kansas, where about two thousand specimens were obtained, bringing the total number of insects in the Harvard collection from this locality to some forty-three hundred.

Most of this new material, however, was collected in a layer of limestone which has not previously been worked for insects. The layer was first found during the summer of 1931 by Mr. C. B. Read, of the United States Geological Survey, who very kindly furnished me with a complete description of its position. It is exposed in a short gully running from east to west about four hundred feet northwest of the quarry which yielded the Harvard collection of 1927 (see plate 1 of Part 1 of this series of papers, 1930). With the exception of this new limestone bed the stratigraphy here is essentially the same as indicated by Dunbar in his diagrammatic section of the region (1924). In figure 1, plate 1, I have included a photograph of part of the 1932 quarry, showing the position of the new layer in relation to the rest of the Elmo limestone. The floor of this quarry is the top of the stump layer (D). Resting upon this floor is the white, blocky limestone (C) which at other exposures has produced the insects comprising the Sellards, Yale, and earlier Harvard collections. This layer is exposed in small gullies scattered irregularly over about ten acres, although it is not everywhere fossiliferous. In all these other gullies this blocky layer of limestone is overlain by several feet of shaly, thin-bedded limestone, unfossiliferous except for fragments of plants, as indicated in Dunbar's diagram of this area. But in the 1932 gully, shown on plate 1, this blocky layer is surmounted by about two inches of soft calcareous shale (B), which in turn is overlain by another blocky layer of limestone (A), the one found by Mr. Read. Above this is the shaly limestone found elsewhere in the field.

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The new layer of limestone is apparently not very extensive. At the head of the gully (the part shown in fig. 1, pl. 1), the layer is best developed, but it is not very fossiliferous there. On the south side of the gully (the right of figure 1), the layer is well developed and most fossiliferous; I worked it back to about ten feet from the edge of the gully, at which point it became thin and much fractured. On the bank opposite the mouth of the gully, not more than twenty-five feet west of the exposure shown in figure 1, the layer is completely absent, its place being taken by the usual shaly limestone. On the north side of the gully (left side of figure 1), not more than eight feet from the opposite side, the layer disappears abruptly, apparently because of a fault with a throw of about eight feet, the bottom of the stump layer being as the level of the surface of the ground. Just how far the layer extends eastward, i. e., in the direction in which the photograph in figure 1 was taken, I did not definitely ascertain. About ten feet eastward, where I made a small section, the layer was absent, but it may be present a few feet either side of this spot. Even with this uncertainty, however, it is clear that the layer is decidedly lenticular and limited to a very small area in comparison with that covered by the rest of the Elmo limestone.

The new layer of limestone closely resembles the lower, but differs in certain particulars. It is much harder, requiring the use of a heavy hammer to break it up; at the blow of the hammer sparks are frequently seen and there is a noticeable odor of sulfur. The rock is also somewhat heavier than that of the lower bed, the average specific gravity of the upper layer being about twelve per cent higher than that of the lower bed. Finally, the mineralogical nature of the two layers is somewhat different. The new one contains spots of limonite and numerous dendrites of combined limonite and manganese oxide, the resultant compound being dark gray in color. These dendrites are so abundant in the rock that they are visible in nearly every fragment of the upper limestone. This provides a means of recognizing fragments of the layer, if they should happen to become mixed with pieces of the lower layer.

Although it is impossible to set a definite figure for the amount of time which elapsed between the deposition of the lower layer and the upper layer, I believe that we are justified in concluding that at the most this could not have been more than a few hundred years, which from a geological viewpoint means that the faunas contained in the two layers of rock were practically contemporaneous. But what is also significant is that these two layers of limestone are separated by a dissimilar type of rock,—unfossiliferous and of a different structure. Such a condition suggests a change in the environment, possibly of a climatic nature. This is also indicated by the differences in the two limestone layers, already mentioned above. In view of these facts we would be justified in expecting to find the faunas of the two layers somewhat different, for although the species in each layer might easily have existed during the several hundred years consumed in the deposition of the two layers, the environments were apparently not entirely alike. It was with the hope of finding this to be the case that I made plans to visit the Elmo locality again after Mr. Read had described to me this new limestone. As the field work progressed, it soon became evident that there was a noticeable difference in the composition of the faunas of the two layers. These differences cannot be adequately discussed until all species of both layers have been studied; but a general idea can be formed from the descriptions of the fossils contained in this paper, as well as from the relative abundance of the specimens belonging to the different orders as observed in the field. Since the two layers of limestone were so close together, as shown in the photograph on plate 1, the fossils of both of the beds were collected, which enabled an easy comparison of the faunas by the general run of the specimens. Mecoptera, Protoperlaria, and Plectoptera were much less abundant in the upper layer than in the lower, whereas Psocoptera and Homoptera were decidedly more numerous. Most significant of all, perhaps, is the fact that larvae and nymphs were much more abundant in the upper layer. My examination of these new fossils under considerable magnification shows that the body structures of the insects in the upper layer are much better preserved than those in the lower layer, as will be apparent from the descriptions of the fossils in this paper. Since most of the species, of some groups at least, occur in both layers, this fine preservation of the body has enabled us to determine for the first time the complete structure of many of these Permian insects.

In this paper I have taken up first the new fossils belonging to groups which I have already treated in earlier publications devoted to the 1927 material, namely, the Protodonata, Odonata, Megasecoptera, Homoptera and Psocoptera; and following these, I have considered all the fossils in the Sellards and both Harvard collections belonging to the Delopteridae, Protelytroptera, and Plectoptera. In the descriptions of the insects of the Elmo limestone which are contained in this paper and others to follow, the layer which produced the specimens under discussion will be indicated as upper or lower, in reference to the new and old layers of limestone respectively. This will enable us to form an accurate conception of the two faunas when the study has been completed. Of course the Sellards, Yale and 1927 Harvard collections were taken entirely from the lower layer.

Before closing these introductory remarks, I wish to acknowledge the financial aid of the Milton Fund of Harvard University, which made the collection of this additional material possible; and the indispensible assistance of my wife, who collected with me in the field regardless of the difficult conditions, and secured about half of the series of specimens. We are both indebted to Mr. and Mrs. E. E. Bert and Miss Rowena Bert, of Abilene, Kansas, for the kindness which made our stay in Kansas pleasant in spite of the labor of the field work. We are also under obligation to Paul and Joseph Bert, who frequently helped us in the field and found several fine fossils. To Mr. C. B. Read, I owe my deepest thanks for furnishing me with notes concerning the new layer of limestone; and I am sure that all who are interested in the geological history of the insects share this gratitude with me.

Order PROTODONATA.

Seventeen specimens belonging to this order were collected in 1932, but three of these are small fragments which are too incomplete for determination. One other fragment (No. 3255), found in the lower layer of limestone, is so unusual that I have included a figure of it here (fig. 3), although there is not enough of the fossil preserved to justify the erection of a new genus and species. The preserved portion is 35 mm. long and consists of the anterior part of the wing near the apex. Judging from the size of the fragment, I estimate that the whole wing would be a little smaller than that of *Typus permianus*, probably around 10 cm. The peculiar features of the wing are the thickening of R1 in the region that would correspond to the pterostigmal area of the true Odonata, and the abnormal cell-formation below this thickening. Nothing of this nature has been found in the described Protodonata, so in all probability the fossil represents a new family. It is very unfortunate that more of this strange wing is not preserved. Another fragment of a wing (No. 3256), collected in the upper layer of limestone, belongs to *Oligotypus tillyardi* Carp., which has previously been known only from the type; it consists of the middle portion of the wing. The remaining specimens of Protodonata, except the two described below, belong to *Typus permianus* Sellards, which is now so well known that these additional fossils require no further comment.

Family MEGANEURIDAE.

Typus readi, new species.

Figure 1.

Length of preserved part of wing, 102 mm.; maximum width, 32 mm. Sc extending well beyond the origin of Rs, probably to the apex of the wing; R1 close to Sc distally, almost touching it beyond the origin of R3; R4 + 5 originating somewhat before the middle of the wing, remote from R3 and separated from it by several rows of cellules; first branch of MA arising slightly distad of the origin of Rs, Cu1 terminating beyond the middle of the wing; 1A well developed, remote from the hind margin; anal crossing undeveloped.

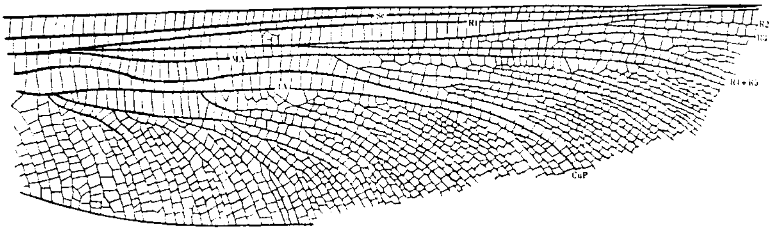

FIGURE 1. *Typus readi*, n. sp., holotype, No. 3253ab, Mus. Comp. Zoology.

Holotype: No. 3253, Museum of Comparative Zoology; collected by the writer with the aid of Joseph Bert, in the upper layer of the limestone at Elmo. I have named this species for Mr. C. B. Read, of the United States Geological Survey.

This species is much larger than the genotype, *permianus* Sellards. Comparing the preserved part of *readi* with the corresponding portion of *permianus*, I estimate the whole wing of the former to be about

140–150 mm. long, that of *permianus* being only about 110 mm. long. This new fossil possesses many interesting characteristics which require us to modify our definition of the genus Typus in at least one respect. The subcosta of *readi* extends very nearly to the apex of the wing, if not actually to the apex, as in Megatypus; whereas in *permianus* it terminates near the middle of the wing. I do not believe that this difference is sufficient to justify the separation of *readi* into a distinct genus, since in other respects the venation is very similar to that of *permianus*. There is no specialized anal crossing at the base of the wing, as in Megatypus, and R4 + 5 originates at about the middle of the wing, not in the more distal portion as in Megatypus.

Megatypus vetustus, new species.

Figure 2.

Length of preserved portion of wing, 60 mm.; width, 20 mm. Sc terminating somewhat before the apex; R1 close to the anterior margin distally; R4 + 5 arising well beyond the middle of the wing; first branch of MA originating just before the origin of Rs.

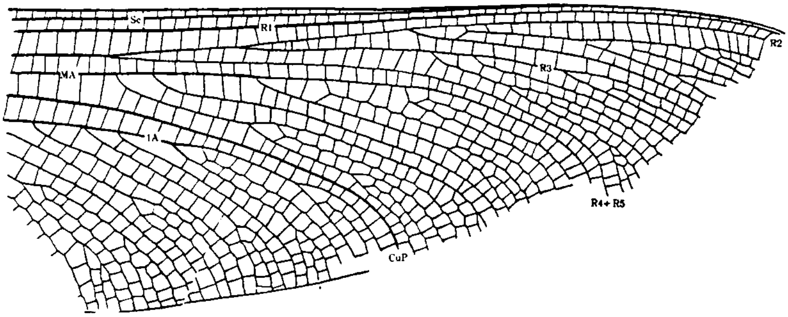

FIGURE 2. *Megatypus vetustus*, n. sp., holotype, No. 3254ab, Mus. Comp. Zoology.

Holotype: No. 3254, Museum of Comparative Zoology; collected by the writer in the upper layer of the limestone at Elmo. This fossil consists of the distal half of a finely preserved wing, but there is enough to show definitely what its affinities are. The wing is much smaller than that of *schucherti*, for the size of the preserved part indicates that the whole wing would be about 110–115 mm. long, about 60 mm. shorter than that of *schucherti*. The late origin of R4 + 5, much beyond the middle of the wing, places the insect in Megatypus, even though the structure of the anal crossing is not known. The wing differs from that of *schucherti* in two obvious features: the subcosta terminates before the apex, and the first branch of MA originates proximad of the origin of Rs.

Before leaving the Protodonata, I wish to call attention to Martynov's recent paper (1932a) on the Meganeuridae, in which he proposes to separate this family into a distinct order, Meganisoptera. This order he distinguishes from the true Protodonata, as represented by Protagrion, by the difference in the structure of the cubitus. For he considers that MP in the Meganeuridae became attached to MA and degnerated into one or more of the pectinate branches of MA; also that the convex vein CuA (or Cu1) changed to a concave vein, and is therefore the vein usually regarded as CuP (or Cu2). Similarly, the vein CuP (Cu2), which was originally concave, became convex and is hence the vein ordinarily identified as 1A. In a subsequent paper (1932b), however, after considering my description of the Protodonata in the Museum of Comparative Zoology, Martynov revises his conclusions and accepts the usual view that the convex and concave veins mentioned are CuA and CuP respectively. He still regards the Meganeuridae as a distinct order, though admitting that there is little to support this contention.

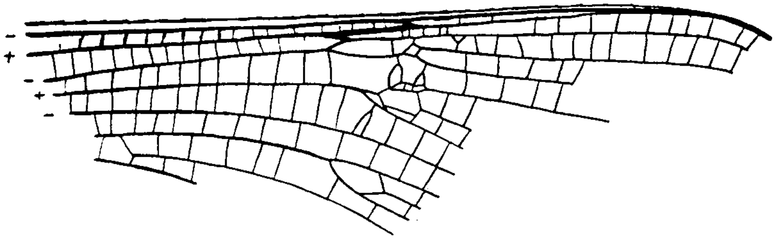

FIGURE 3. Fragment of wing of Protodonatan (No. 3255ab) in Mus. Comp. Zoology.

Order ODONATA.

Three specimens of Odonata were found in 1932, two of which belong to the Protozygoptera, while the third represents the Protanisoptera.

Suborder PROTOZYGOPTERA.

Family KENNEDYIDAE.

Kennedya mirabilis Tillyard.

Tillyard, 1925, Amer. Journ. Sci., **10**: 66; figs. 10–12.
Carpenter, 1931, Amer. Journ. Sci., **21**: 116; fig. 4.

One specimen of this species (No. 3220), found in the upper layer of limestone, consists of the basal two-thirds of a wing, from the very base to a little beyond the origin of IR3. The preserved portion of the wing is 17 mm. long and 4 mm. wide, so the whole wing would probably be about the same size as the type. This is the only specimen of *mirabilis* which has been found in addition to the types and it is the first one to show the cubitus and 1A complete. These veins were partly missing in the types (Nos. 5005, 5006a, Peabody Museum), but were assumed by Tillyard to be similar to those of Permagrion, from the Australian Permian. This new fossil shows that Cu2 and 1A were essentially as he supposed them to be. 1A extends to a point nearly below the origin of R3, sending a series of eight veinlets to the posterior margin.

There are a few points of difference between the new fossil and the types, but I believe that these are either only of an individual nature or are due to normal differences between fore and hind wings. The subcosta is longer in the new fossil, extending almost to the point of origin of R4 + 5. Also there is a cross-vein extending from R + M to Cu + 1A just basal of the origin of Rs + M which is not present in the types.

Progoneura minuta Carp.

Carpenter, 1931, Amer. Journ. Sci., **21**: 120–122; fig. 4.

The type specimen of this minute species consisted of the basal part of a wing. In the new material there is a nearly complete wing (No. 3221), found in the lower layer of limestone, which undoubtedly belongs to this species. The wing is somewhat smaller than that of the type, the preserved part (including all but the very apex of the wing) being only 15 mm. long. The entire wing could not have been more than 17 mm. long at the most and my estimate of the length of the type specimen was 21 mm. The new wing is also narrower than the type at the base, just beyond the petiole. But in Recent Zygoptera, at least in the Agrionid groups, the hind wings are noticeably shorter and narrower at the base than the fore wings, so that I am led to believe that this new fossil is the hind wing of *minuta*. Aside from these differences in size and shape, the preserved part of *minuta* is identical with the corresponding part of the new wing.

Inasmuch as the fossil is more complete than the type we are now able to add to our knowledge of the specific features of this insect. The base of the wing has already been described in detail in the original description of the species (1931), so I shall confine my discussion to the parts of the wing now known for the first time. Unfortunately, the cross-veins are not clearly preserved and parts of the posterior margin are broken away. Two post-nodals are visible, one just beyond the end of Sc and the other at about the middle of the wing. The pterostigmal area is preserved, but the pterostigma itself is not discernible. Rs, as suggested even in the base of the wing of the holotype, is surprisingly remote from R1 for its entire length. This is one of the out-standing features of the wing and may rank as a generic characteristic. R4 + 5 originates slightly distad of the termination of Sc and it terminates on the posterior margin quite remote from the apex of the wing. MA, of course, also terminates on the posterior margin, just beyond the middle of the wing, at a point far more proximal than in Kennedya; and Cu2 apparently ends below the origin of R4 + 5, much nearer the base than in any of the other known Protozygoptera. The space between Cu2 and MA is very wide distally.

Suborder PROTANISOPTERA.

Family DITAXINEURIDAE.

Ditaxineura cellulosa, new species.

Figure 4.

Length of preserved part of wing, 15 mm.; width, 6 mm. Wing pointed; three cross-veins between R1 and R2 + 3; two cross-veins between R2 and the pterostigma; three cross-veins between the consecutive branches of Rs.

Holotype: No. 3222, Museum of Comparative Zoology; collected by the writer in the upper layer of the limestone at Elmo, 1932. The type consists of the distal half of a very clearly preserved wing.

This species, the second in the genus, is close to *anomalostigma* (see Carpenter, 1931, fig. 4), but is larger and possesses more cross-veins and consequently more cells. The corresponding part of the wing of *anomalostigma* is 11 mm. long; if the wing of *cellulosa* was the same shape as that of *anomalostigma*, and there is no evidence to the contrary, then the whole wing would have been about 28 mm. long, 7 mm. longer than that of *anomalostigma*. The difference in the arrangement of the cross-veins is rather striking. In *anomalostigma* there are only two cross-veins between the consecutive branches of Rs and between R4 + 5 and MA, whereas in *cellulosa* there are three in all cases. In the preserved part of *cellulosa* there are consequently 41 cells; whereas in the corresponding part of *anomalostigma* there are only 29 cells.

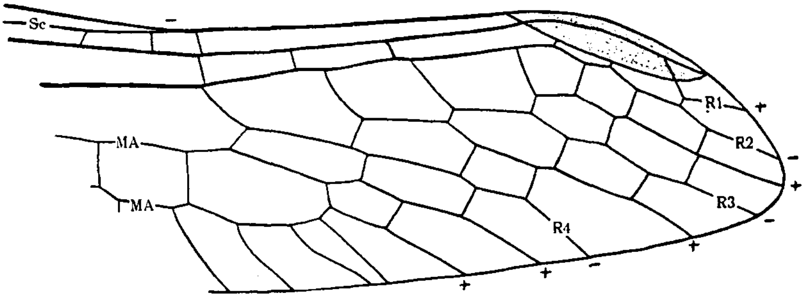

FIGURE 4. *Ditaxineura cellulosa*, n. sp., holotype, No. 3222ab, Mus. Comp. Zoology.

THE WING VENATION OF THE ODONATA.

Since the publication of my previous paper on the Lower Permian Odonata (1931), several additional accounts have appeared in print describing other Permian Odonata and discussing the evolution of the wing venation. Only a few weeks after the publication of my paper mentioned above, in which I established the suborder Protanisoptera, Martynov (1931a) published a description of a new fossil from the Permian of Russia, which he called Permaeschna. He assigned this genus to a new suborder, which he also named Protanisoptera, not knowing that I had already used this term. In a subsequent note (1931b), after discovering this repetition of names, Martynov substituted the term Permanisoptera for his Protanisoptera. In a still later paper (1932) he discussed more completely the relationship between Ditaxineura and Permaeschna and concluded that the two fossils and hence the suborders Protanisoptera and Permanisoptera were only remotely related and that whereas the Permanisoptera were the ancestors of the true Anisoptera, the Protanisoptera were an aberrant, side-branch. This conception of the affinities of these fossils is founded upon Martynov's interpretation of the venation of the anisopterous forms, an interpretation which differs widely from that employed by Lameere, Tillyard, and others, as well as previously by Martynov himself.

Since Martynov's explanation of the venation of the Odonata is based mainly on his conception of the venation in Permaeschna, let us first examine the fossils belonging to this genus. Two such specimens have been found and they have been placed by Martynov in separate species, *dolloi* and *proxima*. The latter consists only of the very distal part of a much distorted wing, including the area between the apex and the base of the pterostigma. This specimen therefore reveals none of the important phylogenetic features of the wing and may be dismissed without further comment. *Dolloi* consists of a rather poorly preserved wing, lacking the entire basal portion including the arculus, and having the more distal part considerably distorted. Certain areas in the central part of the wing are broken away or are obscure. Now it can be seen from this account of the nature of the fossils that the most important area of the wing, the arculus region, on which the classification of the Odonata is chiefly based, is completely unknown, so that whatever conclusions one may reach on the affinities of these fossils can only be regarded as controvertible until the nature of the arculus region has been determined. So far as the veins present in the wing are concerned, Martynov believes that CuA and MP are present in addition to CuP and MA, which have usually been recognized. He contends that CuA, however, has lost its basal connection with the stem of Cu and arises from M near the middle of the wing; and that MP is a short vein arising more or less below the nodus. But since the base of the wing is missing, there is absolutely no evidence to support the idea that CuA has become attached to the stem of M. So far as the preserved portion of the wing is concerned, we have as many reasons for assuming that it has *not* fused with M as for assuming that it *has* fused, especially since no Odonata are known which show such fusion. And as for the distal branch of MA, there are no more reasons for regarding this as MP or any other longitudinal vein than there are for considering the other secondary veins (such as the radial supplement) to be derived from a main vein. Or to put the last idea differently, since secondary veins are known to exist in the wing of Permaeschna, then there seems to be no evidence indicating that the distal branch of MA is more than a secondary vein.

I am therefore led to believe that Martynov's interpretation of the venation of Permaeschna is without foundation, simply because the important regions of the wing are not preserved, including the basal connection of the main veins. His conclusions may be correct, but at present there does not seem to me to be any evidence in this direction; and it is hardly justifiable to discard the now widely used interpretation of the venation in the Odonata without substantial proof.

About the same time that Martynov's description of Permaeschna appeared in print, Zalessky (1931a, 1931b) published a description of another odonate fossil, *Philidoptilon camense,* from a slightly different locality, but from the same horizon. He placed this fossil, which consisted of a single wing lacking the apex and most of the posterior part, in a new order, which he termed the Permodonata. From his description and discussion of this fossil it is at once obvious that Zalessky had only a very vague idea of the venation of the Odonata; he employs a somewhat modified form of the Comstockian system of nomenclature, assuming that the radial sector arises in the region of the nodus and crosses over part of the media. In a later paper (1932) Zalessky, apparently after investigating the literature on the venation of the Odonata, reconsiders the venation of Philidoptilon and proposes another interpretation of the evolution of the wings of the Odonata, as well as of the Plectoptera. Zalessky also advocates a new system of terminology for the veins, which is as confusing as it is misleading. He still contends in this second paper that Rs crosses over a part of the media, but he coins a set of new names for most of the other veins. He designates M1, M2, and M3 of Comstock (Rs of Lameere, Tillyard, etc.) as the *mediana anterior* (M. a.); M4 of Comstock as the *mediana posterior* (M. p.); a part of Comstock's Cu as the *cubitus anterior* (Cu. a.); and Cu2 of Comstock as the *cubitus posterior* (Cu. p.). As Zalessky himself has pointed out his *mediana anterior* is very different from the *anterior media* (MA) of Lameere, Tillyard, etc.; his *mediana posterior* different from the *posterior media* (MP) of Lameere; and so forth for the branches of the cubitus. It is difficult to understand why Zalessky, if he wished to coin new terms for these veins, should have employed ones which were identical in all but word order with those now in general use! In addition to the confusion caused by the use of such similar terminology much obscurity is thrown over Zalessky's arguments by the frequent occurrence of inaccurate and contradictory statements. He states that Redtenbacher and Brongniart designated concave veins by odd numbers and convex veins by even ones, although these authors actually did the reverse! Zalessky's paper on the venation of the Odonata is such a combination of misunderstandings and inaccuracies that his whole discussion is valueless.

As Martynov has already pointed out (1932a, p. 36), the figure of Philidoptilon given by Zalessky shows that the fossil is not only a member of the order Odonata, but is referable to the suborder Protanisoptera. In fact, Ditaxineura and Philidoptilon are strikingly similar in all important features, especially in view of the fact that the latter specimen is considerably distorted. Philidoptilon has a larger number of cells than Ditaxineura and it may also have had a differently shaped pterostigma; Zalessky shows the latter just distal of the nodus, but this is so far from the apex of the wing and the structure of the pterostigma so unusual as drawn by him, that I suspect that the real pterostigma is beyond the preserved part of the wing. The anal area of Philidoptilon is of particular interest; by linear arrangement of the sides of certain cells, directly below 1A, a longitudinal vein is formed which becomes fairly straight towards the margin of the wing. Now if we compare this area of Philidoptilon with that of Permaeschna, bearing in mind the presence of this secondarily formed vein in the former, we note a very close resemblance. In fact it would not be at all difficult for us to imagine the basal part of the wing of Philidoptilon attached to the preserved part of *Permaeschna dolloi*. The concave vein which Martynov figured below 1A in the latter could easily be formed by the linear arrangement of the sides of certain cells, as indicated nearer the base in Philidoptilon. This of course is only a supposition and too much weight should not be attached to it; but I believe that when the whole wing of Permaeschna is found, it will be very similar to that of Philidoptilon and Ditaxineura.

Order MEGASECOPTERA.

Suborder PROTOHYMENOPTERA.

Twenty-seven specimens belonging to this group were collected in 1932, aside from fragments too small for further study. Although most of these fossils are members of the family Asthenohymenidae, some excellent Protohymenidae are also included, which have enabled us to obtain for the first time a definite conception of the general habitus and body structure of these remarkable insects.

Family PROTOHYMENIDAE.

Protohymen permianus Till.

Protohymen permianus Tillyard, 1924, Amer. Journ. Sci. (5) **8**: 114.
Tillyard, 1926, ibid., **11**: 58.
Carpenter, 1930, Psyche, **37**: 349; fig. 1, 11.

Two specimens (No. 3264ab and 3265ab, Mus. Comp. Zool.) were found in the lower layer of limestone, and another (No. 3261ab), consisting of the basal part of a wing, was taken in the upper layer. One of the former specimens (No. 3264) consists of a finely preserved hind wing and is the best wing of this insect which I have seen. The venation is very nearly the same as shown in figure 1 of my earlier paper (1930), but differs in three respects: MA separates from Rs immediately after contact with it; the first cross-vein between R4 + 5 and MA is missing; and there are two cross-veins between Cu1 and MP. These are undoubtedly only individual variations.

Protohymen elongatus Carp.

Protohymen elongatus Carpenter, 1930, Psyche, 37: 353; fig. 2.

This species was previously known only from the holotype, which lacked a small part of the wing below the pterostigma. In the new material there are two more specimens. One of these (No. 3263), found in the upper layer, consists of the distal part of a wing, finely preserved. This of course shows the area of the wing missing in the type; it is essentially as I had restored it in my figure, with the cross-veins distributed as in *permianus*. The preserved part of the wing is 8 mm. long, which indicates that the whole length of the wing was about 20 mm. long, the same size as the type. There is one venational feature of this specimen which does not occur in the type, although it is unquestionably an individual characteristic; MA coalesces with Rs close to the origin of the latter, so that the fused portions of these veins are much longer than in the type. This new specimen of *elongatus* is of considerable importance, because it has enabled us to ascertain the nature of the apical region of the wing and to distinguish between *elongatus* and the new species described below.

The second specimen of *elongatus* (No. 3262ab), also from the upper layer, consists of all four wings and the abdomen, including the cerci. Three of the wings are more or less twisted, but the fourth, a hind wing, is spread out flat, and although a part of the base is broken away, the wing is 18 mm. long. This specimen is most interesting because of the preservation of the cerci, which have not previously been found in the Protohymenidae, although I have already described them in the Asthenohymenidae. The cerci, which are perfectly preserved, are 16 mm. long and are broken off at that length, so that in all probability they were some longer. There are about fifty segments in the preserved part of the cerci, shaped like those of *Asthenohymen dunbari.*

Protohymen readi, new species.

Figure 5.

Fore wing: length, 19 mm.; width, 3.5 mm.; very slender and pointed; R1 slightly diverging from the costal margin in the pterostigmal region, but the space between the costal margin and R1 is uniformly thickened and chitinized; between R1 and R2 + 3, just proximal of the pterostigma, there is a long heavy cross-vein; the first cross-vein between R1 and MA is proximal of the fork of Rs; Cu1 diverges upwards after its separation from Cu2, and immediately coalesces with the stem of M; MP separates from Cu1 close to the base of the wing, before the first cross-vein between Cu1 and Cu2; there are three cross-veins between Cu1 and Cu2, and three between Cu2 and 1A; 1A is joined to the posterior margin by two cross-veins, one midway along 1A and the other just before its termination.

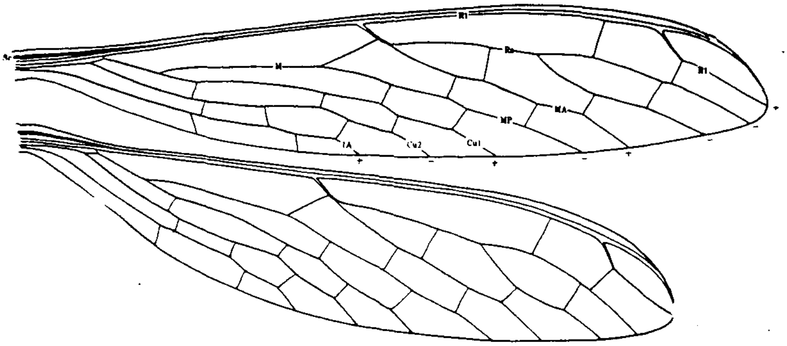

FIGURE 5. *Protohymen readi,* n. sp., fore and hind wings; holotype, No. 3257ab, Mus. Comp. Zoology.

Hind wing: length, 17 mm.; width, 4 mm.; slender and pointed; venation similar to that of the fore wing, except that the pterostigmal cell is somewhat shorter.

Length of body, not including cerci, 19 mm.; width of thorax, 2.8 mm.

Holotype: No. 3257ab, Museum of Comparative Zoology; collected by the writer in the upper layer of limestone at Elmo, in 1932. This species is named for Mr. C. B. Read, of the United States Geological Survey.

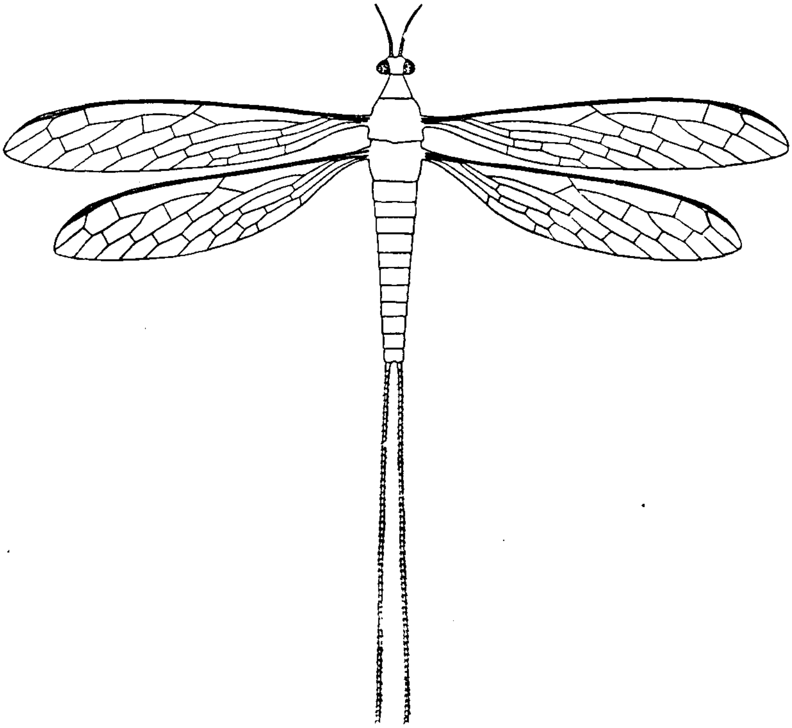

FIGURE 6. Reconstruction of *Protohymen readi*, n. sp., based on specimen No. 3257ab, with the addition of the cerci present in *Protohymen elongatus* Carp. (No. 3262ab). The legs are unknown.

This specimen consists of all four wings outspread, in a fine state of preservation, together with the head, thorax, and abdomen. Only the first few segments of the cerci are preserved, but undoubtedly the cerci were similar throughout all the Protohymenoptera. The antennae of this specimen are also preserved, but unfortunately the segments are not clearly separated; in all probability the antennae of the Protohymenidae were much like those of the Asthenohymenidae, already described in detail. The head is rather broad and possesses large compound eyes, which appear in the fossil to project posteriorly slightly, though this may be due to the position of preservation. The prothorax, which is clearly preserved, is rather long and is narrowed anteriorly. The prothorax is thus very similar to that described by Martynov (1932) in Pseudohymen, from the Permian of Russia. The mesothorcic and metathoracic segments in *readi* are rather large in proportion to the rest of the body, but that may be due in part to the flattening of the body during preservation. The segmentation of the terminal part of the abdomen is somewhat obscure, but this again was probably more or less the same in all the members of the family.

The wings of *readi* resemble those of *elongatus* closely and to a smaller degree those of *permianus*. From both of these, however, they can be distinguished by the presence of the long cross-vein between R1 and R2 + 3 before the pterostigma, as well as by minor characteristics, such as the earlier branching of MP from Cu1, and the presence of the third cross-vein between Cu2 and 1A.

This specimen of *readi* is especially interesting because of certain resemblances to the Protohymenoptera which have recently been described by Martynov from the Permian of Russia (1932). Four species were found by him in these beds, two of which, *Pseudohymen angustipennis* Mart. and *Ivahymen constrictus* Mart., belong to the family Protohymenidae. The other two, *Kuloja paurovenosa* and *K. cubitalis* Mart., belong to a very different family, Kulojidae. In the wings of the specimens of Pseudohymen and Ivahymen there is one vein which is not present in any of the previously described Kansan fossils,—the long cross-vein between R2 + 3 and R1, just proximad of the pterostigma. But in the specimen of *readi* this cross-vein shows in both pairs of wings. In other respects also *readi* approaches Pseudohymen and Ivahymen more closely than any of the other Kansan Protohymenoptera. The most obvious difference between these two Russian genera and *readi* is the number and arrangement of minor cross-veins. It should be noted that many of these cross-veins which Martynov has figured are shown as broken lines, indicating that their presence is not altogether certain. I think that we may also conclude that other cross-veins may exist in the wings of the Russian fossils, such as between the hind margin and 1A, but are simply not preserved. This lack of distinct preservation of the venation, as indicated by the broken lines in Martynov's figure of Pseudohymen may in fact account for his conclusion that M, Cu and 1A are coalesced with R1 at the base of the wing, which was the characteristic which he designated as the diagnostic feature of the genus Pseudohymen. If my suspicions are correct then in all probability Pseudohymen is much more closely related to *readi* and to the genus Protohymen than might be supposed at first from Martynov's figure. Certainly if the specimen of *Pseudohymen angustipennis* had been found in the Elmo limestone it would very likely have been placed in Protohymen; but naturally the remoteness of its real locality exaggerates the differences between these genera.

Since the type specimen of *angustipennis* has the body partially preserved, we are able to compare the body structure of this insect with that known in *readi*. Of particular interest is the similarity in the shape of the thorax, especially the prothorax, which is long and tapering in both insects. It is slightly longer in the Kansas fossil but the general form is remarkably close. The head and eyes are almost the same, although Martynov's fossil does not have the eyes shaped quite the same as in *readi*.[1] The abdomen is obviously much better preserved in the specimen of *angustipennis* than in that of *readi*, where it is curved to one side near the middle. The legs are not preserved in Martynov's fossils and only the femur of the right fore leg is visible in the type of *readi*, so we do not yet know anything of the leg structure of the Protohymenidae; but in all probability the legs of the members of this family were much the same as in the Asthenohymenidae, in which they have previously been described.

Family ASTHENOHYMENIDAE.

Asthenohymen dunbari Till.

Asthenohymen dunbari Tillyard, 1924, Amer. Journ. Sci. (5) **8**: 117.
Doter minor Carpenter (nec Sellards), 1930, Psyche, **37**: 360; figs. 2, 12, 13, 14; plate 2.
Doter minor Carpenter, 1931, Amer. Journ. Sci. (5) **22**: 127; fig. 8.
Asthenohymen dunbari Carpenter, 1932, ibid., **24**: 21.

Six good wings of this species were taken in 1932, three of which came from the upper layer of limestone. None of these specimens consists of more than a single wing, however, so that we are not yet able to distinguish between the fore and hind wings. It does not appear possible to estimate these differences in the pairs of wings from those which are known to occur in the Protohymenidae, for in the latter the differences between the fore and hind wings are in shape and the wings of Asthenohymen are so unlike those of Protohymen and Permohymen in form that we are unable to assume similar differences.

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[1] Tillyard has described the eyes of *Permohymen schucherti* as being very small (1926, fig. 1), but I have been unable to discern them at all in the poorly preserved specimen which he figured.

In my former paper on the Protohymenoptera (1930) I included a figure of the wing of a specimen of *Asthenohymen dunbari* in which I indicated that the wing was very narrow at the base and that the anterior margin near the base bulged outward slightly. Among the new specimens I find an interesting series of modifications of these two features. In specimen No. 3266 and 3267, for example, the hind margin is more or less smoothly rounded basally so that the wings are not nearly so narrowed at the base as shown in my figure; in specimen No. 3269 the hind margin is also somewhat curved, but the wing is fully as narrow basally as indicated in my figure. In specimen No. 3268 there is a noticeable bulge in the hind margin but the base is not nearly so narrow as in the foregoing fossil. I am therefore led to the conclusion that either the variation in both of these features is due to individual instability, or the variation in one of them (i. e. either the narrowness of the base or the bulge in the posterior margin) is due to the difference in structure of the fore and hind wings. At any rate since the fossils show no fixed association of these two features, it is clear that the differences in shape are not of a specific nature.

The new material has also enabled me to distinguish a certain variation in the form of Cu2. In the figure mentioned above, I represented this vein as being smoothly curved, approaching Cu1 at the base, but diverging from it distally. It now appears that while this form of Cu2 does occur, it is an extreme condition and not nearly so frequent as where Cu2 is straight, as in the specimen shown in figure 12 of my 1930 paper.

Asthenohymen pusillus Tillyard

Figure 7.

Asthenohymen pusillus Tillyard, 1926, Amer. Journ. Sci. (5) **11**: 68; fig. 7.

Doter pusillus Carpenter, 1930, Psyche, **37**: 362.

Although a rare species in the lower layer of the limestone, only three specimens having been found in a total of about 6,000 fossils, this insect is apparently not uncommon in the upper layer, 8 specimens having been taken in 1932. Since *pusillus* has not been correctly figured, I include here a drawing of the venation. The wing is smaller and broader than that of *dunbari*, and has several distinctive venational characteristics, such as the very short 1A, and the presence of only one cross-vein between MP and Cu1. Apparently there is a weak cross-vein between Cu2 and 1A.

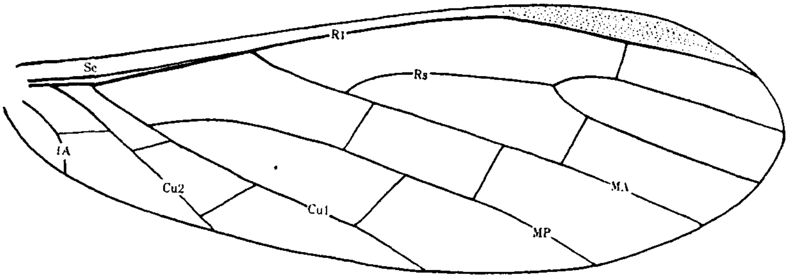

FIGURE 7. *Asthenohymen pusillus* Tillyard, fore(?) wing, based on specimens No. 3272ab and 3275ab, Mus. Comp. Zoology.

Asthenohymen affinis Tillyard.

Asthenohymen affinis Tillyard, 1926, Amer. Journ. Sci. (5) **11**: 65; fig. 2.
Asthenohymen stigmatazans Tillyard, ibid., 66; fig. 4.
Doter affinis Carpenter, 1930, Psyche, *37:* 361.
Asthenohymen affinis Carpenter, 1932, Amer. Journ. Sci. (5) **24**: 21.

This insect is also more abundant in the upper layer of limestone than in the lower layer, seven specimens having been collected there in 1932 and only three specimens having been found among all the fossils from the lower layer. Like *pusillus* this species has only one cross-vein between MP and Cu1, but it also lacks the basal cross-vein between Cu1 and Cu2 present in *pusillus*.

THE AFFINITIES OF THE PROTOHYMENOPTERA.

In my earlier discussion of the Protohymenoptera (1930) I concluded that the available evidence indicated that the Protohymenoptera were not ancestral to the Hymenoptera as originally advocated by Tillyard, but were merely a suborder of the Megasecoptera, which apparently became extinct in the Permian without giving rise to any existing orders. This view of the affinities of the Protohymenoptera seems to be generally accepted,[2] although Tillyard asserts (1932, p. 23) that he cannot accept my "supposed proof of it." In this connection the new fossils, Pseudohymen and Ivahymen, described by Martynov from the Permian of Russia, and the specimens of Protohymen in the Harvard collection secured last summer are especially interesting since they serve to substantiate this widely recognized view. In figure 6 I have included a drawing of the whole specimen of *Protohymen readi,* n. sp., with the cerci added, as they are found in *Protohymen elongatus.* As already pointed out this insect bears a striking resemblance to *Pseudohymen angustipennis* Mart. and both approach closely the Carboniferous Megasecoptera. All the body structures which are known in the Protohymenidae, such as the head, thorax, abdomen, eyes, antennae and cerci are identical with those of the Megasecoptera. The similarity of the venation in the Protohymenoptera to that of the Megasecoptera has already been discussed in detail in connection with the 1927 Harvard collection, but *Ps. angustipennis* and *P. readi* approach the Carboniferous Megasecoptera in general form of wing even more closely than the previously known Protohymenoptera. Of course the wings of some species of the latter, as in Permohymen Tillyard, are much broader than those of the members of Protohymen and the Carboniferous Megasecoptera, but in these broad winged species the venation is identical with that of the extremely petiolate types, except of course for the distance between veins.

Although Megasecoptera are not uncommon in the European Carboniferous, only a very few specimens have been found in American rocks of that horizon and these are so aberrant that their assignment to the Megasecoptera has been very uncertain.[3] While I was first studying the Kansan Protohymenoptera in 1928 I was struck by the resemblance between the venation of these fossils and that of one of the aberrant megasecopterous forms, *Raphidiopsis diversipennis* Scudder (1893), from the Upper Carboniferous of Rhode Island. But there were a few important points of difference in venation and not having the opportunity at the time to examine the type of Raphidiopsis at the National Museum, I made no reference to the fossil in my discussion of the Protohymenoptera. More recently, however, I had occasion to visit the National Museum and while there took advantage of the opportunity to study Scudder's fossil, which was kindly placed at my disposal by Dr. Bassler. Unfortunately the base of the fore wing is not preserved and the basal half of the hind wing is also obliterated; but I believe that enough is present to enable us to determine the relationship between this fossil and the Protohymenoptera. In figure 8 I have redrawn the fore wing of Raphidiopsis, and although the venation in general is the same as that given by Scudder, there are a few discrepancies to which I wish to call attention. Of course Scudder drew the wing just as it was preserved on the rock, including all the irregularities which were obviously due to the uneven surface of the matrix,—such as the indentation in the middle of the anterior margin of the fore wing. In my figure I have eliminated these obvious imperfections, just as one would omit from a figure of a recent insect such defects as wrinkles or tears in the wing. By completing the outline of the wing in this fashion, we find that the shape of the wing is more characteristic of the Megasecoptera than is suggested by Scudder's figure.

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[2] See for example Crampton, 1931, Ent. News, **42:** 136; Martynov, 1932, Trav. l'inst. Paléozool. de l'Acad. Sci. URSS, **1:** 2; Carpentier, 1932, Revue Quest. Sci., (4) **21:** 483.

[3] A typical Megasecopteron, belonging to the family Brodiidae, has recently been found in the Carboniferous of Kansas. The description of this insect is now in press.

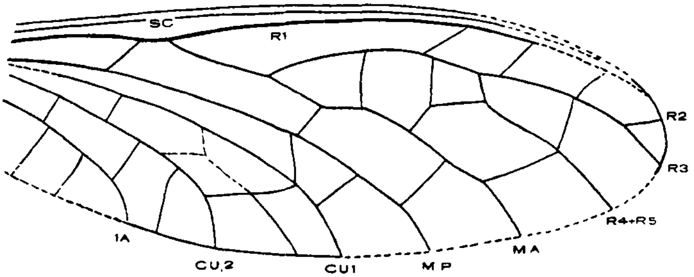

FIGURE 8. *Raphidiopsis diversipenna* Scudder, fore wing (original, drawn from holotype in U. S. Nat. Mus.). Upper Carboniferous, Rhode Island.

The subcosta can easily be recognized as a concave vein running more or less parallel to the costal margin and terminating in the apical third of the wing. Directly below the subcosta is the convex radius, which is more remote from the subcosta than the latter is from the costal margin; just before the origin of Rs the radius bends downward slightly, but soon curves upwards again and continues to the apex, almost parallel to the anterior margin, without forking. Scudder was uncertain of the origin of Rs in the fore wing, as shown in his figure, where he connected it to the media by a broken line. But in the hind wing the origin of the radial sector is more clearly preserved and was correctly represented in his figure of that wing. In the fore wing also, I believe that this structure can be discerned with careful examination. At any rate, the venation of the fore and hind wings is known to have been the same in the Megasecoptera, so far as the main veins are concerned. It seems clear, then, that the radial sector originates directly from R near the middle of the fore wing,[4] and shortly gives rise to two main veins, the anterior one being the true radial sector and the posterior one the vein which I have designated MA. The radial sector itself divides into R2 + 3 and R4 + 5, the former forking slightly at its very tip. The media divides before the origin of Rs, according to my interpretation, into the anterior and posterior branches (MA and MP); MA is directed upwards and fuses with R, but the posterior media continues without branching to the hind margin. The oblique vein proximal to Rs, which I consider the stem of MA, was apparently not observed by Scudder, although it is very strongly preserved in the fore wing. The origin of the next vein (convex) is obscure; as far as it can be traced proximally it is independent of all other veins and its convexity and position shows that it is the first cubitus (Cu1). Between this vein and the next one proximal there is perhaps a short interpolated vein, which is very faintly indicated and seems to be formed by the alignment of two cross-veins; there is some doubt that this vein actually exists, for the wing in this region is almost obliterated. The next longitudinal vein, Cu2, is clearly preserved, except at the base; and the last vein which runs almost parallel to the inner margin of the wing is likewise distinct. The latter vein, of course, is one of the anals, presumably 1A. Since only one anal vein is usually all that is present in the Megasecoptera, it is fairly certain that no others existed nearer the base in the wing of Raphidiopsis.

A comparison of this wing with that of a more typical Megasecopteron, as Psilothorax, demonstrates clearly its close relationship to that order. In Psilothorax the forking of R2 + 3 is much deeper than in Raphidiopsis, and the longitudinal veins are closer together, especially in the apical part of the wing; but the most obvious difference lies in the fact that Rs arises far more distad in Raphidiopsis than it does in the other Megasecoptera, and that the origin of MA is more distinct. These are the features that set Raphidiopsis off from the more primitive and typical Megasecoptera, for which I have proposed the subordinal name Eumegasecoptera. Now if we also compare the wing of Raphidiopsis with that of the Protohymenoptera, as Protohymen, we shall note several interesting similarities. The subcosta and R1 are the same in both, except that they are more widely spaced in Raphidiopsis; and in the hind wing at least there is no pterostigma in that genus, although this structure is very weakly developed in Protohymen and Pseudohymen and consequently is not essential to the Protohymenoptera. R1 in Protohymen diverges away from the costal margin at the apex of the wing, a feature which is not found in Raphidiopsis; but it is also not found in Asthenohymen. The radial sector arises in the same manner in both insects; and, exclusive of the slight twigging of R2 + 3 in Raphidiopsis (which may be merely an individual feature) the branching of Rs is strikingly similar. Most interesting of all, perhaps, is the origin and formation of the anterior media (MA), including the oblique basal portion. In the Lower Permian Protohymenoptera this oblique vein joins Rs rather than R, as in Raphidiopsis; but the latter has the more primitive structure. The posterior media (MP) is the same in the two wings, although of course we do not know whether MP is independent or is fused with Cu1 at the base of the wing in Raphidiopsis. Cu2 and the anal veins of the latter are more like those of Permohymen than of Protohymen, because the subpetiolate wings of the members of Protohymen have naturally required a tenuity of the anal and cubital veins.

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[4] According to my interpretation, of course, this stem of the radial sector really consists of Rs + MA.

So close are the wings of Raphidiopsis and those of the Lower Permian Protohymenoptera that it is difficult for us to find points of variance. The suggestion of an interpolated vein between Cu1 and Cu2 in Raphidiopsis is the only difference that is particularly obvious. There are also more cross-veins in Raphidiopsis than in the Protohymenidae: 18 in Raphidiopsis and 10–14 in the Protohymenidae. Of course there may be some other differences in the base of the wing which are not preserved in Raphidiopsis; but it seems clear that this insect is more closely related to the Protohymenoptera than to the more typical Megasecoptera of the Carboniferous (Eumegasecoptera). In view of these facts I believe that Raphidiopsis should be regarded as a Carboniferous member of the Protohymenoptera.

Since the base of the wing of Raphidiopsis is not preserved, we are unable to determine whether or not it was petiolate or broad at the base. The Permian Protohymenoptera possessed both of these types of wing-form in the extreme, and it is interesting to note that the shape of the wings seems to have developed along similar lines in both the Protohymenidae and Asthenohymenidae. In the former family, the wings of Protohymen and Pseudohymen are petiolate, whereas those of Permohymen are very broad, especially at the base. In the Asthenohymenidae the wings of *Asthenohymen dunbari* are distinctly subpetiolate whereas those of *A. pusillus* are nearly as broad as in Permohymen. The shape of the wing, then, appears to have been a very minor factor in the evolution of the Protohymenoptera. In all probability the ancestral Carboniferous forerunner of the Protohymenoptera possessed subpetiolate wings, like most of the Carboniferous Megasecoptera. In connection with the broadening of the wings at the base, it is interesting to find that the fore and hind wings were not expanded to an equal degree. The Carboniferous Megasecoptera, so far as known, possessed homonomous wings, as did also Protohymen and Pseudohymen; whereas Permohymen had the hind wing much broader than the fore wing, especially in the region of the anal area. This was also probably true of *A. pusillus*. In both wings, however, the venation was the same, except for the space between the veins.

Returning to the question of the relationship between the Protohymenoptera and the existing insects, Tillyard has suggested (1932, p. 23) that if the Protohymenoptera are descended from the Megasecoptera, then we may have to reconsider the view originally proposed by Handlirsch that the Neuroptera were derived from them also. This suggestion of Tillyard's was made because of the presence of a certain insect, Martynoviella, in the Elmo limestone, which he considered to be neuropterous and to have a wing venation close to that of the Protohymenoptera. This question will be discussed fully in the account of the Neuroptera.

Order HEMIPTERA.

Over fifty specimens of Homoptera were collected in the Elmo limestone in 1932, about thirty being well preserved. Of this series of fifty fossils all but one (No. 3280, *Permopsylla americana* Till.) were found in the upper layer of the limestone. Since this is about twice as many Homoptera as have previously been found in the lower layer in the Sellards, Yale, and 1927 Harvard collections combined, it is apparent that the order is more abundantly represented in the new limestone layer than in the lower one. The new material is as follows:

Lithoscytinidae, new family.

Tegmen: Sc parallel to R and R1, except at the base of the wing; R1 forked distally; M independent of R at the base of the wing; Cu dividing into Cu1 and Cu2 very near the base of the wing, *Cu1 continuing almost straight to the posterior margin, not fusing with M or even directed towards it;* Cu1 forked distally; Cu2 unbranched; 1A and 2A present.

Lithoscytina, new genus.

Tegmen: anterior margin straight or nearly so; R slightly arched at its base; M also arched at the base and parallel to R; Cu2 straight, remote from Cu1 distally; 1A remote from Cu2; 2A close to inner margin of the wing. Hind wing unknown.

Genotype: *Lithoscytina cubitalis,* n. sp.

Lithoscytina cubitalis, new species.

Figure 9.

Tegmen: length of preserved part (base to end of R1), 5 mm.; estimated length of whole tegmen, about 5.5 mm.; greatest width, 1.8 mm.; costal space relatively narrow; pterostigma slender; Rs originating at about mid-wing, at first directed posteriorly, but curving anteriorly near the apex; M dividing distad of the origin of Rs; Cu1 forked just before the division of M1; cell between Cu1a and Cu1b high; 1A terminating midway between Cu2 and 2A.

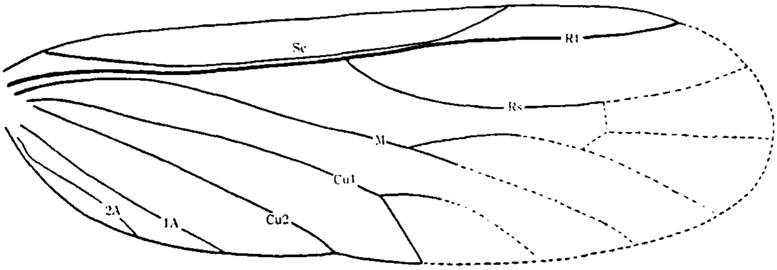

FIGURE 9. *Lithoscytina cubitalis,* n. sp., fore wing; holotype, No. 3289ab, Mus. Comp. Zoology.

Holotype: No. 3289, Museum of Comparative Zoology; collected by the writer in the upper layer of the limestone at Elmo, Kansas, 1932. The fossil lacks the apex and the distal part of the posterior margin; in figure 9 I have restored this part of the wing as it is in the Archescytinidae.

The family Lithoscytinidae is the third in the Homoptera now known from the Lower Permian of Kansas. The distinctive characteristic of this family is found in Cu1, which is free from M and not fused with it or even arched towards it as in all the other known species of Permian Homoptera. In this respect Lithoscytina is by far the most primitive Homopteron that has been found. The structure of the cubitus represents one more step towards the condition in the psocids. Aside from this peculiarity, Lithoscytina is similar to the Archescytinidae, except that 2A is much shorter and closer to the inner margin.

Family ARCHESCYTINIDAE.

Genus *Permopsylla* Tillyard.

Twenty-four specimens belonging to this genus were secured in 1932. Fifteen of these are too faintly preserved to show satisfactory specific characteristics, though in all probability they are either *permiana* or *americana*, the commonest species of the genus. Three tegmina of *americana* were found, of which No. 3282, taken in the upper layer of the limestone, is especially well preserved. In my earlier figure of this species (1931, fig. 3), I did not indicate the free piece of Cu1 basal of the junction with M because it was not clear in any of the specimens which had been found; but in this new fossil (No. 3282), as well as in another specimen (No. 3280, lower layer), this part of Cu1 is clearly preserved. It is essentially the same as in *permiana* (1931, fig. 4). The latter species is also represented in the new material from the upper layer by three tegmina, of which Nos. 3283 and 3284 are especially good. The following new species are also present:

Permopsylla grandis, new species.

Figure 10.

Tegmen: length, 6.5 mm.; width, 2.7 mm.; pterostigma short, R1b arched; R + M strongly arched at the base of the wing; Rs, after arising from R at a point nearer to the pterostigma than to the origin of M, curves posteriorly and then bends anteriorly near the apex of the wing, the arched formed being parallel to that of R1b; area between M and Rs very wide; M with three branches; M3 + 4 originating below the base of the pterostigma, remote from M2; distal cell of Cu1 about as high as long; 1A close to Cu2 and almost parallel to it; 2A remote from 1A and slightly sigmoidal in shape; inner margin of wing abruptly curved.

Holotype: No. 3287, Museum of Comparative Zoology; collected by the writer in the upper layer of limestone, 1932. This specimen is well preserved, showing all the veins clearly and lying smoothly on the rock. *Grandis* is much larger than any of the other described members of the genus and is also distinguished by the upward curvature of R1b and Rs.

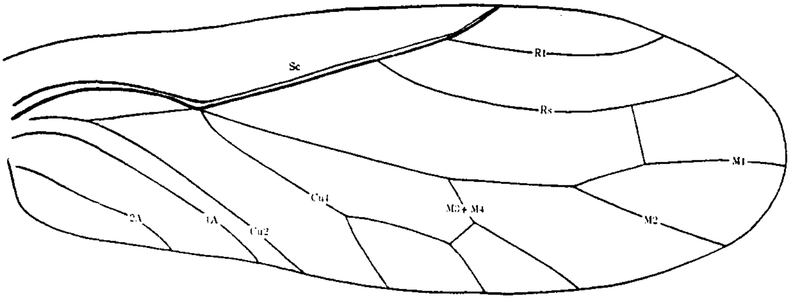

FIGURE 10. *Permopsylla grandis*, n. sp., fore wing; holotype, No. 3287ab, Mus. Comp. Zoology.

Permopsylla anomala, new species.

Figure 11.

Tegmen: length, 3 mm.; width, 1 mm.; pterostigma long; R1b slightly arched; Rs arising from R at a point much nearer to the origin of M than to the base of the pterostigma, the area between Rs and R1b being unusually wide; M with three branches; M1 diverging away from Rs distally; Cu1 with three branches, Cu1a apparently forked distally; cell between Cu1b and the lower branch of Cu1a about twice as high as broad; Cu2 smoothly curved; 1A close to Cu2 and converging towards it distally; 2A remote from 1A, slightly curved sigmoidally; inner angle of wing sharp.

Holotype: No. 3288, Museum of Comparative Zoology; collected by the writer in the upper layer of limestone at Elmo, 1932. The specimen is complete and shows the veins clearly.

Anomala is a very small species, being about the same size as *minuta* Carp. It is easily separated from the other species of Permopsylla by the presence of three branches on Cu1. It should be noted, however, that the extra branch may possibly be a part of the media, making the latter vein 4-branched; but the connection between this extra branch and Cu1 is very distinct, whereas that between it and M is very faint and is much like a cross-vein in appearance. Possibly the 3-branched condition of the cubitus may require the erection of a separate genus for this species. A 3-branched and even 4-branched cubitus has been found in other Permian Homoptera, however, as Orthoscytina, but has not been regarded as a generic characteristic.

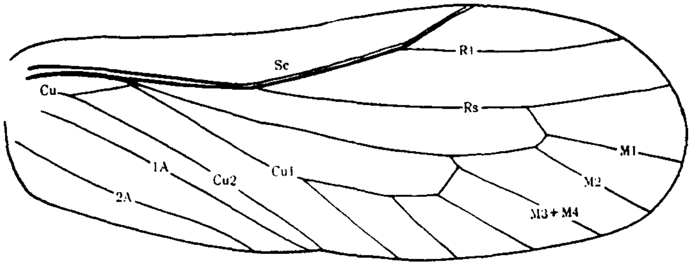

FIGURE 11. *Permopsylla anomala*, n. sp., fore wing; holotype, No. 3288ab, Mus. Comp. Zoology.

EVOLUTION OF THE HOMOPTERA.

Up to the present time a total of thirty genera of Homoptera have been described from the Permian, of which six are from the Kansas locality (Lower Permian), nine from Russia (Upper Permian) and fifteen from Australia (Upper Permian). Most of these fossils are known only from the tegmina; but the hind wings have been found in some of the Australian and Russian fossils. The body structure is known mainly from a single specimen of *Archescytina permiana* in the Museum of Comparative Zoology (Carpenter, 1931), although the general habitus of the body of an undetermined species of Scytinopteridae from Australia has been figured by Tillyard (1926). As I have already shown, the body structure of Archescytina demonstrates that the members of the family Archescytinidae, at least, cannot be referred to either the division Sternorrhyncha or Auchenorrhyncha, but require a separate division which I called Paleorrhyncha. Whether the other Permian families of Homoptera, of which there are nine described, belong to this same division cannot be ascertained until various body structures, especially the antennae, are known.

From a comparison of the fore wings or tegmina of the Upper Permian and the Lower Permian Homoptera, I believe that we can form a fairly accurate idea of the evolution of the tegmina during this period of time, and perhaps reach some conclusions regarding the venation of the fore wings in the Carboniferous ancestor of the Homoptera. The tegmina of the Upper Permian Homoptera are more specialized than those of the Lower Permian species in the following respects:

1. The thickness and granulation of the membrane. That the membrane of the Lower Permian species was tough and heavy is indicated by the intensity of the convexity of R1, as well as by the apparent texture of the wing surface; but there is no sign of the granulation or pitting which is so clearly preserved in the Scytinopteridae of the Russian and Australian fossils.

2. The division of the tegmen into a tough basal part and a softer, delicate distal portion. The tegmina of some of the Prosbolidae[5] of the Upper Permian show a distinct division into these two parts, the dividing line extending from about the termination of R1a to the end of Cu1b. Correlated with this division is the formation of the break or bend in the smooth contour of the anterior margin, forming a nodus somewhat comparable to that of the Odonata. In none of the Lower Permian Homoptera is there an indication of such a nodal structure or of the division of the membrane.

3. The intensification of the claval suture. In the majority of the Upper Permian Homoptera the claval suture, which lies along Cu2, is very deeply cut so that most of the specimens lack the anal area, which has broken off along this suture. Although the latter is also present in the Lower Permian species, it is not very deep; I have not seen one specimen of the Homoptera from the Elmo limestone which had the anal area broken off along this suture.

The fact that none of the Lower Permian Homoptera lacked the specializations mentioned above led Tillyard to believe that the archetype of the Homoptera could be formulated from the characteristics of the family Archescytinidae. In most respects this view appears to be correct, but the new fossil described above as *Lithoscytina cubitalis,* which I placed in a separate family, seems to contain certain venational features even more primitive than those of Archescytina; and in my opinion the media of some of the Upper Permian Homoptera is more primitive than that of the Kansas forms. So far as the subcosta is concerned there can be no doubt that the condition represented in Archescytina is by far the most primitive; it is entirely independent of the radius and terminates on the costal margin at a point rather remote from the end of R1a. The simplest condition of R is probably that in Permoscytina and Lithoscytina, where it is very little arched at the base and does not have an abrupt bend basally, as in Permopsylla. In all these Lower Permian forms R1 sends a small branch anteriorly, forming the posterior side of the pterostigma, while the main stem of R1 is continued nearer to the apex of the wing. Rs had already attained an unbranched condition and a late origin from R1; undoubtedly the condition which most nearly approaches that of the original Homopteron wing is present in *Permopsylla anomala,* where Rs arises from R somewhat nearer the base than in the other species. So far as the orgin of the media is concerned, I believe that the structure present in Lithoscytina· is the most simple; in this fossil it is entirely independent of R at the base, whereas in all other Permian species it is fused with R for about a quarter of the wing-length. In all the Lower Permian Homoptera the media has three branches, M1, M2, and M3 + 4; but in many genera of the Upper Permian family Scytinopteridae (Stenoscytina, Homaloscytina, Elliptoscarta and some species of Orthosctina) the media has four equal or subequal branches; and in at least one species of Orthoscytina the media has three branches, consisting of M1 + 2, M3, and M4,—the opposite of that present in the Archescytinidae. I therefore believe that in the archetype of the Homoptera the media consisted of four subequal branches.[6] The simplest condition of the proximal part of the cubitus is unquestionably represented in Lithoscytina; in this genus alone, of all the Permian Homoptera known, Cu1 is entirely unattached to M and does not even bend towards it basally. The question of the original number of branches on Cu1 is perhaps not so easily disposed of, however. In all the Lower Permian species, except *Permopsylla anomala* it forms a short distal fork, similar to the *areola postica* in the psocids. In *P. anomala,* however, it appears to be three-branched, and in many species of the Upper Permian Orthoscytina and Scytinoptera it is also three-branched. Tillyard (1926) argues that the two-branched condition is the original one and the three-branched state a specialization, and he is probably correct; but of course there is also the possibility that the Kansan forms merely represent a line of descent in which the cubitus was reduced to a two-branched condition from an original three-branched one. Cu2 is unbranched and strongly concave in all the Permian Homoptera, and both 1A and 2A are present and unbranched.

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[5] Regarded by Tillyard as directly ancestral to the Tropiduchidae and possibly to other families of the Fulgoroidea (Tillyard, 1926).

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[6] Tillyard, however, regards the four-branched media as a specialization in the Homoptera, rather than a primitive trait; this is also true of the three-branched cubitus (1926).

In figure 12 I have drawn a hypothetical wing of a Homopteron, with the combined primitive venational features present in the Permian Homoptera. If our deductions are correct, as I believe they are in all important respects, then this wing gives some idea of the common ancestor of the Homoptera, which must have existed back into the Carboniferous. This hypothetical wing again emphasizes the striking resemblance, already noted by Tillyard and Crampton, between the Palaeozoic Homoptera and Psocoptera. The only outstanding difference between the venation depicted in figure 12 and that of the Permian Dichentomidae is in the structure of Rs, which is forked in the psocids, but unbranched in the Homopteron. When we compare the hind wings of the Permian representatives of these two orders, however, we find great differences. The hind wings of the Lower Permian Homoptera are not known, but in the Upper Permian species they had an expanded anal area similar to that in recent Homoptera. The hind wings of the Permian psocids, on the other hand, were very nearly identical with the fore wings (see figure 13). In

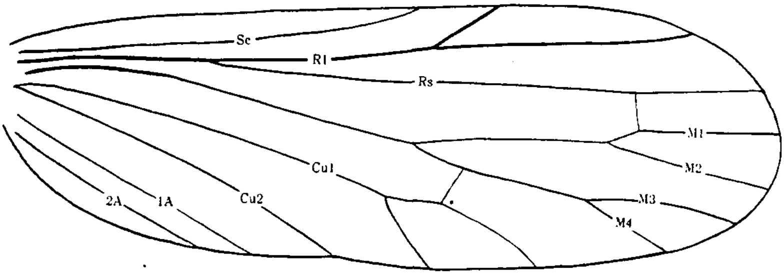

FIGURE 12. Hypothetical fore wing of common ancestor of the Permian Homoptera. For explanation, see text.

subsequent geological times, of course, the hind wings of the psocids were modified along lines directly opposite those followed by the hind wings of the Homoptera: they were greatly reduced in venation and in size. Since the homonomous condition of the wings in the Permian psocids was apparently a primary one and not a secondary state, and since the venation of the fore wings of the Homoptera was so very close to that of the psocids, it is very probable that the two orders arose from a common ancestor which had homonomous wings, possessing a venation much like that depicted in figure 12, with the addition of a fork on Rs.

Order PSOCOPTERA.

The psocids were taken abundantly in 1932, especially members of the genus Dichentomum. About 150 specimens were found, slightly more than half of which occurred in the upper layer of limestone. But among all these fossils taken in the upper layer, only one belongs to the family Permopsocidae, which apparently indicates that the Permopsocids had greatly decreased in numbers during the interval between the deposition of the two layers of limesone, at least in that vicinity where the rock is now exposed. The new collection of psocids is especially rich in specimens which show the body structure, those from the upper layer revealing many minute details which have not previously been known. A study of this material has added much to our knowledge of Permian psocids, the oldest members of the order known.

Family DICHENTOMIDAE.

Dichentomum tinctum Tillyard.

Figure 13.

Dichentomum tinctum Tillyard, 1926, Amer. Journ. Sci. (5) **11**: 320.
Dichentomum tinctum Carpenter, 1932, ibid., **24**: 7; fig. 1 (complete synomymy given here).

About 125 specimens of this species were collected in 1932. Most of these consist of single fore or hind wings and add nothing to our knowledge of the insect; but there are a few which are more completely preserved than any that I have previously seen. My figure of the fore wing of *tinctum* already published (1932, fig. 1) although correct for the specimen drawn, does not represent the average or commonest form of the wing. In most cases the fore wing is not quite so slender and it has a slight bulge in the anterior margin near the

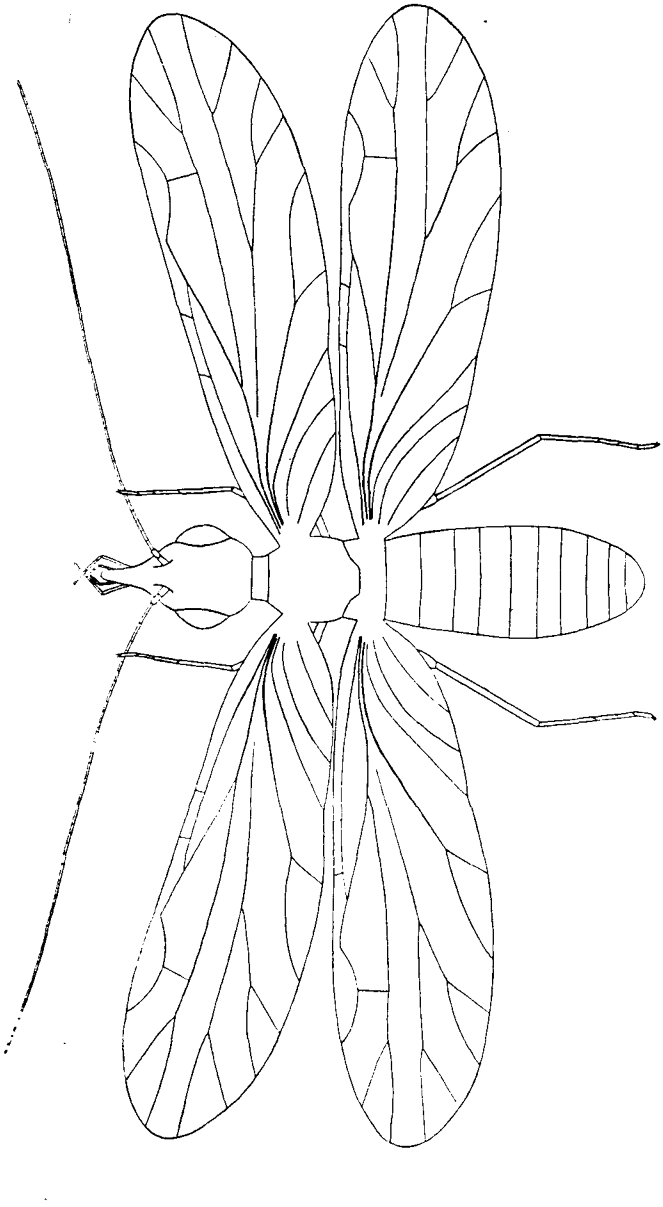

FIGURE 13. Reconstruction of *Dichentomum tinctum* Till. based on several specimens in the Museum of Comparative Zoology.

base. The more typical form of the wing is shown in figure 13. In none of the large number of specimens found is the basal connection of the media visible and I am convinced that this connection had already disappeared by the Lower Permian. In my previous account of these psocids, I stated that the hind wings were essentially the same as the fore wings, not smaller and with a reduced venation as described by Tillyard. This conclusion was based upon several specimens in the 1927 Harvard collection, although as I stated at the time in all of these fossils the hind wings were partially covered by the fore wings. In the new material now at hand there are a few specimens (e. g. No. 3316) which have the fore wings directed forwards and entirely clear of the hind wings. In these we see more distinctly that the fore and hind pairs were alike, the latter being only a little shorter and having a slightly lower areola postica.

The specimens of *tinctum* found in the upper layer furnish us with a nearly complete knowledge of the body structure, as a result of a preservation that is truly remarkable in view of the minute size of the insects. Tillyard (1926, p. 327) has given a restoration of the lateral view of this insect, based upon the type of *Ps. kansasense* (No. 5077, Peabody Museum), and has also presented a more detailed figure of the thoracic and abdominal structure, based upon the type of *Ps. robustum* (No. 5074, Peabody Museum).[7] The latter specimen I examined carefully at the Peabody Museum but was unable to distinguish some of the details which he shows in his figure, such as the small eye situated at the top of the head. The former specimen I was not able to locate in the Yale collection when my previous paper was being written, and I consequently accepted Tillyard's statements concerning the structural features which he described, such as the 5-segmented tarsi and the size and position of the eyes. On a very recent visit to the Peabody Museum I was able to examine this fossil and found that the head and tarsal structure was not nearly so distinct as I had expected. In the new material secured in 1932, however, there are many well preserved specimens which show these structures clearly and also differently from the way in which they

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[7] Tillyard figures (1926, p. 325) certain structures which he supposed to be terminal abdominal appendages in this species (based on the type of *Ps. abnormale*, No. 5073, Peabody Museum), but I have already given reasons (1932, p. 2) for regarding these as something entirely foreign to the psocid in question. In none of the new specimens have any such appendages been found.

were depicted by Tillyard. The eyes are obviously very large, and the fore tarsi, at least, are only 4-segmented. These particular specimens in the Harvard collection are so well preserved and the fossils in the Yale collection on which Tillyard based his description so indistinct that I believe there is no doubt of the true condition of these parts in the species under discussion. To show how clearly these portions are preserved in the new fossils, I have included here a photograph showing the head and fore legs as preserved in specimens No. 3324 and 3348 (plate 1). The former specimen is a dorsal view, with the eyes and the palpi splendidly preserved; the latter is a lateral view, the fore legs outstretched and parallel, the tarsi being very distinct. Other specimens show various portions of the body, the following being the most important: No. 3321ab, showing the head, abdomen, and antennae; 3323ab, with clearly preserved legs and abdomen; 3331ab, a fine complete insect, with distinct antennae and fore legs; 3346ab, with excellent antennae; 3347ab, with all legs; 3349ab, the whole insect in lateral aspect; 3353, showing the antennae; 3354ab, with the tarsi (fore?) preserved; and 3357ab, with the antennae.

After a study of these specimens and others not so important, I have prepared a reconstruction of a whole specimen of *Dichentomum tinctum*, by combining these various structures into one individual. The several parts of the insect have been drawn to the same scale by careful micrometric measurements and I believe that the resulting figure gives an accurate idea of the appearance of this insect (figure 13).

There are several features of unusual interest in this insect to which I wish to call particular attention. The head is rather large in proportion to the rest of the body, approaching in this respect the condition in recent psocids. The eyes are also large and prominent, as in the existing members of the order. The antennae are inserted just before the eyes, their insertions being most clearly indicated in specimen No. 3348ab. The first (and possible the second) antennal segment is rather stout and short, but the remaining segments are smaller. The antennae are long and filamentous, being very thin distally, and consist of about 50 segments.[8] The most remarkable feature of the head structure is the presence of what appears to be a

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[8] In my previous paper I described the antennae as consisting of about 27 segments. This description was based upon one antenna of a single specimen; in the new material there are several specimens with much longer antennae, which appears to indicate that in the earlier specimen the antennae were broken off near the middle.

short rostrum. This is preserved in so many specimens (see Plate 1) that there can be no doubt of its existence. Such a structure is unknown in other psocids. That this rostrum is not formed by long mandibles shut together is demonstrated by the fact that in a dorsal or ventral view of the insect there is no line of separation visible along the middle. The maxillary palpi are long and apparently three-segmented; the labial palpi are smaller and the segments have not been distinguished. The details of the thorax are still more or less incompletely known; in all specimens the thorax as a whole has the appearance of being crushed or flattened, so that the divisions between the segments are obscure. The length and width of the thorax is easily estimated, but the shape of the different segments has not been ascertained with certainty. I believe, however, that the prothorax is short and more or less in the form of a collar; the mesothoracic segment is clearly the widest and largest, and it appears to have a broad posterior prolongation; the metathorax is rather short and not markedly separated from the abdomen. As for the attachment of the wings, some fossils (e. g. No. 3324) show that the hind wings are attached to the thorax rather far back, as in most recent psocids. At any rate, the hind wings are so broad at the base that they would naturally be placed as far as possible from the fore wings in order to prevent overlapping. The abdomen is well preserved in many specimens; it is rather short, being only about as long as the head and thorax combined, but it is quite similar to that of recent psocids in that respect. Ten segments are visible. I can find no traces of cerci or genital appendages. Certain species show in ventral view a large subcircular disc at the very apex of the abdomen which may be the female genitalia, but no definite structure can be distinguished. As to the legs, I have not seen any specimen which shows the coxae distinctly, this being due to the crushing of the thorax previously mentioned. The femur is slender, and the tibia and tarsus even more so; no tibial spurs are discernible, but they may have been present. I have been able to see the tarsal segmentation with absolute clearness in the fore legs only. The tarsus is long, the metatarsus being about twice as long as the other segments; a pair of small but sharply curved claws is present. The middle tarsus is preserved in some of the fossils, but the segmentation cannot be made out satisfactorily. The hind tarsus is not known to me at all, since the hind legs are usually folded up under the abdomen. In all probability the middle and hind tarsi were similar to the front ones, and I have so drawn them in my figure.

The habitus of *D. tinctum*, as represented in this reconstruction contains many resemblances to that of the recent psocids,—the large eyes and head, short abdomen and particularly the wing venation; but at the same time the homonomous nature of the wings gives an appearance of generalization foreign to the existing Psocoptera.

Dichentomum grande, new species.

Figure 14.

Length of wing: 6 mm.; width, 2 mm.; wing broad, the anterior margin convex, and the inner margin of anal area abruptly curved; costal space broad basally, the anterior margin without the bulge usually present in *tinctum;* Sc giving rise to three veinlets leading to the margin (this number may be variable); R1 abruptly diverging anteriorly at the origin of Rs; pterostigma very faint; M dividing near the middle of the wing; fork M1 + 2 deeper than fork M3 + 4; Cu1 remote from M at the base, but arched towards it near the origin of Rs, then curved towards the posterior margin; areola postica about twice as long as high; Cu2 smoothly curved, remote from Cu1; 1A close to Cu2 and nearly papallel to it, slightly divergent distally; 2A remote from 1A, not strongly arched.

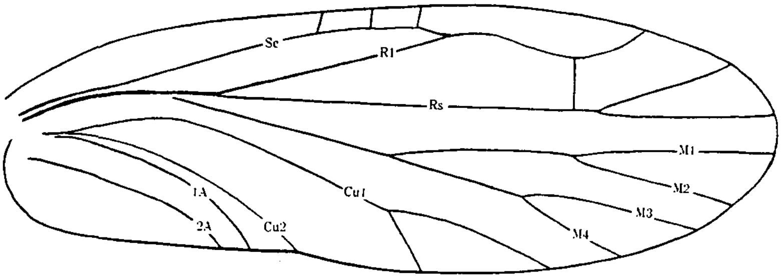

Figure 14. *Dichentomum grande*, n. sp., fore wing; holotype, No. 3358ab, Mus. Comp. Zoology.

Holotype: No. 3358ab, Museum of Comparative Zoology; collected by the writer in the upper layer of limestone at Elmo, 1932. The shape of the areola postica indicates that this is probably a fore wing.

This is the largest psocid which has been found thus far in the Elmo limestone. In addition to its size, it is distinguished from the other species of Dichentomum by the strongly arched anterior margin, especially the basal portion, by the remoteness of Cu1 from M at the base, and by the abrupt bend of the inner margin of the wing. The type specimen is finally preserved, the veins being dark brown and the membrane light brown. Several other psocid wings with similar habitus were also collected in the upper layer, but since the venation is not very clear in any of these, they can be placed here only with some uncertainty.

Family PERMOPSOCIDAE.

Progonopsocus permianus Tillyard.

Progonopsocus permianus Till., 1926, Amer. Journ. Sci. (5) **11**: 338; fig. 14.

Progonopsocus permianus Carp., 1932, ibid., **24**: 12; fig. 5.

Four specimens of this species were collected in 1932, one (No. 3361ab) being found in the upper layer. Only one of the fossils is worthy of note: No. 3363ab. This consists of an excellent fore wing, in which Rs has three branches, the extra branch being on R4 + 5. In all other respects the wing is identical with that illustrated in my figure (1932, fig. 5).

Permopsocus latipennis Tillyard.

Permopsocus latipennis Till., 1926 Amer. Journ. Sci., (5) **11**: 340; fig. 15.

Permopsocus latipennis Carp., 1932, ibid., **24**: 13; fig. 6. (Complete synonymy given here.)

Two specimens of this species were collected in 1932. One, taken in the upper layer (No. 3359ab), consists of three wings and parts of the body. The abdomen and thorax are similar in habitus to those of Dichentomum. The head as a whole is not clear, but the two antennae are preserved. They are about the same in size as represented in my earlier paper, so in all probability the antennae which I described there were complete. They are much shorter than those of Dichentomum.

Lithopsocidium permianum Carpenter.

Figure 15.

Lithopsocidium permianum Carp., 1932, Amer. Journ. Sci. (5) **24**: 15, fig. 7.

One well preserved specimen (No. 3365ab) of this insect was taken in the upper layer, consisting of a single wing, 3.9 mm. long. It has the same texture as the type, the membrane being heavy in appearance. All veins, including M3 + 4 are much clearer than in the type and several features not visible in the latter specimen are discernible. There is a series of five long and heavy cross-veins between Sc and the costal margin; this is a characteristic not previously found in the Permian psocids and probably represents an ancestral condition in which the subcostal veinlets were even more numerous. The base of Cu1 is more remote from M than I supposed it to be in my description of the type specimen. In my figure of the latter I showed the indentation of the hind margin at the end of 1A; the new specimen demonstrates that this is incorrect, the apparent indentation in the type being due to an inperfection in the preservation. The indentation really is situated at the end of Cu2, as in other psocids. In figure 15 I have redrawn the wing with the addition of the several structures not previously given and with the above correction made.[9]

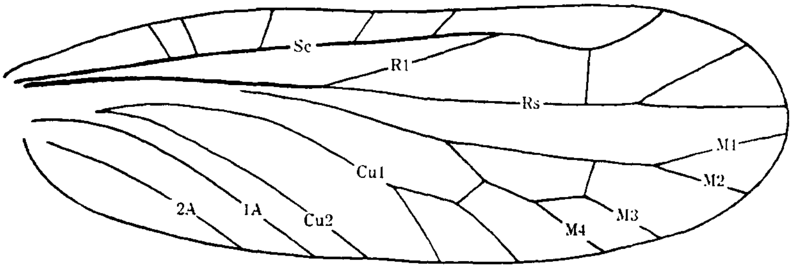

FIGURE 15. *Lithopsocidium permianum* Carp., fore wing, drawn from specimen No. 3365ab, Mus. Comp. Zoology.

Family DELOPTERIDAE.

Among the insects which Dr. Sellards collected in the Elmo limestone thirty years ago there were three closely related species for which he erected the genus Delopterum and the family Delopteridae, and which he placed in the order Protorthoptera. These insects were represented only by fore wings, aside from one or two obscure and twisted portions of hind wings. In the Yale collection from this formation Tillyard found twenty-nine specimens of the family, including two species which he described as new. From this material he was able to demonstrate that the hind wings were nearly similar to the fore wings and did not possess the expanded anal area characteristic of the Protorthoptera. A careful study of the venation also led Tillyard to the conclusion that the Delopteridae were most closely related to the Psocoptera (Copeognatha) and he established for them the suborder Embiopsocida. About the same time Martynov published an account of some allied fossils from the Permian of Russia, and after proposing a widely different interpretation of the venation, he designated them as typical representatives of a new order, Miomoptera. In the present paper I shall first describe the new fossils from the Kansas Permian and then discuss the evidence for and against the two views which have been advocated.

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[9] This concludes the description of the new (1932) fossils belonging to groups which have previously been treated by the writer in the account of the Sellards and 1927 Harvard collections. The following pages deal with the insects contained in all three collections.

In the Harvard collection of insects from the Elmo limestone there are fifty-five more specimens of this family, some of these showing the body structure, which has not previously been known.[10] These fossils enable us to form a more exact picture of the members of this group than has previously been possible, and to correct some erroneous ideas of the classification.

All the Kansan species of Delopteridae are very closely related and agree in the number and arrangement of the main veins.[10] In the case of the cubitus, however, this uniformity of structure has not previously been recognized. Sellards described Cu1 of *minumtum* as "apparently simple," although it was forked distally in his other two species, *latum* and *elongatum*. Tillyard found one specimen (No. 5385)[11] of a fore wing of *minutum* in the Yale collection and he figured and described it also as having an unbranched Cu1. In the Harvard collection there are twenty-six specimens identical with *minutum,* except that Cu1 possesses the distal fork present in all the other Delopteridae. When I examined the Yale fossil figured by Tillyard, I was able to see clearly this same fork of Cu1, as shown in the photograph on Plate 1, figure 4. The Yale specimen is therefore identical with the twenty-six in the Harvard collection, and I believe that we are justified in interpreting Sellards' phrase "apparently simple" as indicating that he was not certain of the structure of Cu1. It seems clear then that the Sellards type, the Yale specimen, and the Harvard fossils belong to the same species, which possessed a forked Cu1.

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[10] Except in the genus Permembia, which, for reasons that will be presented later, I do not consider a member of the family Delopteridae.

[11] Tillyard has designated this specimen as a *paratype* of *minutum* Sellards, but in so doing he has employed that term in a very unusual manner. According to all usage that I have ever encountered, a paratype must be a member of the original series of specimens in the possession of the author of the species when the description is made. In the above instance the specimen designated by Tillyard as a paratype was not even collected until fifteen years after Sellards published his description of the species! This is true of all those specimens in the Yale collection which Tillyard has designated as paratypes of the following of Sellards' species: Psocoptera, *Delopterum minutum*, *D. elongatum*, *D. latum;* Protodonata, *Typus permianus;* Protoperlaria, *Artinska clara*, *A. ovata*, *Paraprisca frigilis*, and *Lecorium elongatum*. I feel that it is necessary to call attention to this peculiar use of the term paratype in order to avoid confusion on the part of future workers, who, not knowing the history of the collections, would naturally assume that all specimens designated paratypes of Sellards' species were part of his collection and were seen by him.

The specimen in the Yale collection (No. 3394) which Tillyard described as a new species (*dunbari*) is also *minutum,* as I shall show later. Now unfortunately Tillyard, believing that *minutum* possessed an unbranched Cu1, restricted the genus Delopterum to that species and *dunbari* and erected the genus Delopsocus for the other Delopteridae, in which he recognized the presence of a forked Cu1. As we have seen above, however, *Delopterum minutum* really possessed a similar Cu1, so that the distinction between these genera does not exist, making Delopsocus identical with Delopterum.[12]

With these modifications, and exclusive of the genus Permembia, the Lower Permian Delopteridae include a series of small insects, having a wing expanse of ten or fifteen millimeters; the body was short, the wings extending far beyond the abdomen at rest; the antennae were prominent and rather stout, with 12 or perhaps 13 segments, each about twice as long as broad; the tarsi were 4-segmented, and terminated in a pair of prominent claws. In the fore wing, R1 was feebly forked distally in the region of the weak pterostigma; Rs included three branches, R2, R3, and R4 + 5; M was fused with Cu1 for some distance proximally and possessed two long branches (M1 + 2, M3 + 4); Cu1 was forked to form a distinct areola postica; Cu2 and the two anal veins were unbranched. The hind wing was similar to the fore except that M was fused with Cu1 at the very base of the wing only; and that Cu2 was coalesced for its entire length with Cu1, the compound vein appearing as a concave vein, in place of a convex Cu1, as in the fore wing. The areola postica of the hind wing was a little higher and shorter than that of the fore wing. Cross-veins were few in both pairs of wings, but the humeral cross-vein (hm) was always distinctly formed.

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[12] If the genus Delopsocus Tillyard (1928) is the same as Pseudomantis Martynov (1927), as Martynov contends (1930), then Pseudomantis now becomes synonymous with Delopterum Sellards (1907).

Genus *Delopterum* Sellards.

Delopterum Sellards, 1909, Amer. Journ. Sci., (4) **27**: 168.
Pseudomantis Martynov, 1927, Zool. Anz., **52**: 105.
Delopterum Tillyard, 1928, Amer. Journ. Sci. (5) **16**: 471.
Delopsocus Tillyard, ibid., p. 474.

Fore wings: Sc usually ending before the middle of the wing and connected by a distal cross-vein to R1; Rs arising at about one-fourth the wing length from the base; R4 + 5 originating near the middle of the wing; M separating from Cu1 just before the origin of Rs; M1 + 2 and M3 + 4 diverging at the middle of the wing. Hind wing: similar to the fore wing, with the exceptions noted for the family.

Genotype: *Delopterum minutum* Sellards.

Delopterum minutum Sellards.

Figure 16; plate 1, fig. 4.

Delopterum minutum Sell., 1909, Amer. Journ. Sci. (4) **27**: 168; fig. 16.
Delopterum minutum Tillyard, 1928, ibid. (5) **16**: 472; fig. 1.
Delopterum dunbari Tillyard, 1928, ibid., p. 473; fig. 2, 3.

Fore wing: length, 4.3–4.6 mm.; width, 1.4, mm.; Sc usually terminating at about one-third the wing length from the base; R2 + 3 joined to R1 by an oblique cross-vein; M1 + 2 and M3 + 4 separating below the origin of R4 + 5; areola postica usually about twice as long as high. Hind wing: length, 4.3; width, 1.8 mm. Anterior margin slightly concave basally; Rs directed away from R1 at the very base; cross-vein between R2 + 3 and R1 as in fore wing; areola postica about as high as long; 1A and 2A straight, slightly diverging distally. Length of body, 3.4 mm.; thorax, 1.2 mm.; abdomen, 1.6 mm. The holotype of this species was No. 64 in the Sellards collection, but unfortunately that specimen has been lost. As the neotype of the species I designate specimen No. 3295ab, in the Museum of Comparative Zoology; this consists of all four wings and the body, showing the head, thorax, and abdomen. It was collected by the writer in the upper layer of limestone in 1932.

I have already shown above that Cu1 of this species is forked as in other members of the family and I believe that there is no doubt that *dunbari* is identical with *minutum.* The distinctions on which Tillyard based this species, such as the apparent absence of cross-veins, depend entirely on preservation. The main characteristic which he emphasized, the double forking of Cu1, does not exist; Cu1 has only the normal areola postica, shaped as in *minutum.* The hind wing of this species is also not quite as depicted by Tillyard; the fossil on which he based his figure lacked the posterior margin proximal of the termination of M. The wing is really a little broader than the fore, with a more prominent anal area, although the latter is not expanded. The cross-veins of both wings are usually only faintly preserved and are frequently indicated merely by slight bends in the longitudinal veins which they connect. Certain of the cross-veins, such as the subcostal cross-vein and the one between R2 + 3 and R1, are always distinctly preserved and were undoubtedly more strongly developed than the others. In the Harvard collection there are twenty-six specimens of *minutum*, collected by the writer in the Elmo limestone during 1927 and 1932; only two of these (No. 3295ab and 3304ab)

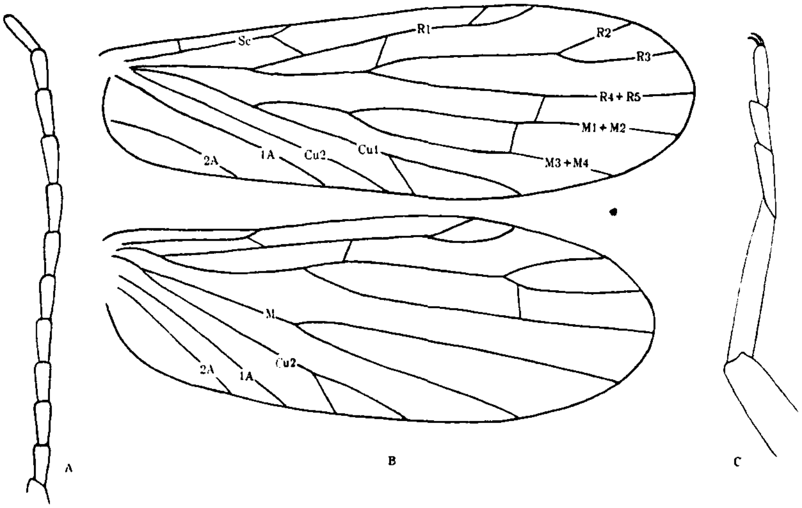

FIGURE 16. *Delopterum minutum* Sell. A, antenna, drawn from specimen No. 3209ab; B, fore and hind wings, based on specimens No. 3203ab; C, hind (or middle) tarsus, drawn from specimen No. 3201ab.

were found in the upper layer. Specimens No. 3203ab is an excellent fore wing, the cross-veins and origins of the main veins clearly indicated. No. 3204ab, 3205ab, 3206ab, 3207ab, 3299ab and 3303ab are complete fore wings, well preserved. No. 3202ab, 3298ab, 3301ab, and 3302ab are good hind wings, although Sc is faintly preserved in the first of these. No. 3208 shows a hind wing and parts of the body; No. 3209ab includes the hind wings and parts of the body, even the antennae; No. 3200ab shows parts of all wings and body, including portions of the antennae; No. 3295ab shows the four wings and body, without the antennae. No. 3201ab is a lateral view of the whole insect; most of the fore and hind wings were originally showing, but parts of these were later removed in order to reveal the body structure; the bases of both antennae and one whole antenna are preserved, although the segmentation is not so clear as in specimen No. 3209ab; the tarsus of a fore leg and a tibia and tarsus of a hind (possibly middle) leg are also present. The latter is particularly well preserved, showing four distinct tarsal segments and the apical claws. The metatarsus is more than twice as long as the others, but the outlines of the other segments are so distinct I do not believe that there is any possibility that this basal segment really consists of two separate segments.

Delopterum elongatum Sellards.

Delopterum elongatum Sellards, 1909, Amer. Journ. Sci. (4) **27**: 168; fig. 20.

Delopsocus elongatus Tillyard, 1928, ibid. (5) **16**: 475; fig. 4, 5.

Delopsocus fasciatus Tillyard, 1928, ibid. p. 476; fig. 6, 7.

Fore wing: length, 5.5–6.5 mm.; width, 1.8–2 mm. Sc usually terminating at about the middle of the wing; M dividing almost below the origin of R4 + 5; areola postica shaped as in *minutum*. Hind wing: length, 5.8 mm.; width, 1.8 mm. The oblique cross-vein between R1 and R2 + 3 is absent.

The holotype of this species was No. 61 in the Sellards collection, but that having been lost, I designate specimen No. 3210ab in the Museum of Comparative Zoology as the neotype. A photograph of Sellards' type, which I made in 1927, is in the Museum of Comparative Zoology.

This species is very similar to the preceding one, but it is larger. The only venational differences are the longer subcosta and the fewer cross-veins. The anterior margin of the fore wing is nearly straight, as in *minutum;* this is true of the specimen in the Yale collection, which Tillyard has figured as having a distinctly convex anterior margin. The venation of the hind wing is like that of *minutum,* Cu and M originating as in that species, not as represented by Tillyard in his figure.

D. fasciatum Till., which was based upon the presence of a brown color pattern, is unquestionably synonymous with *elongatum,* in my opinion. Indications of the color pattern in various degrees of intensity can be seen in several specimens of *elongatum.*[13] Tillyard's figures indicate that the areola postica in *fasciatum* is quite differently shaped from that of *elongatum,* but the figure of the latter is incorrect in this respect.

In the Harvard collection there are five specimens of *elongatum,* of which only one (No. 3305) is from the upper layer of limestone: No. 3210ab, 3305, 3306ab, 3307ab, fore wings; No. 3212, a hind wing, with the very base missing. Specimen No. 3211ab, a fore wing, may also belong here, but it is rather small (length, 5.2 mm.).

Delopterum latum Sellards.

Delopterum latum Sellards, 1909, Amer. Journ. Sci. (4) **27**: 169; fig. 17.
Delopsocus latus Tillyard, 1928, Amer. Journ. Sci. (5) **16**: 475; fig. 8, 9.

Fore wing: length, 6 mm.; width, 1.9 mm.; Sc usually terminating before the middle of the wing; M dividing proximad of the origin of R4 + 5; areola postica a little higher than in the two preceding species; anal area broad; 2A sigmoidally curved and possessing a short posterior veinlet (possibly a cross-vein). Hind wing: length, 5 mm.; width, 1.7 mm.; venation apparently as in other species. I am unable to discern the diverging branches of 2A in the hind wing which Tillyard figured.

The holotype of this species was originally No. 94 in the Sellards collection, but since that has been lost, I designate as the neotype specimen No. 3214ab (lower layer of limestone) in the Museum of Comparative Zoology. This species is easily recognized by the broad fore wing. In other respects the wing is shaped like those of *minutum* and *elongatum,* the convexity of the hind margin of the specimen figured by Tillyard being due to a deep depression in the matrix at this point. The cross-vein leading from 2A is a distinctive feature not found in the other species.

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[13] That the presence or absence of the color pattern in fossil insects from the Elmo limestone may be a matter of preservation is shown most clearly in the case of the Palaeodictyopteron *Dunbaria fascipennis,* some specimens of which show a very definite color scheme, while others possess only the faintest traces of coloration.

In the Harvard collection there are five specimens of *latum*, all from the lower layer: No. 3214ab, a complete fore wing, showing the humeral cross-vein; No. 3213ab, 3215, and 3308ab, complete fore wings, No. 3215 showing the cross-vein between M3 + 4 and Cu1a very clearly; No. 3216ab, a good hind wing, showing the costal space narrowed basally. Three other specimens, No. 3309ab, 3218ab, and 3217, probably belong here also, but they are not well preserved.

The Relationships of the Delopteridae.

In addition to the Kansan species above, insects belonging to the family Delopteridae have been found in the Upper Permian of Russia. As far back as 1904 Handlirsch described a fore wing of an insect from these beds under the name of *Palaeomantis schmidti*, and he also gave a description of a badly preserved hind wing, with an expanded anal area, which he considered to be the hind wing of the same species. From a study of these fossils he came to the conclusion that they belonged in the Orthopteroid series, related to the Mantids, and he erected for them the family Palaeomantidae. In 1927 and 1928, nearly a year previous to the publication of Tillyard's account of the Kansan Delopteridae, Martynov described two related genera from the Permian of Russia, Pseudomantis and Miomatoneura, placing them, together with Delopterum in a new order, Miomoptera. In a later paper (1930) he published a more complete account of the Palaeomantidae, which he regards as typical of the order Miomoptera.

Some confusion of names has been introduced into the literature pertaining to these insects, partly because Tillyard did not see Martynov's 1927 paper before the publication of his work on the Kansan Delopteridae:

1. Martynov contends that the family Delopteridae is identical with the Palaeomantidae, the latter name having priority. This, of course, would be the case if Palaeomantis and Delopterum belonged to the same family; but I do not believe this is supported by the facts. The venation of the fore wing of Palaeomantis is decidedly different from that of Delopterum, even as represented in the revised figure published by Martynov (1930; pl. 13, fig. 1). a) In this genus there is a series of veinlets between Sc and the front margin of the wing and also between R1 and Sc in the basal half of the wing; these veinlets

are absent in Delopterum. b) Cu1 in Palaeomantis has three well developed branches, but in Delopterum it has only two. c) 1A and 2A arise as distal branches of a common vein in Palaeomantis, but do not do so in Delopterum. These features seem to me to be sufficient to require the separation of Palaeomantis and Delopterum into distinct families. Martynov has described and figured (1932) a fossil which he considered to be the hind wing of *Palaeomantis schmidti*, but he made this determination simply because the hind wing in question was approximately the same as in Delopterum. It differs distinctly from the fossil which Handlirsch supposed to be the hind wing. I therefore believe that Palaeomantis and Delopterum belong to different families, Palaeomantidae and Delopteridae.

2. Martynov also claims that the genus Pseudomantis (1928) and Delopsocus Tillyard (1928) are identical and in this I believe he is correct. The fore and hind wings of the specimens of Pseudomantis are so similar to those of Delopsocus that if they had been found in the same beds they would probably been placed in the same species. But as I have already shown above, Delopsocus Tillyard is really identical with Delopterum Sellards (1907), the venation of the latter having been misunderstood by Tillyard. Consequently, Pseudomantis Martynov is synonymous with Delopterum Sellards.

To recapitulate then, the family Delopteridae, as defined above, consists of two genera:

Delopterum Sellards (1907) (= Pseudomatis Mart., Leptoneurula Mart.,[14] Delopsocus Till.), known from the Lower Permian of Kansas and the Upper Permian of Russia.

Miomatoneura Martynov (1927), from the Upper Permian of Russia.

As already stated Tillyard concluded that the Delopteridae were most closely related to the Psocoptera and he established for them the suborder Embiopsocida. Martynov, on the other hand, was led to the belief that they were not related to the psocids but to the Lemmatophoridae of Sellards. The latter were previously designated by Tillyard as typical members of the new order Protoperlaria, allied to the Perlaria. Martynov, however, regards the Lemmatophoridae, together with the Delopteridae, Probnisidae, and certain other Permian families, as constituting the order Miomoptera, with very different affinities. Martynov's conclusions were based upon the following points:

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[14] Leptoneurula Martynov (1928) was synonymized with Pseudomantis by Martynov himself (1930).

1. The coalescence of M and Cu1 is present in the fore wing of the Delopteridae and is frequently found in the Lemmatophoridae, but is not known in the psocids.

2. The wing surface of the Delopteridae and Lemmatophoridae is covered with hairs, whereas that of the Permopsocida is devoid of hairs. In this connection it should be borne in mind that Tillyard was mistaken in his belief that the wing surface of the Delopteridae was the same as that of the Dichentomidae and Permopsocidae (Permopsocida). The hairs are plainly visible in many specimens of Delopterum.

3. Rs originates more distally in the hind wing than in the fore wing in the Delopteridae and the Lemmatophoridae, but not in the psocids.

4. M is fused with Cu for a shorter distance in the hind wing than in the fore wing of the Delopteridae and Lemmatophoridae, which is not true of the psocids.

5. In *Delopterum latum* 2A divides into two branches as in the Lemmatophoridae, but this is not found in the psocids.

6. CuP (Cu2) is lacking in the hind wing of the Delopteridae and also, according to Martynov's interpretation, of the Perlaria and Embiaria, but not in the hind wing of the psocids.

7. The reduced anal area of the Delopteridae, which seems at first glance to separate these insects from the Lemmatophoridae and their allies (Perlaria) may have been secondarily acquired, as in the case of some Recent perlids, as Chloroperla.

8. In both fore and hind wings of the Delopteridae MA is fused with Rs basally and arises from Rs as a branch of that vein near the middle of the wing. This coalescence of Rs and Ma occurs commonly in some species of Lemmatophora, but is unknown in the psocids.

Now at the time when Martynov's paper was written, his knowledge of the hind wing of the Permian psocids was based upon Tillyard's description, which was far from correct, however. As a consequence the 3rd and 4th arguments listed above lose their weight. The 5th argument is also without significance for the anal vein in *Delopterum latum* which Martynov supposed to be 2A is merely a cross-vein leading to the hind margin; it is at most a specific characteristic and of no phylogenetic value. The 7th argument is of a negative nature only; that is, it merely attempts to explain away an obstacle between the Delopteridae and the Lemmatophoridae, but it does not disprove the possibility of relationship between the Delopteridae and the psocids. As to the 2nd argument, the trichiation of the wings, although it is true that the Permian psocids, Dichentomidae and Permopsocidae, had no hairs on the wings, many species of Recent psocids, especially of the family Amphientomidae, have both scales and hairs on the wing membrane. Obviously then the presence of hairs in the Delopteridae has no weight in eliminating them from the psocids. Finally, in considering the 8th argument, we must bear in mind that there is no evidence whatever that MA has fused with Rs in the Delopteridae, as Martynov contends. There is no indication of the divergence of MA from MP at the base of the wing. It seems to me that the only reason why Martynov supposed this fusion to have taken place in the Delopterids, was because it has done so in the Lemmatophoridae. This argument is therefore worthless, depending upon the assumption of the conclusion to be proved.

Apparently, then, from this series of arguments against the idea that the Delopteridae are members of the Psocoptera, only two have any significance, the 1st and the 6th. These alone are hardly sufficient to disprove such a relationship. Furthermore, it is not difficult to find characteristics of equal or even greater significance which differ in the Delopteridae and the Lemmatophoridae. In the latter, for example, Cu1 (or CuA of Martynov's nomenclature) is obviously 3-branched, the first branch arising so close to the base of the wing that it appears to be an independent vein. This condition is entirely opposite to that in the Delopteridae. And regardless of Martynov's attempt to explain away the difference in the structure of the anal area of the hind wing of the Delopteridae and the Lemmatophoridae, I believe that this is still the main objection to the thesis that these two groups of insects are closely related. In the Delopteridae the veins in the anal area of the fore wing are the same as those in the anal area of the hind wing, there being only the two unbranched veins 1A and 2A. In the Lemmatophoridae as well as all other Protoperlaria and nearly all Perlaria, the anal area of the hind wing is much expanded and contains numerous branches of the anal veins which are absent in the fore wing and which were apparently developed to support the extra membrane. Now if such an enlarged anal area were reduced to a small one, as Martynov contends to have been the process which produced the small anal area of the hind wing in the Delopteridae, then we should expect to find in the reduced wing some traces of the extra branches of the anal veins. Such is the case, for example, in the perlid Nemoura, which Martynov cites as an example of this reduction. But as I have already stated, in the Delopteridae the anal veins are unbranched and similar in both pairs of wings.

The body structure of the Delopteridae is much more like that of the Psocids than like that of the Lemmatophoridae. The new fossils described in this paper show that cerci were absent in the Delopteridae and in the Permian Psocoptera, although they were well developed in the Protoperlaria. In both the Delopteridae and the Permopsocida the abdomen was short and the tarsi were 4-segmented; whereas in the Protoperlaria the abdomen was long and the tarsi were 3-segmented.

In closing this account of the Delopteridae, I therefore believe that we are justified in concluding that there is more evidence that the Delopteridae were closely related to the Psocoptera than that they were allied to the Lemmatophoridae and the other families which Martynov included in the order Miomoptera. I am still inclined to accept Tillyard's view that the Delopteridae were an aberrant branch of the Psocids which died out without giving rise to any other group of insects known to us at the present time. Perhaps the unusual nature of the hind wing, with the complete fusion of Cu1 and Cu2, was the result of a tendency to narrow the hind wing, for such a line of descent was obviously followed by the Psocoptera which have persisted through the geological ages and up to the present era. The thorax of the Delopteridae as well as that of the Permopsocida was hardly large enough to bear two homonomous wings which the Dichentomidae and Permopsocidae possessed. Even in the Recent psocids, in which the hind wings are much reduced, they are attached far back on the thorax, in many families seeming to arise from the abdomen. It is not unreasonable to assume that the Delopteridae represent one line of descent of the Psocoptera in which the hind wing had already begun to be narrowed in a unique method, but which was not able to persist for as long a time as some other lines of development.

Order PROTELYTROPTERA.

Probably the most remarkable insects of the entire fauna of the Elmo limestone are the elytrophorous species to which Tillyard has applied the ordinal name Protelytroptera.[15] These fossils, although not common in the limestone, are by no means as rare as some other groups. In the Yale collection Tillyard found seventeen specimens; in the Sellards collection I have identified one specimen and in the Harvard material fifty-six specimens belonging to this order. Most of these fossils consist of a single elytron, which resembles closely a coleopterous elytron with a feeble venation; but a few specimens have been found which show the structure of the hind wings and certain portions of the body.

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[15] In his paper on these fossils (1931) Tillyard used the term Elytroptera as well as Protelytroptera in reference to the order. Although the former would perhaps have been more suitable, the latter is the one which he employed first, in the definition of the order, and therefore has priority.

Through the courtesy of Professor Dunbar, I have been able to examine the Yale specimens in the Peabody Museum, which were studied by Tillyard. From my examination of the entire series of seventy-four specimens, I believe that it is necessary to revise somewhat Tillyard's classification of these insects and to modify certain of his statements regarding the wing and body structure. These changes are only to be expected, of course, inasmuch as I have worked on more than four times as many specimens as were seen by him.

Although there are some structural features, such as the mouth-parts, on which we are entirely uninformed, we do know the general habitus and a few details of the body. The head was small but broad; the eyes prominent, convex, situated well back on the sides of the head; the antennae, in Protelytron at least, were rather short and thick, consisting of about twenty-two loosely jointed segments; the pronotum was longer and wider than the head and shaped like an oval disc, closely resembling that of some Carboniferous Protorthopteroids. The fore legs are not known, but the middle and hind legs possessed five tarsal segments, the fifth being the longest; the hind legs in both Protelytron and Acosmelytron were robust and had very short tarsal segments; the distal part of the tibiae bore a series of spines and teeth, which were probably used to assist in the folding of the hind wings, as is the case with many beetles. The middle legs in Parablattelytron and probably all other members of the order were more normal and slender, with the last tarsal segment the longest. The abdomen was a little shorter than the elytra and was covered by them at rest. The fore wings were typical elytra, strongly convex, with a prominently widened costal area (ce) at the very base, which was not observed by Tillyard in the Yale fossils. In the more generalized forms Sc, R1, Rs, Cu1, Cu2 and three anal veins were present, and there was also a submarginal thickening nearly paralled to the posterior border, forming the sutural margin (sm); in the more highly modified species only the bases of R1, Sc, and Cu2 are discernible.

All the veins in the elytra were convex, including the subcosta, except Cu2. At the base of the wing the subcosta is margined anteriorly by a deep furrow, which resembles and may be homologous to the epipleural fold of the coleopterous elytra. The hind wings were much enlarged, with an expanded and folded anal area, though the usual longitudinal veins were reduced and confined to the anterior part of the wing.

As far as the structure of the wings is concerned, there is every indication that the Protelytroptera had already differentiated into several distinct families by the time of the lower Permian. The following families were recognized by Tillyard:

1. Protelytridae, including those species in which the sutural margin is broad and evenly developed along the whole of the posterior border. In this family Tillyard placed two genera, Protelytron and Protelytropsis.

2. Permelytridae, including those species having the sutural margin broad from the base to about half way along the posterior border, then narrow or obsolete for the remaining length of the elytron. Two species and genera were placed here, *Permelytron schucherti* and *Megelytron robustum*. After examing the type of *schucherti* (No. 5411ab, Peabody Museum), I suspected that the distal part of the sutural margin was not actually narrowed as described by Tillyard, but only covered up by a part of the rock in which the elytron was preserved. I therefore used a fine needle on this part of the fossil and found that my suspicions were correct: *the sutural margin was uniformly developed along the entire posterior border*, as in all of the other Protelytroptera in which it is present. With the elimination of this supposed family characteristic the genus Permelytron becomes a member of the Acosmelytridae, as I shall show later. In the type of Megelytron, the elytra are folded back over the abdomen, the posterior portions are broken away, for some distance distad of the middle of the wing (as shown in Tillyard's figure), and the apical part is very obscure. I do not believe that a true sutural margin was present in this insect at all, but at any rate Megelytron differs so markedly from the other Protelytroptera that it certainly requires a distinct family for its reception.

3. Blattelytridae, including the species in which the venation is more or less atrophied and reduced, but always showing the basal part of Cu2 (*vena dividens*). In this family Tillyard placed two genera, Blattelytron and Parablattelytron. It is unfortunate that

the genus Blattelytron, from which the family name is derived, is known only from a single fragment of an elytron lacking the proximal two-thirds of the anterior margin, and the whole base, and being very poorly preserved. From my examination of this specimen I am inclined to believe that a sutural margin was present in this insect. Tillyard states that there is "no clear indication of a sutural margin," but if we bear in mind that the fossil is very poorly preserved and fragmentary, and that poor specimens of Protelytron show no signs of a sutural margin, I think that the explanation of the apparent absence of the sutural margin on the grounds of poor preservation is acceptable. With this assumption, it is clear that Blattelytron and Parablattelytron are members of the same family, as they were regarded by Tillyard. But if Blattelytron should turn out to have lacked a sutural margin, then it would require a distinct family.

4. Acosmelytridae, including those species with an atrophied venation, not even Cu2 being visible. A single genus, Acosmelytron, with three species, was placed here. From my study of the Yale types I am convinced the apparent absence of the basal part of Cu2 is merely the result of poor preservation of these three specimens. In two of these, *acutum* and *elongatum*, I could distinguish the basal part of Cu2 as clearly as the basal parts of R and Sc which Tillyard described; and in the third species, *delicatum*, I was unable to see any veins at all. However, even if we assume that Cu2 in Acosmelytron is less developed than that in Parablattelytron and Blattelytron, I do not consider that difference sufficient to separate these genera from the Blattelytridae.

In my conception, then, the previously described Protelytroptera from the Lower Permian can be grouped into the three families Protelytridae, Megelytridae (new family), and Blattelytridae. But in the new material constituting the Harvard collection there are in addition representatives of two other families, as widely divergent from one another and from the three former families, as the latter are from each other. One of these, Archelytridae, is characterized by having a 3-branched Cu1 in the elytron; the other, Elytroneuridae, is distinguished by having a much narrowed anal area, a remarkably broad costal region, and no sutural margin. I have also found in this material a new genus, Permelytropsis, which is characterized by the absence of Cu2 and which I have placed in the Protelytridae.

The classification used in this paper is given in the following outline:

Family Protelytridae.
  Genus Protelytron Tillyard.
  Genus Protelytropsis Tillyard.
  Genus Permelytropsis, new.
Family Megelytridae.
  Genus Megelytron Tillyard.
Family Blattelytridae.
  Genus Blattelytron Tillyard.
  Genus Permelytron Tillyard.
  Genus Parablattelytron Tillyard.
  Genus Acosmelytron Tillyard.
Archelytridae, new family.
  Genus Archelytron, new.
Elytroneuridae, new family.
  Genus Elytroneura, new.

Family PROTELYTRIDAE.

Elytron: Sc, R1, Rs, M, Cu2, 1A, 2A, and 3A present and unbranched; Cu1 present except in Permelytropsis, new genus; costal area broad; sutural margin well developed.

Genus *Protelytron* Tillyard.

Protelytron Tillyard, 1931, Amer. Journ. Sci., **21**: 239.

Elytron: costal margin convex, the costal expansion (ce) at the base being very prominent; Sc strongly developed at the base, but becoming weak near the middle line of the elytron and apparently disappearing at this point; costal area broad, with patches of short, stout spines, at least near the subcosta; Rs arising at the middle of the wing or beyond; M arising from the base, free from Rs and Cu1; Cu1 and Cu2 unbranched, diverging at the base of the wing; 1A, 2A, and 3A straight and nearly parallel; a fourth anal vein may be present; inner margin of the anal area rounded.

The head is wide, with prominent eyes, and probably small mouthparts; antennae with about 24 loosely jointed segments, the basal one being short and cylindrical; body flattened, the pronotum wider than the head, with rounded margins. The fore and middle legs are unknown in this genus, but the hind legs are stout, the trochanter being spherical; the femur is thick, being only about three times as long as wide; the tibia is narrow basally but wide distally, with several teeth on the outer distal margin; the first four tarsal segments are thick and short, and each bears at least two spines distally; the last tarsal segment is about twice as long as the others and is more slender; there are two large terminal claws.

Genotype: *Protelytron permianum* Till.

In this genus Tillyard placed two species, *permianum* and *dunbari.* The former was based upon a single finely preserved specimen consisting of one elytron, both hind wings, and portions of the body. *Dunbari* was based upon three specimens: No. 5418ab (Peabody Museum), the holotype, consisting of two elytra, more or less complete, and the distal part of one hind wing; No. 5405ab (paratype), consisting of two elytra lying nearly parallel along the abdomen, and faintly preserved portions of the hind wings; and No. 5417 (paratype), consisting of two elytra folded back over the abdomen, completely concealing the hind wings. I believe that the former paratype (No. 5405) does not belong to the same species as the holotype and the other paratype (No. 5017), and that the species (*dunbari*) represented by the latter fossil is the same as *permianum.* Tillyard separates his two species of Protelytron on the following points of difference:

1. The shape of the apex of the hind wing. Tillyard states that in *dunbari* the apex of the hind wing is widely and bluntly rounded, as shown in his figure 4.[16] On examining the holotype specimen, I came to the conclusion that the real apex of the wing was probably still covered up by the matrix; by chipping away the matrix at this part, I found that this was in fact the case, and that *the apex of the hind wing was in reality shaped exactly like that of permianum.* A comparison of the hind wings of the holotypes of *permianum* and *dunbari* now showed no differences whatever. In the paratype No. 5405ab the venation of the hind wing is very obscure and the apical part is not even preserved. In paratype No. 5017 none of the hind wings are visible.

2. The size and shape of the elytra. Tillyard characterized the elytra of dunbari as being narrower, smaller, and having a narrower

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[16] According to the caption under this figure the drawing was based on specimen No. 5405ab (paratype), but I think it more than likely that it was really based on the holotype (No. 5418ab). The venation in the hind wing of the former fossil is very indistinct and is not even preserved in the apical part, which is the portion figured. In the holotype, on the other hand, the apical part of the wing is very clear and the contour of the apex before I cleaned the specimen was exactly as shown in Tillyard's figure. The present discussion, however, does not depend upon the assumption of such an error in labelling the figure.

costal area than those of *permianum*. The elytra of the holotype and paratype No. 5417 of *dunbari* are very similar to each other and so far as I can see similar to those of *permianum*. They are 5.5 mm. long, only .3 mm. less than those of the latter species; the dimensions of the width are not given by Tillyard, but I find them to be about the same as those of *permianum*, so that the elytra are shaped as in the latter insect. I am unable to distinguish any differences in size or shape of the costal field. The other paratype (No. 5404ab), however, has very different elytra; they are much smaller in proportion to the size of the entire insect, so that the abdomen extends beyond them.

3. Minor venational features, such as the indistinct subcosta and a slight difference in the place of termination of R1. The degree of distinctness of the subcosta or any other vein which was originally weakly chitinized is unquestionably dependent upon preservation to a large extent; an examination of a series of protelytropterous elytra shows a variation of the clearness of the veins in proportion to the degree of preservation of the whole elytron. Also the point of termination of R1 and of the other main veins in these elytra seems to be subject to slight variation in various individuals.

For the reasons given above, I consider the holotype and paratype No. 5017 of *dunbari* to be identical with *permianum*, and accordingly regard *dunbari* as a synonym of the latter species. The other paratype, No. 5405ab, apparently belongs to a distinct species, which will be described below as *tillyardi*.

Protelytron permianum Tillyard.

Figures 17, 18 (A, B).

Protelytron permianum Tillyard, 1931, Amer. Journ. Sci., **21**: 240; fig. 1–3; 14, 15.

Elytron: length, 5.3–6 mm.; width, 1.7–2 mm. Costal area very broad; R and R1 nearly straight; Rs usually arising just beyond the middle of the wing; M straight, terminating on the sutural margin near its termination; Cu1 and Cu2 and the anal veins straight; sutural margin wide; the patch of spines on the costal area is large, extending to about the middle of the wing; another cluster of stout spines is located on the basal part of the sutural margin. Hind wing: length, 8 mm.; width, 6 mm. In figure 17 I have shown the complete venation of the hind wing, based mainly on specimen No. 3366ab in the Museum of Comparative Zoology, although the type specimen

No. 1019a in the Peabody Museum was also used to verify certain doubtful points. It will be seen that my figure differs somewhat from that given by Tillyard. This is mainly due to the fact that the hind wings are more clearly preserved in the Harvard fossil than in the specimen which Tillyard studied. Of course the venation of any wing which is adapted for folding, as these hind wings were, is naturally more difficult to interpret in the fossil state than the venation of a wing which is normal and has no lines of folding or creasing. The chance or error in working out such a venation obviously decreases in inverse ratio to the number of specimens available for study, and it is therefore only logical to expect that certain corrections will be made as more and more material is accumulated. So far as the anterior part of the wing is concerned, i. e., that section including

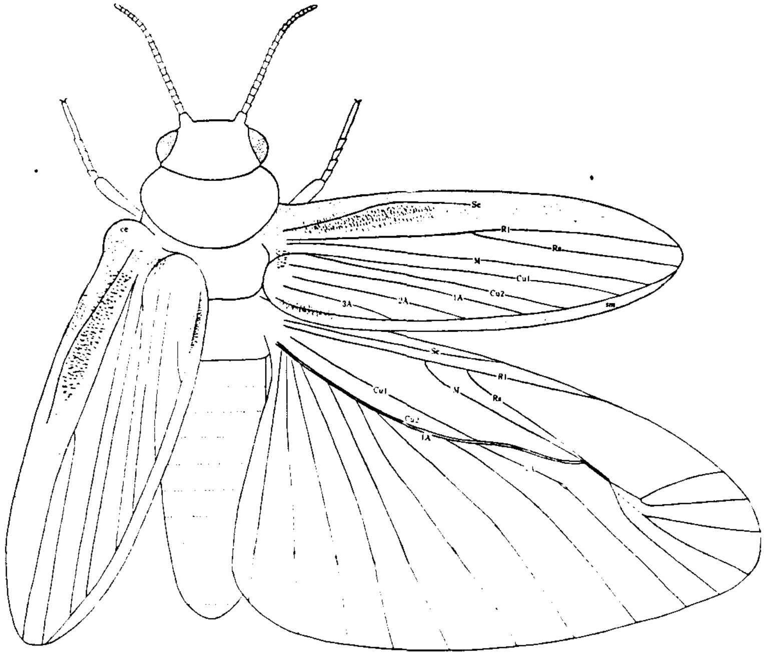

FIGURE 17. *Protelytron permianum* Till. Reconstruction, chiefly based on specimen No. 3266ab in the Museum of Comparative Zoology. For explanation, see text.

the main veins, I believe that we now have a fairly accurate picture of the venation; but I am not certain of the exact number of radiating veins in the anal fan. The principle differences between Tillyard's figure and mine lie in the number of branches leading from R and from the fulcrum; and the identity of M and 1A. In both wings of the Harvard fossil two branches can be seen leading from R1; the more distal of these is presumably Rs, which leads to the fulcrum, passes beyond and forms two branches which bend abruptly upwards; the second vein arising from R is apparently M, which leads to the fulcrum and seems to end there; but just beyond that point there is a distinctly forked vein which I assume to be the end of M, though it may be another branch of Rs. In the area between the origin of Rs and the branches R2 and R3, i. e., directly above the fulcrum, I can discern no veins whatever. In this area Tillyard shows several more or less oblique and uncertain veins, but these are no more than wrinkles or creases in the membrane. In the distal part of the wing Tillyard shows a series of pectinate veins, directed towards the apex; this part is poorly preserved in the Yale specimen but is very clear in the Harvard fossil. It is the one part of the wing about which there can be no doubt as to the nature of the venation. Below M another vein can be distinguished, but this is well down near 1A; it leads from the base of the wing and terminates on 1A just beyond the origin of Rs, well before the fulcrum. This vein is convex and I consider it to be Cu1. Parallel and contiguous with 1A is a concave vein which is probably Cu2. 1A is a strong vein, the strongest in the wing; at first it curves posteriorly but is then directed anteriorly and finally terminates on the fulcrum. Between M and the concave vein which I have called Cu2, Tillyard figures two veins, both arising from a short stem at the base and then coalescing again distally, thus forming a closed lanceolate cell. The anterior of these veins he designated as Cu1 and the lower as Cu2. Tillyard characterized the former of these as "weakly chitinized" and the second as "a deep furrow, barely if at all chitinized." From my examination of the Yale fossil I am inclined to consider that neither of these structures represent true veins but only lines of folds or creases, which are numerous in the specimens and some of which are indicated by dotted lines in Tillyard's figure. In the Harvard fossil I cannot make out the two weakly chitinized veins described by him, whereas the veins which I have indicated as Cu1 is a strongly chitinized structure; since this is a true vein, then there is not space in the same area for the veins which are depicted by Tillyard. In both the Harvard and Yale fossils a vein can be seen leading from the distal end of the fulcrum and terminating below the apex. This is probably the end of 1A, or at least a branch of that vein. The remaining veins in the hind wing, all of the radiating type, are in all probability composed of 2A, 3A, and perhaps some auxiliary veins.

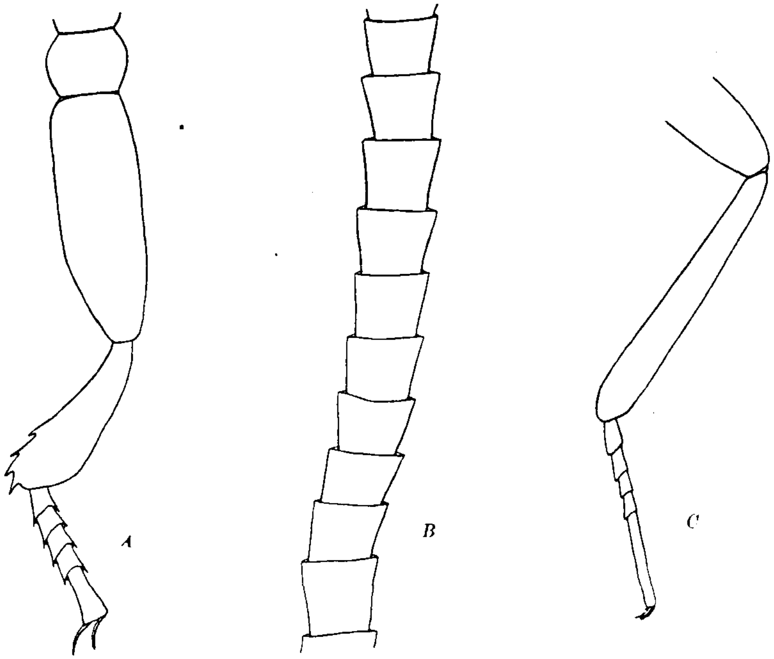

FIGURE 18. A, hind leg of *Protelytron permianum* Till. drawn from specimen No. 3266ab, Mus. Comp. Zool.; B, part of antenna of same specimen, showing the shape of several segments; C, middle leg of *Parablattelytron subincisum* Till., based on specimen No. 3374ab, Mus. Comp. Zoology.

Holotype: No. 1019a, Peabody Museum; counterpart, No. 1019b, in Dr. Tillyard's collection.

There are five good specimens of this insect in the Harvard collection, in addition to numerous poorly preserved elytra which do not merit discussion. No. 3366ab, which was taken in the lower layer and has already been described above, consists of all four wings and portions of the body; the two elytra are spread out on each side, but one of the hind wings is bent over to the opposite side so that this pair is on the same side of the body. Originally, of course, one hind wing was partly covered by the other, but with the aid of a fine needle I removed all of the uppermost wing and exposed completely the bottom wing. The elytra of this specimen are splendidly preserved and show signs of coloration. Almost the whole costal area is very dark brown and the basal part of the sutural margin is similarly colored. One middle and one hind leg are clearly preserved, as well as the head and one antenna. In all probability the head of this specimen was in a normal position when the insect was entombed, that is, the longitudinal axis of the head was at right angles to that of the thorax, so that the view shown in figure 17 represents the back of the head. This would explain the apparent absence of mouth-parts, for the anterior part of the head as preserved in the fossil is really only the anterior border of the vertex. It would also explain the position of insertion of the antennae. Specimen No. 3369ab, lower layer, is a very good elytron, showing the veins distinctly and the hairs with surprising clearness. No. 3370ab, also from the lower layer, is likewise a complete and well preserved elytron, showing the subcosta better than most specimens. No. 3367ab, lower layer, is a fairly well preserved elytron. Specimen No. 3368ab, the only one from the upper layer of limestone, is also a good elytron, though preserved on an uneven surface of the rock.

In two respects the preservation of the elytra of this insect and of the Protelytroptera in general is most unusual. The well-preserved elytra of all fossil Coleoptera and Dermaptera which I have seen, including material from the Florissant, Green River, and Oeningen shales, have distinct indications of the original color of the wing; they are usually uniformly black or brown, or occasionally show some other type of coloration, as maculations. None of the Solenhofen (Jurassic) Coleoptera which I have examined show any color in the elytra, the latter being the same color as the matrix; but these Solenhofen insects are notoriously poorly preserved, which readily accounts for the lack of coloration. The insects of the Elmo limestone, however, are usually very well preserved, the elytra of the majority of the Protelytroptera showing minute spines on the costal area; yet none of them give the slightest indication of ever having been uniformly black or brown. Almost all of these elytra are white like the matrix, though a few specimens of Protelytron show patches of brown pigment on the costal area. Since the coloration on the wings of many of the Elmo insects is unusually clear (e. g., Dunbaria, Lemmatophora, Agetopanorpa, and the cockroaches), it seems very probable that the elytra of the Protelytroptera were not black or brown, but almost colorless, perhaps yellowish, having at most a patch of brown or black in the costal space.

Another interesting aspect of the Protelytroptera is the actual mode of preservation. Tillyard has already pointed out that when a membraneous wing of an insect is preserved as a fossil, the impression of the upper surface of the wing is the negative of the upper surface in the original wing, so far as the topography of the wing is concerned, that is, the convexity and concavity of the veins. Similarly, the impression of the under surface of the wing is the negative of the lower surface, the topography being like that of the upper surface of the original wing. On the other hand, the surface markings, such as the coloration and hairs, of each surface of the wing is preserved in the corresponding impression, so that any hairs on the upper surface of the wing, for example, are preserved in the impression of that surface, although the topography of that half of the fossil is like that of the under side of the original wing. This situation is obviously due to the structure of the wing, which is essentially in the form of a flattened, hollow sac, the upper and lower surfaces separating as the wing becomes preserved. But apparently as the result of its curvature and thickness, the elytron of the Protelytroptera was not preserved in this manner. An examination of a series of obverses and reverses of these elytra shows that although the surface markings and spines of the upper surface may be visible in the half of the fossil in which the elytron is concave, they are always better preserved in the half containing the convex elytron. Hence, in these elytrophorous insects the half of the fossil which has the topography like that of the upper surface of the elytron also shows the markings of that side. Since it is hardly justifiable to entertain the notion that the spines and coloration were on the under surface of the elytron, I have come to the conclusion that the elytra were more or less replaced by mineral matter during the process of preservation, although enough organic material persisted to effect the limestone chemically, producing a print of the surfaces, analogous to a photographic print, on the exterior. That half of the fossil elytron bearing a concave elytron may therefore be either the mould of the upper surface of the elytron, or the inner surface of the lower side of the elytron, itself; and the convex half may be either the upper surface of the elytron, or the cast of the inner side. By means of the spines on the costal area it is possible to determine into which category any fossil may belong; if, for example, the elytron is convex and shows the spines, then it is obviously the upper surface preserved. Probably all strongly convex elytra, such as those of most Coleoptera, are preserved in a similar manner. Professor P. E. Raymond, with whom I discussed the matter, informs me that this type of preservation also occurs in the trilobites, in which the external skeleton of the dorsal part of the body was thick and convex.

Protelytron tillyardi, new species.

Figure 19.

Elytron: length, 5 mm.; width, 1.9 mm.; costal area very broad; R straight; R1 slightly arched proximally; Rs arising at about the middle of the wing; M straight, close to R proximally; Cu1 and Cu2 and the anal veins are nearly straight; sutural margin unusually broad.

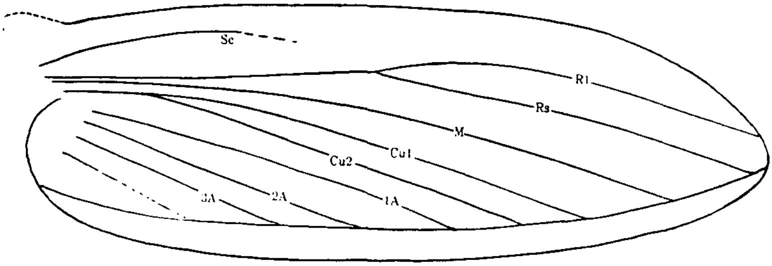

FIGURE 19. *Protelytron tillyardi*, n. sp., elytron; holotype, No. 3372ab, Mus. Comp Zoology.

Holotype: No. 3372ab, Museum of Comparative Zoology; collected by the writer in the lower layer of limestone at Elmo, 1932. The type consists of a complete elytron, with all veins very clear except A2 and A3. This species is distinguished from *permianum* by its proportionally shorter and broader elytron, and by the wide sutural margin. Specimen No. 5405ab in the Peabody Museum, which was designated a paratype of *dunbari*, seems to belong to this species.

Protelytron angustum, new species.

Figure 20.

Elytron: length, 6 mm.; width, 1.8 mm.; costal margin not quite so convex as in *permianum;* apex very pointed; costal area a little narrower than in *permianum;* Sc close and parallel to the costa basally; Rs arising just beyond the middle of the wing; M parallel to R for about one-third the length of the elytron, then diverging away parallel to Cu1; stem of Cu almost touching R at the base; 1A, 2A, and 3A about equidistant; a short anal vein, probably 4A, is visible near the inner angle of the wing; sutural margin very narrow for its entire length, only about one-half as wide as that of *permianum.*

Holotype: No. 3371ab, Museum of Comparative Zoology; collected by the writer in the upper layer of the limestone, in 1932. Although this elytron is about the same size as that of *permianum,* it is distinguished by having a pointed apex, by the submarginal Sc, the presence of a fourth anal vein, and particularly by the narrow sutural margin. The type is very well preserved, except for the very basal part of the costal area, which is broken away as in a great many of the Protelytroptera.

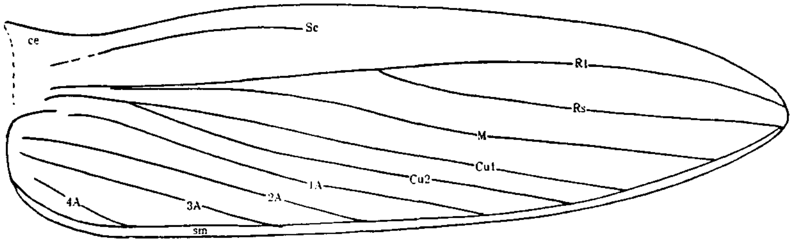

FIGURE 20. *Protelytron angustum,* n. sp., elytron; holotype, No. 3371ab, Mus. Comp. Zoology.

Permelytropsis, new genus.

Elytron: costal margin convex, the costal expansion prominent; Sc strongly developed at the base, terminating on the costal margin just basad of the middle of the wing; Rs arising scarcely beyond the middle of the wing; M arising from the base, free from R or Cu; Cu well developed but unbranched; anal veins not parallel; sutural margin well developed; spines on the costal area and sutural margin at the base.

Genotype: *Permelytropsis cubitalis,* n. sp.

Permelytropsis cubitalis, new species.

Figure 21.

Elytron: length, 6 mm.; width, 2 mm.; apex moderately pointed; costal area very broad; Sc closer to margin than to R, slightly sigmoidal in shape; Rs nearly straight; M sigmoidally curved; Cu strongly convex, diverging away from M at the base, approaching 1A near the middle of the wing, then diverging anteriorly and finally terminating near M; 1A slightly curved away from Cu near the middle of the elytron; 2A remote from 1A basally but converging towards it distally; 3A short, subparallel to 2A; sutural margin nearly the same width for its entire length.

Holotype: No. 3373ab, Museum of Comparative Zoology; collected by the writer in the upper layer of limestone at Elmo, 1932.

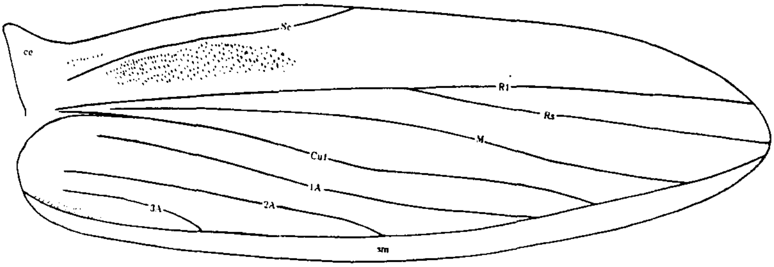

FIGURE 21. *Permelytropsis cubitalis*, n. sp., elytron; holotype, No. 3373ab, Mus. Comp. Zoology.

The outstanding feature of this genus is the unbranched cubitus. It is obvious from the structure of the basal part of this vein that it really consists of Cu1 and Cu2 coalesced; the narrowness of the area between Cu1 and 1A shows that the absence of the branches of Cu is not merely due to lack of preservation. The specimen described is very well preserved, exceptionally so, in fact; the short spines on the costal area, which extend from the base to almost the middle of the wing, are very clear, as well as those at the base of the sutural margin.

Genus *Protelytropsis* Tillyard.

Protelytropsis Tillyard, 1931, Amer. Journ. Sci., **21**: 245; fig. 5.

This obscure genus is known only from a single species (*grandis* Till.) and specimen (No. 5426, Peabody Museum), consisting of only the very distal end of an elytron. It shares with Megelytron the possession of a faint meshwork of cells, although in Protelytropsis the meshwork is restricted to a much smaller area. The apex of the elytron is more pointed than that of Megelytron. Since only a small fragment of this insect is known, it is not possible to determine its affinities with certainty; the basal part of the elytron, when found, may show features which will require a separate family for this species. Tentatively, however, I believe it fits into the Protelytridae better than into any other described family.

Megelytridae, new family.

Elytron: Sc, R1, Rs, M, Cu1, Cu2 and probably 3 anal veins present; Cu1 apparently branched; costal area broad; no true sutural margin; Rs arising in apical part of wing; distal half of wing with a delicate reticulation.

Genus *Megelytron* Tillyard.

Megelytron Tillyard, 1931, Amer. Journ. Sci., **21**: 247; fig. 7, 8.

This genus is also known from a single species and specimen (*robustum* Till., No. 5406ab, Peabody Museum), no additional representatives having been found in the Harvard collection. The type consists of a more or less whole specimen, with the elytra folded back over the abdomen. I do not believe that a sutural margin is present. In the proximal half of the elytron there is a vein running parallel to the posterior border, forming a margin which superficially resembles the sutural margin, except that it is on the same level as the rest of the wing, and continues the convexity of the elytron as a whole. In the other Protelytroptera the true sutural margin is in the form of a flat shelf. There are apparently no true branches of the main veins, except Cu1, but the distal part of the wing is filled with a meshwork which may have been derived from a complete venational system or may be a secondary formation associated with the development of a coriaceous texture to the wing. Cu1 is probably branched, but even this is not certain, for as Tillyard states the posterior portion of the elytron is covered by the decomposed abdomen. The costal expansion is apparently broken away in the fossil but was probably present in this species as in the others. This fossil is of much interest because it shows a part of the thorax; the prothorax is shaped as I have described it in Protelytron. I do not believe, however, that Tillyard has correctly interpreted the eyes, which he shows as being very small;

only a minute piece of the head is preserved, so that it is not possible to check one side of the head with the other. In the specimen of Protelytron in which I have described the eyes, the whole head is preserved and shows by the symmetry of the parts just what the eyes were like.

Archelytridae, new family.

Elytron: Sc, R1, Rs, Cu2, 1A, 2A and 3A present and unbranched; M present and apparently forked, the anterior branch coalescing with Rs; Cu1 with three long and well developed branches; Cu2 markedly concave, for its entire length; costal area broader than in the Proelytridae; sutural margin well developed. Hind wing unknown.

Archelytron, new genus.

Elytron: costal margin strongly convex, the costal expansion at the base small but distinct; Sc weakly developed at the base, but well formed distally; Rs originating well beyond the middle of the wing; M forked just before the origin of Rs, the anterior branch coalescing with Rs.[17] Cu1 and Cu2 diverging at the very base; Cu1 forking dichotomously near the middle of the wing, the anterior branch forking again before the division of M; Cu2 and the anal veins subparallel and unequally spaced.

Genotype: *Archelytron superbum*, n. sp.

Archelytron superbum, new species.

Figure 22.

Elytron: length, 6 mm.; width, 2 mm.; costal area very broad; apex smoothly rounded; Sc weakly developed at its base; diverging anteriorly and fuses with the costa for a short distance, then gradually leads away and terminating on R1 just about at the origin of Rs; R and R1 straight; M straight up to its point of forking; M3 + 4 terminating at the very end of the sutural margin; Cu2 diverging from Cu1, its point of termination being very remote from that of the latter; 1A close to Cu2, but 2A remote from 1A. The sutural margin is apparently uniform in width for its entire length. There appear to be neither cross-veins nor a meshwork of cells; the entire surface of the elytron is rough and when examined under light parallel with the longitudinal axis of the elytron, a series of short ridges can be seen over the whole surface.

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[17] There is of course the possibility that this oblique vein between M and Rs is a modified cross-vein; but I have been led to regard it as a branch of M because it diverges from M at a dichotomous fork, and because it is strongly convex.

Holotype: No. 3389, Museum of Comparative Zoology; collected by the writer in the upper layer of limestone at Elmo, 1932. The specimen is an observe, splendidly preserved. This species is the most generalized of any of the known Protelytroptera and is the only one in which there are branches on several of the main veins. The phylogenetic significance of this fossil will be discussed later.

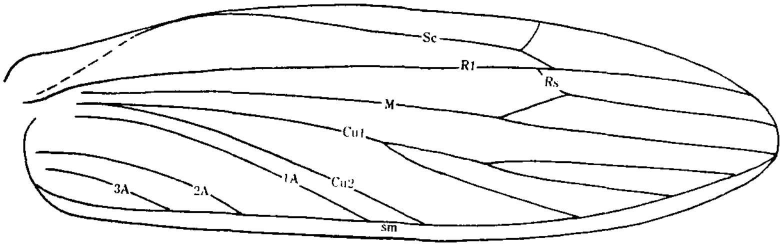

FIGURE 22. *Archelytron superbum*, n. sp., elytron; holotype, No. 3389ab, Mus. Comp. Zoology.

Elytroneuridae, new family.

Elytron: nearly flat, not convex as in the other families of the Protelytroptera. R1, M, Cu1, Cu2, 1A, 2A and probably Rs present and unbranched;[18] 3A absent; Sc present and forked dichotomously before the middle of the wing; R very remote from the costal margin, the costal space being extremely broad; M is fused with Cu1 for some distance basally; Cu2 well developed, markedly concave; sutural margin entirely absent; the short spines already described on the costal space in the Protelytridae are also present in this family.

Hind wing unknown.

Elytroneura, new genus.

Elytron: costal margin strongly convex except at the base; costal space (between Sc and margin) broad, especially at the base of the elytron, where the margin bulges slightly; Sc well developed, though weak basally; Rs (if present) arising in the distal third of the wing; M fused with Cu1 to about one-third of the length of the elytron; Cu1 and Cu2 diverging away from each other from the base of the wing; Cu2 and the anal veins unequally spaced and not parallel.

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[18] Inasmuch as the apical third of the elytron on which this family is based is missing, Rs is not preserved; but since it is present in all other species of the order, it probably existed here as well; and since Rs obviously does not arise before the distal third of the wing it was in all probability unbranched.

Genotype: *Elytroneura permiana*, n. sp.

Elytroneura permiana, new species.

Figure 23.

Elytron: length of preserved part, 9 mm.; width, 3.5 mm.; Sc weakly formed at the base, but apparently arising from near R and arching towards the anterior margin; then curving away just before forking; the anterior fork of Sc is directed obliquely towards the costal margin, the posterior one continuing in almost a straight line the main stem; R strongly developed and nearly straight, directed more or less anteriorly; at the base it is nearer to the posterior margin of the wing than to the anterior margin; M and Cu apparently arising from R at the very base; M and Cu1 straight; M subparallel to R; 1A close to Cu2, 2A remote from 1A; anal area narrowed basally.

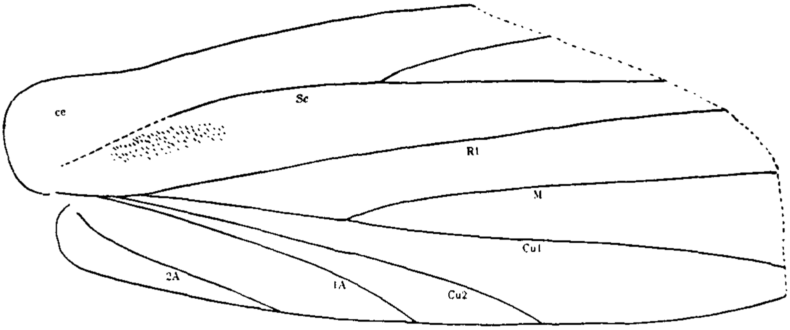

FIGURE 23. *Elytroneura permiana*, n. sp., elytron; holotype, No. 3388ab, Mus. Comp. Zoology.

Holotype: No. 3388ab, Museum of Comparative Zoology; collected by the writer in the lower layer of limestone at Elmo, 1927. The type specimen consists of a single elytron, minus the apical third, the preserved part of the wing being very clear. The whole elytron was probably about 12 mm. long.

This is one of the most peculiar of all the Protelytroptera. The elytron is distinctly flattened, not convex like that of the other species; yet the wing was obviously hard and coriaceous. In this species the widening of the area between R1 and the costal margin reaches an extreme condition, which together with the narrow anal area produces an unbalanced appearance. The dichotomously forked Sc is probably a primitive feature lost in the other species of the order; but the coalescence of M and Cu1 is a specialization not developed in the others.

Family BLATTELYTRIDAE.

Elytron: strongly convex; the main veins obsolescent in at least the distal half; anal veins absent; sutural margin well developed. Hind wing: in the genus Acosmelytron, at least, similar to that of the Protelytridae.

Genus *Blattelytron* Tillyard.

Blattelytron Tillyard, 1931, Amer. Journ. Sci. **21**: 250; fig. 9.

This genus includes the single species *Blattelytron permianum* Tillyard, and since that is known only from a poorly preserved fragment of an elytron, it is not possible for us to give as complete a list of the diagnostic characteristics as for the other genera. The base of the elytron being one of the missing portions, the nature of the main veins, Sc, R and M, is not known; it seems almost certain, however, that these veins are weakly developed in the rest of the elytron and simulate those of Parablattelytron. As I have already stated I do not believe that this fossil really lacks the sutural margin, as was thought by Tillyard. The elytron is so badly preserved and so fragmentary that the absence of the margin in the specimen seems almost certainly due to that condition; and under a very oblique light, directed from the posterior border, the faint outline of a sutural margin is, I believe, visible. The type specimen of *permianum* is so poor that I doubt if additional specimens of this species will be recognized. There are no representatives in the Harvard collection so far as I can determine.

Genus *Acosmelytron* Tillyard.

Acosmelytron Tillyard, 1931, Amer. Journ. Sci., **21**: 253.

Elytron: costal margin convex, the costal expansion at the base well developed; Sc, R, and Cu1 are visible at the base of the wing only, Cu1 being more or less straight after its origin; there is no definite Cu2.

Genotype: *Acosmelytron elongatum* Till.

Acosmelytron elongatum Tillyard.

Acosmelytron elongatum Tillyard, 1931, Amer. Journ. Sci. **21**: 253; fig. 12.
Acosmelytron acutum Tillyard, ibid., p. 254.

Elytron: length, 7.8 mm.; width, 2.4 mm. The basal parts of Sc, R, M, Cu1 and even the anal veins are discernible. The sutural margin is broad, more so than in any of the other Protelytroptera. It should be borne in mind that in Tillyard's figure the basal expansion of the costal margin is not illustrated, and that the sutural margin is shown narrowed basally, which is not the case.

Holotype: No. 5415, Peabody Museum. In the Harvard collection there are two fine specimens of this species: No. 3387 and No. 3386ab, both collected by the writer, the former in the lower layer, the latter in the upper layer of the limestone. No. 3386 shows the veins clearly as well as the cluster of spines on the costal space near Sc. *Acutum* was based upon a poorly preserved elytron (No. 5414, Peabody Museum), 7.5 mm. long, but lacking the apex. This fossil shows Cu1 somewhat more clearly than the type of *elongatum*, but this is unquestionably a matter of preservation. In such wings as these, in which the veins are only faintly indicated at best, the degree of perfection of preservation makes a great difference in the appearance of the veins.

Acosmelytron delicatum Tillyard.

Acosmelytron delicatum Tillyard, 1931, Amer. Journ. Sci., **21**: 254; fig. 13.

Elytron: length, 6 mm.; width, 1.6 mm. The type specimen of this species (No. 5416, Peabody Museum) consists of portions of the body and the elytra, fairly well preserved. The fossil may be a small *elongatum*, since there seems to be no differnce except in size. However, the specimen is of much interest since it shows part of the hind leg. In Protelytron, as I have already shown, the hind legs are peculiarly modified and armed; in *Acosmelytron delicatum* they are apparently similar in segmentation and armature. Tillyard has given a figure of part of the leg (fig. 13) but he did not include the complete tarsus, for he did not remove the limestone matrix which covered the four terminal segments. With the permission of the authorities of the Peabody Museum, I removed this matrix and exposed the entire tarsus, which turns out to be like that of Protelytron.

Genus *Parablattelytron* Tillyard.

Parablattelytron Tillyard, 1931, Amer. Journ. Sci., **21**: 251.

Elytron: costal margin convex or at least strongly curved; costal expansion at the base well developed; Sc, R, and Cu2 are visible at the base only, there being no trace of the convex Cu1.

Genotype: *Parablattelytron subincisum* Til.

Parablattelytron subincisum Tillyard.

Parablattelytron subincisum Tillyard, 1931, Amer. Journ. Sci., **21**: 251; fig. 10.

Elytron: length, 7 mm.; width, 2 mm.; slender; R and Cu2 are visible to about the middle of the wing, but Sc can be seen only at the base; sutural margin rather narrow but uniform in width for its whole length; the costal expansion, which is not shown in Tillyard's figure, is shaped as in *Parablattelytron latum*, n. sp. The apex of the elytron is rather pointed, due largely to the curvature of the anterior margin.

Holotype: No. 5401ab, Peabody Museum. The type specimen is not well preserved and a portion of the elytron just distal of the middle of the wing is broken away. In the Harvard collection there are five specimens of this species, as follows: No. 3376ab, consisting of the elytra, parts of the body, and both hind wings; the venation of the latter is not very clear, but enough can be seen to enable us to know that the wing structure is similar to that of the Protelytridae. No. 3377ab, a complete and well preserved elytron. No. 3378, a complete specimen, with the elytra folded over the abdomen; No. 3379ab, a complete and good elytron; No. 3375, also a complete elytron; and No. 3374ab, a nearly complete specimen, the hind wings hidden under the elytra; one middle leg is very well preserved (figure 18c) and shows a rather slender and normal tarsus and tibia; the terminal tarsal segment is as long as all the others combined. Of these fossils, No. 3379ab and 3375 were found in the upper layer of limestone.

Parablattelytron latum, new species.

Figure 24.

Elytron: length, 7.5 mm.; width, 2.8 mm.; very broad; Rs, Cu2 extend to about the middle of the wing; Sc present only at the base; sutural margin broader than in *subincisum;* apex bluntly rounded, the costal margin not so curved distally as in the latter species.

Holotype: No. 3380ab, Museum of Comparative Zoology; collected by the writer in the lower layer of the limestone at Elmo, 1932. This species, which is based upon a complete and well preserved elytron, is readily distinguished by the greater width of the elytron and its bluntly rounded apex.

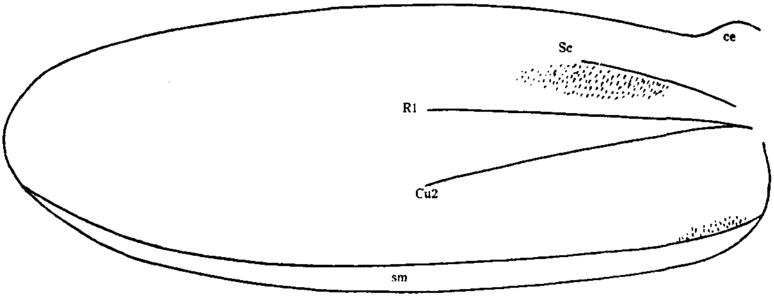

FIGURE 24. *Parablattelytron latum*, n. sp., elytron; holotype, No. 2338ab, Mus. Comp. Zoology.

Parablattelytron rectum Tillyard.

Parablattelytron rectum Tillyard, 1931, Amer. Journ. Sci. **21**: 252; fig. 11.

Elytron: length, 5.2–5.5 mm.; width, 1.5 mm.; Cu2 extending to the middle of the wing; Sc and R present only at the base, R extending a little further than Sc; sutural margin narrow; costal margin and the apex shaped as in *subincisum;* the costal expansion, not shown in Tillyard's figure, is similar to that of the other members of the genus.

Holotype: No. 5413ab, Peabody Museum. In the Harvard collection there are five specimens of this species: No. 3381, 3382, are good elytra, from the lower layer; No. 3384 is a good elytron from the upper layer; No. 3388 and 3383 are poorer elytra from the upper layer. There is one specimen of this species, consisting of an elytron, in the Sellards collection (No. 1417). The elytron of *rectum* is similar to that of *subincisum,* but differs mainly in its smaller size.

THE RELATIONSHIPS OF THE PROTELYTROPTERA.

Tillyard has already presented a very thorough discussion of the possible relationships between the Protelytroptera and the Coleoptera, and between the Protelytroptera and the Dermaptera. As he points out, this estimation of the affinities of these remarkable insects must be based almost entirely upon the structure of the hind wing, since the fore wings have been reduced along similar lines in all three orders. The conclusion which he reached was that the Protelytroptera had no genetical relationship with the Coleoptera, but were probably ancestral to the Dermaptera; and that the Protelytroptera known to us are a highly reduced remnant of a more extensive group. The new material which has been described in the foregoing pages does not contribute much to our understanding of this particular question. The pentamerous segmentation of the tarsi of the Protelytroptera has no weight in the discussion, for although pentamerous tarsi do occur among some Coleoptera and only trimerous tarsi among the Dermaptera, the pentamerous condition may have been present in the primitive Dermaptera. That this was in fact the case is shown by the Jurassic Dermapteron *Protdiplatys fortis* Mart., from Turkestan, which possessed pentamerous tarsi.

The few changes which I have made in the interpretation of the hind wings of the Protelytroptera do not effect Tillyard's conclusions, for the latter were based not upon the homology of the venation as upon the similarity of folding of the wing in the Dermaptera and Protelytropsis, the condition in the latter apparently representing an elementary stage of folding which has been brought to a higher degree of specialization in the modern Dermaptera. But although I accept Tillyard's conclusion regarding the relationship between the Coleoptera and the Protelytroptera, I believe that more evidence is needed before we are justified in concluding that the Dermaptera are lineal descendents of the Protelytroptera. To my mind there still remains the possibility that the Protelytroptera were an end branch of insects which developed the elytrophorous condition as independently from the Dermaptera as did the Coleoptera; for I am not entirely satisfied that we have certain knowledge of the method of folding of the hind wings in the Protelytroptera. But it is obvious that we know now as much about the structure of the Kansan Permian Protelytroptera as we shall ever know, in so far as the estimate of the relationships of these insects is concerned. New and better specimens may be found, but the facts added will almost certainly be of such a nature as to be unable to contribute materially to our understanding of the phylogenetic position of the order. The evidence to substantiate or to disprove our present views of the affinities of the

Protelytroptera will come when we have a more complete series of insects from the upper Permian and the Mesozoic. Unfortunately, the hind wings of the Jurassic Prodiplatys are folded under the elytra; if we knew the structure of these wings, we would be in a better position to decide more definitely on the relationships of the Protelytroptera with the Dermaptera.

Although the fossils in the Harvard collection do not aid materially in our search for descendents of the Protelytroptera, they do contribute a great deal of information needed in considering the question of the ancestors of these insects. It is clear that so far as the elytra are concerned the order had already attained a long history by the time of the Lower Permian; for the families described above represent several lines of descent which had already reached wide divergence. The hind wings, of course, are not known in enough of the families to enable us to utilize their structural features. But by selecting the more generalized structural characteristics of the elytra in these families, I believe that we can form some idea of the fore wing in the common ancestral group which gave rise to this series of fossils.

We are justified in concluding, I believe, that the evolution of the Protelytroptera as a whole has been in the direction of the perfection of the elytrophorous condition, that is, by the increase of the hardness and convexity, and the atrophy of the venation. That evolution can clearly be seen in the known families. The fore wing in the Elytroneuridae is especially interesting because it is hardly an elytron at all; it is obviously coriaceous and thickened, but it is perfectly flat, not convex as are the elytra of the other families. This type of elytron I consider to represent the more generalized form in the order. The subcosta is shortened, but it is dichotomously forked; this again is certainly less specialized than that of the other members of the order. In all the species R and R1 are unbranched. M is apparently branched in Archelytron, though this branch does coalesce with Rs. In the same genus Cu1 has two dichotomous forks, although in the other species it is unbranched. Cu2 is unbranched in all species; and this is also true of the three anal veins and of the fourth anal vein in *Protelytron angustum*. In Elytroneura the sutural margin is completely absent, and in view of the flattened condition of this wing, I believe that this is a primitive trait. In figure 25 I have included a drawing of a hypothetical wing consisting of the more generalized characteristics. All of the veins, except Cu2 and the basal part of Sc, are convex, so we are unable to determine whether M is MA, MP, or both combined. This wing bears a strong resemblance both in general form and venation to the fore wings of some fossil cockroaches, in particular those of Scutinoblattina Scudder, Nepioblatta Handl., and Brephoblatta Handl. Scudder called attention in his description of these fossils (1885, 1886) to the pronounced convexity of the fore wings, which are in fact true elytra. The types of these genera are now in the Museum of Comparative Zoology, and on comparing them with specimens of Protelytroptera, I was greatly impressed with the remarkable similarity in the form of the elytra, including the wide costal space, and in the general arrangement of the venation. It is interesting to note in this connection that the elytrophorous condition has apparently arisen several times in the cockroaches.[19] Some of the elytra described by Scudder from the Permian of Colorado and placed by him and Handlirsch in the Blattaria are so close to the elytra of the Protelytroptera that it is not improbable that they may prove to be members of the latter order. I venture to suggest, then, that the Protelytroptera are descended from the ancient cockroaches, probably arising in the lower part of the Upper Carboniferous. Tillyard has already called attention to the resemblance between the Protelytroptera and the insects which he has described from the Upper Permian as Protocoleoptera, and the new fossils described above, especially Elytroneura, serve to intensify this similarity. He suggests that these Proto-

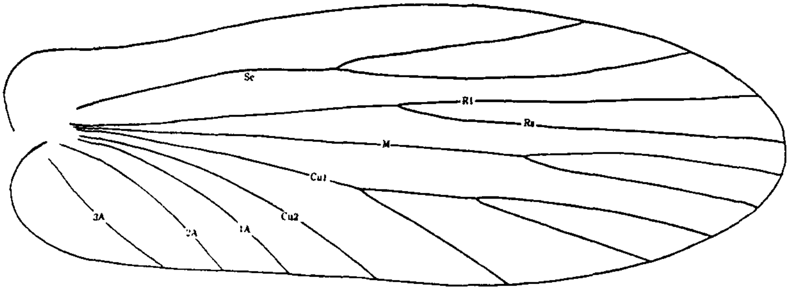

FIGURE 25. Hypothetical elytron of common ancestor of the Permian Protelytroptera; for explanation, see text.

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[19] As in the Epilampridae, for example.

coleoptera may prove to be an archaic remnant of the older Protelytroptera, at a stage when the elytron was still tegminous and more or less flat. The family Elytroneuridae, in which the elytra were flat and tegminous, may have been a survival of that very group which was the common ancestral stock of the Protocoleoptera and the more distinctly elytrophorous Protelytroptera.

Order PLECTOPTERA.

The may-flies constitute a very prominent part of the fauna of the Elmo limestone. In the Sellards collection there are nearly forty specimens, of which about a dozen are well preserved. In the Yale collection there are ninety-two specimens, of which one-third were estimated by Tillyard to be in a good state of preservation. In the collection at the Museum of Comparative Zoology there are more than one hundered and fifty specimens, about one-half of which are well preserved. This amounts to a total of more than 280 specimens of may-flies from this single formation. The importance of this fine series of fossils in the study of the geological history and the phylogeny of these insects can be realized if we bear in mind that they constitute the earliest record of true Plectoptera; and with the exception of a single wing from the Permian of Russia and a few doubtful nymphal forms, they comprise the sole representatives of the group prior to the Jurassic.

The specimens in the Sellards collection were studied by Dr. Sellards himself, who published preliminary descriptions of twelve species, based on the best specimens in the material at his disposal (1906). These type specimens I was able to examine and photograph in 1927 in Austin, Texas, although a few of the types could not be found at that time. Unfortunately, a short time after my study of them, all the types were accidently destroyed during the process of renovation of the building in which they were deposited, but the remainder of Sellards collection has been loaned to the Museum of Comparative Zoology. The may-flies in the Yale collection were subjected to an intensive study by Tillyard (1932), who has published a detailed account of his conclusions, together with the descriptions of four new species. These specimens, now at the Peabody Museum, I have also been able to examine.

In his account of the Yale collection Tillyard covered completely certain phases of the phylogeny of the may-flies which consequently require no further discussion. He thoroughly treated the complicated question of the homology of the veins of the Plectoptera, showing that the evidence provided by these fossils supports in every way the conclusion reached by Lameere in 1922 that the media is represented by both MA and MP, and the cubitus by CuA and CuP. Much of Tillyard's discussion was concerned with an attempt to disprove the view advanced by Martynov in 1923 that the media was only represented by MA; but Martynov had already abandoned that idea and accepted Lameere's, as he states in a paper published in February, 1932, the same month as Tillyard's.[20] In this paper I have therefore employed without change this system of nomenclature.

The conclusions which Tillyard reached on the synonymy of certain of Sellards' genera and species very nearly coincides with those to which I was led in my survey of the types in 1927. But Tillyard was not able to ascertain the differences between the fore and hind wings of these Permian may-flies and consequently he confused certain of these differences with true specific characteristics. Fortunately the new material in the Harvard collection contains specimens which enable us to distinguish satisfactorily between the two pairs of wings. Also, Tillyard was not able to determine much about the body structure of these insects and although he presented a reconstruction of *Protereisma permianum,* his figure is incorrect in many respects and gives an erroneous idea of the habitus of the Permian may-flies. Again we are fortunate in having in the Harvard collection a series of fine specimens which show clearly the body structure of these ancient insects.

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[20] Tillyard's remarks concerning my translation of Martynov's Russian paper are somewhat misleading. He states (1932, p. 100) that it was very unfortunate that I "did not see fit" to include Martynov's figures in the translation; and that I did not realize the importance of having the figures to use in connection with the text. But in the introduction to the translation, I stated that "the twelve figures accompanying the original article are essential for a complete understanding of the problems under discussion"; and also pointed out that the nature of the original figures was such that they would have required retouching before they could have been copied. The publication of the translation without the figures cost me personally not less than a hundred dollars, and the reproduction of the illustrations would of course have greatly increased this amount. I naturally assumed that anyone who was interested in the figures in Martynov's paper and who did not have access to the article in the original journal (Rev. Russe Ent. **18:** 145–174), would secure a copy of the issue containing the paper from the entomological society which published it.

Family PROTEREISMATIDAE.

Protereismephemeridae, Sellards, 1907, Amer. Journ. Sci. **23**: 345.
Protereismidae, Handlirsch, 1919, Denkschr. Akad. Wiss. Wien Math. Naturw. Kl. (96) **82**: 63.
Protereismatidae, Tillyard, 1932, Amer. Journ. Sci., **23**: 237.

Fore wing: elongate-oval in shape, somewhat narrowed basally; costa submarginal, forming an arch at the base of the wing; Sc terminating near the apex of the wing; R1 more or less parallel to Sc; Rs originating from R1 near the base, but almost immediately coalescing with MA for a short distance; Rs with three complete triads; R4 + 5 arising a little proximad of the middle of the wing; MA with a single triad arising also at about mid-wing; MP with a single triad originating proximad of the origin of Rs, occasionally closer to the origin of MA; CuA arising from the stem of Cu at the base of the wing, but directed anteriorly and immediately touching M or joined by a short strut to M; CuA with a well developed triad; CuP unbranched; A1 and A2 arising from a thick chitinous stem, joined to the recurved stem of Cu; A1 without a well defined triad.

Hind wing: similar to the fore in shape and size, being only slightly smaller. The venation is also similar to that of the fore wing, but differs in the following points: the costa is closer to the anterior margin in the hind wing than in the fore; CuA does not approach M at the base as closely in the hind wing as in the fore; the hind wing has a well developed triad on A1; and A1 and A2 arise from a common stem at a point where there is a series of thickened cross-veins.

The general habitus of these insects was more or less intermediate between that of the recent Plectoptera and the Palaeodictyoptera. The head was broad, with large globular eyes in both sexes (apparently), and bearing prominent antennae with 11 subequal segments. The thorax was slender and elongate, the prothorax very broad and flattened, having the appearance of possessing small lateral expansions like those found in many Palaeodictyoptera. The legs were nearly homonmous, the fore pair being only a very little longer (4 mm.) than the others, and this difference may have been confined to the male. The coxa and trochanter were very short, but the femur was long and slender; the tibia a little shorter than the femur; the tarsus longer than the tibia and possessing five segments, the metatarsus being the longest and the fourth the shortest. Small tibial spurs were present and also a single small spine at the distal end of each of the tarsal segments. The tarsal claws were prominent. The abdomen was elongate in the male and only slightly shorter in the female. Ten segments are visible in the abdomen from above; the terminal filaments were present in both sexes and in the male were at least as long as the abdomen, containing upwards of 40 segments; the genital claspers of the male were very much like those in existing species.

As Tillyard has already pointed out, these Permian may-flies undoubtedly passed through the subimaginal stage characteristic of existing may-flies,, for the condition of the wings and of the body resembles that of recent may-flies in that state. The structure of the nymphs of these Lower Permian may-flies is very little known, since no well preserved nymphs have been found in the Elmo limestone. One faintly preserved specimen was taken by the writer in 1927; it shows two long filamentous abdominal setae and the general habitus of the recent nymphs, but most of the body is an obscure mass, probably because the delicate gills were folded or twisted over and across the abdomen. Several specimens of may-fly nymphs have been found in the Permian (Upper) of the Old World and in all probability belong to the Protereismatidae or to related families.

Genus *Protereisma* Sellards.

Protereisma, Sellards, 1907, Amer. Journ. Sci. (4) **23**: 347.
Protereisma, Tillyard, 1932, ibid. (5): **23**: 244.
Protechma, Sellards, 1907, ibid. (4): **23**: 349.
Prodromus, Sellards, 1907, ibid., p. 349.
Bantiska, Sellards, 1907, ibid., p. 341.
Rekter, Sellards, 1907, ibid., p. 349.
Pinctodia, Sellards, 1907, ibid., p. 352.
Mecus, Sellards, 1909, ibid., **27**: 151.

Since this is the only genus of the family, it is clearly defined by the characteristics already given under the family. The genotype is *Protereisma permianum* Sellards.

In this genus Tillyard recognized eight species, of which six were originally described by Sellards and placed by him in four genera. I agree entirely with Tillyard's view that all of Sellards' genera except Misthodotes and Esca are identical with Protereisma. Misthodotes is a valid genus, for which Tillyard erected a separate family. The affinities of Esca are not clear; the species included in the genus was not figured by Sellards and when I examined his collection of types in 1927 I was unable to locate the fossil on which he based his description. Since all of Sellards types have subsequently been lost, this genus will probably remain obscure and perhaps should be regarded as a synonym of Protereisma or dropped from the literature altogether.

The classification of these Permian may-flies is rendered difficult by the variability of many venational features, which were supposed by Sellards to be constant in the species. In his classification of the species of Protereisma Tillyard made use of the relative lengths and breadths of the wings, but he also employed certain venational characteristics which he considered constant in the species. From my study of the specimens in the Harvard collection as well as those in the Peabody Museum, I am convinced that some of the venational characteristics with which Tillyard separated the species are subject to much individual variation also, and that others are merely differences between the fore and hind wings; only one specimen (No. 5427) in the entire Yale collection was complete enough to permit a comparison of the wings and from that he was able to ascertain only that the hind wing was shorter than the fore and had a broader anal area. There are numerous specimens of Protereisma in the Harvard collection which have the bases of the fore and hind wings preserved and which have enabled me to ascertain the differences. Of these fossils the following are the most important: No. 3404, 3405, 3420, all *permianum;* and No. 3414, *arcuatum*. Other specimens, although having only one wing preserved, indicate by the position of its attachment to the thorax whether it is a fore or a hind wing, and so enable further comparison. A careful study of these fossils has revealed the following differences between the fore and hind wings:

1. The costa is closer to the margin of the wing in the hind wing than in the fore wing.
2. R1 and Sc are either fused basally or very nearly fused in the fore wing, but in the hind wing they are decidedly remote from each other.
3. CuP always touches or very nearly touches M at the base of the fore wing, but is remote from M in the hind wing.
4. A1 possesses in the hind wing a well defined triad which is absent in the fore wing.
5. A1, IA, and A2 in the fore wing originate from a strong stem arising from the base of the wing; in the hind wing they arise some distance away from the base of the wing at a chitinous node formed by the partial alignment of two or more cross-veins; between this

node and the base of the wing there is a single stout vein apparently formed by the coalescence of the basal parts of A1, IA, and A2.

Now if we consider the characteristics used by Tillyard to distinguish the species of Protereisma we note that he has employed some of these mentioned in the above list. For example, one of the characteristics used in his key to identify *gracile* is that Sc and R are fused basally; but this is now seen to be a feature common to *all fore* wings, and not present in the hind wings of any species. *Sellardsi* was separated from the others by having CuP practically touch M at the base of the wing, which is a characteristic of *all hind* wings. The variation of certain veins in individuals of a species was considered by Tillyard and he noted that many of the basal connections of the veins were unstable, such as the amount of anastomosis between Rs and MA, the length of the costal brace, and the depth of the forks of Rs and MA. From my study of the entire lot of more than 250 specimens of this family, I am convinced that the following venational features are also subject to individual variation, some of which were used by Tillyard in his classification of the species.

1. The shape of the arch formed by the costal vein varies considerably even in the wings of the same individual.

2. The point of origin of the triad of MP is as variable as that of the triads of Rs and MA; it may arise close to the base, almost below the distal end of the costal vein or near to the first forking of MA; and between these extremes there is a complete series of intermediate conditions, as shown in specimens No. 3404ab, 3408ab, 3409ab, and 3390ab of the Harvard collection.

3. The relative widths of the costal and subcostal spaces do not appear to be constant. The majority of specimens have the costal and subcostal spaces subequal in width or the costal space a little wider; but a few wings have been found in which the subcostal space is the broader. This feature does not, however, appear to be associated with any other characteristics that might indicate specific difference, nor does it appear to be confined to either the fore or hind wing. Specimen No. 5438 of the Yale collection, which has a narrow costal space, is a fore wing, as shown by the structure of CuP and the anal veins at the base; but in the Harvard collection there is a wing (No. 3424ab) which also has a narrow costal area, although it is obviously a hind wing, as indicated by its point of attachment to the thorax. Now if we again consider the characteristics on which Tillyard based the species of Protereisma, we note that one of the chief diagnostic

features of *sellardsi* is that the triad of MP arises close to the base, about the level of the costal brace. But the specimens mentioned above in the discussion of this vein show a complete series of gradations from the extreme condition in the type of *sellardsi* to the more normal one. *Gracile* was characterized as having the costal space narrower than the subcostal, which for reasons given above I consider to be only an individual variation.

In view of the evidence supplied by the new material, then, I believe that the number of species in the genus is even less than supposed by Tillyard. *Latum* Sellards and *insigne* Tillyard are at once separated from all others by their large size, the fore wings being more than 25 mm. long. *Arcuatum* Sellards is distinguished by the arched anterior margin, and *elongatum* Sellards by the small number of cross-veins. The remaining species recognized by Tillyard, i. e., *gracile* Sellards, *minus* Sellards and *sellardsi* Tillyard, are in my opinion identical with *permianum*.

Protereisma permianum Sellards.

Figures 26 and 27.

Protereisma permianum Sellards, 1907, Amer. Journ. Sci. (4) **23**: 348; figs. 3, 4.
Protereisma permianum Tillyard, 1932, ibid. (5) **23**: 247; figs. 5, 14.
Protechma acuminatum Sellards, 1907, ibid. (4) **23**: 349; fig. 8.
Prodromus rectus Sellards, 1907, ibid., p. 349; fig. 10.
Scopus gracilis Sellards, 1907, ibid., p. 352; fig. 12.
Protereisma minus Sellards, 1907, ibid., p. 254.
Protereisma gracile Tillyard, 1932, ibid. (5) **23**: 252; fig. 15.
Protereisma minus Tillyard, 1932, ibid., p. 254.
Protereisma sellardsi Tillyard, 1932, ibid., p. 251; fig. 16.

Length of fore wing, 15–20 mm.; width, 4.8–6 mm.; length of hind wing, 17–19 mm.; width, 4.6–5.7 mm. In the specimens which appear to be imagines the membrane of the wings is hyaline and the cross-veins are distinctly brown; in the wings which apparently belong to subimagines the membrane has a uniformly brown tint. Anterior margin straight, not curving until about one-fourth the wing length from the apex; cross-veins numerous; subcostal and costal spaces usually subequal, occasionally one or the other somewhat broader; origin of IMP usually nearly as far distal as the first fork of Rs, but sometimes arising more distad; CuP strongly waved.

Length of antennae, 4 mm.; width of head, 2.8 mm.; length of head, 1.2 mm.; length of thorax, 5.5 mm.; width of prothorax, 3 mm.; length of fore legs, 24 mm.; length of hind and middle legs, 20 mm.; length of abdomen (small specimen), 16 mm.

The holotype specimen, No. 970, was originally in the Sellards collection and was studied and photographed by me in 1927. Since that specimen has subsequently been destroyed along with the other types, I take this opportunity to designate No. 3405ab in the Harvard collection as the neotype. A second specimen, originally in the Sellards collection, consisted only of the body and was placed doubtfully by Sellards in this species or *latum*. This fossil, which has also been lost, is apparently the one to which Tillyard referred as a paratype.

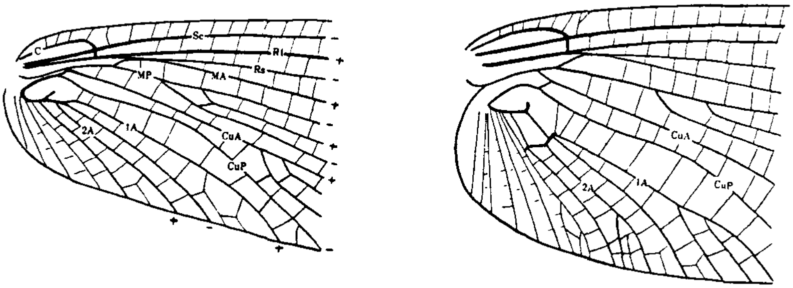


FIGURE 26. *Protereisma permianum* Sell., bases of fore and hind wings, drawn from specimens No. 3404ab, 3405ab, and 3420ab, Mus. Comp. Zoology.

In the Harvard collection there are over forty specimens which can be assigned to this species without question as well as many others which probably belong here. The most important of these fossils are included within the Nos. 3397–3406. Only about a third of the specimens were found in the upper layer of limestone. The restoration of this species, shown in figure 27, is based mainly on the following fossils in the Museum of Comparative Zoology:

No. 3405ab, an excellent specimen, showing the right wings outspread, free from each other, and the two left wings also outspread but slightly overlapping; also showing portions of the body, including one antenna and most of the legs except the tarsi. This fossil shows distinctly the differences between the fore and hind wings.

No. 3404ab, consisting of a fore and hind wing and portions of the body, the legs being especially clear.

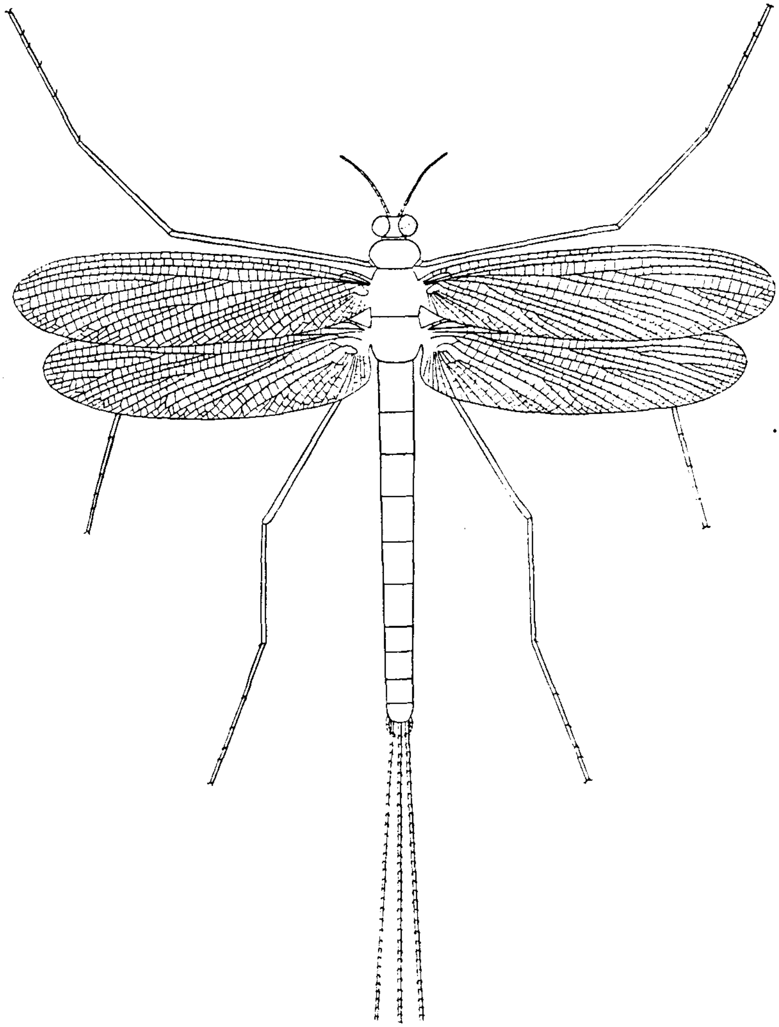


FIGURE 27. *Protereisma permianum* Sell., reconstruction, based on several specimens in the Museum of Comparative Zoology.

No. 3403ab, consisting of the basal halves of two wings and the complete body, showing the antennae and abdomen especially well.

No. 3402ab, a fine lateral view of the whole insect, showing all wings partially overlapped and the legs, including the complete tarsi of two legs.

No. 3400 and 3401, excellent hind wings, showing the anal area unusually well.

Tillyard has already given reasons for synonymizing *acuminatum* and *rectum* with *permianum* and he suggested that *minus* Sellards might also be another synonym. The type of *minus* was somewhat distorted and from my examination of it, I am convinced that the apparent straightness of CuP described by Sellards was due to this distortion. Except for this feature and its slightly smaller size, there are no differences between this wing and that of *permianum*. I have placed *gracile* here also because I believe the characteristics on which Sellards and Tillyard recognized this species were only individual ones. Sellards believed that the wing was distinguished by its slenderness, but this was without definite basis because the posterior part of the wing was missing. Furthermore the apical part of the wing was much wrinkled, as shown by the irregularity of the veins in Sellards' figure; and Sellards himself stated that the wing had suffered "lateral compression," obscuring the radial area at the base. For these reasons, the shape of the type wing cannot be considered as indicative of a similar shape in the species. Tillyard believed that he recognized about a dozen specimens in the Yale collection belonging to *gracile*, which he distinguished by the slenderness of the wing, the fusion of R and Sc at the base, and the broad subcostal space, which was wider than the costal. I have already mentioned above, however, that the fusion of R and Sc at the base is the usual condition if the fore wings of Protereisma, and have pointed out that the relative widths of the costal and subcostal spaces vary even in the same specimen. I therefore regard all specimens in the Yale collection determined by Tillyard as *gracile* to be *permianum*. I have also placed *sellardsi* Tillyard as a synonym of the latter. This species was based upon a single specimen, characterized by having triad of MP arise nearer the base of the wing than in the other species. I have pointed out above, however, that the place of origin of this triad is subject to as much individual variation as the other triads. The species was also based upon the close approximation of CuP and M at the base; but this peculiarity is common to all fore wings of members of the genus, and not confined to any one species.

Protereisma elongatum (Sellards).

Bantiska elongata Sellards, 1907, Amer. Journ. Sci. (4) **23**: 349; fig. 7.
Protereisma elongatum Tillyard, 1932, ibid. (5) **23**: 254.

Hind wing: length, 15–19 mm.; width, 5–5.5 mm.; fore wing probably a little larger. Costal margin straight, curving only in the distal third of the wing; cross-veins fewer than *permianum;* costa (in the hind wing at least) closer to the margin than in *permianum* and other species of Protereisma; CuP markedly waved.

The holotype, No. 976, was present in the Sellards' collection when I studied it in 1927; since it has subsequently been lost, I designate specimen No. 3424ab in the Museum of Comparative Zoology (lower layer) as the neotype. This specimen consists of a fine hind wing and portions of the body. The wing is shown to be a hind wing by the place of attachment to the thorax; it has the same venational features at the base as the hind wing of *permianum*. The basal part of Sc overlaps R at the base, but this is caused by an obvious fold in the wing membrane at this point. I believe that the only definite and constant feature that distinguishes this wing is the smaller number of cross-veins. The proximity of the costal vein to the margin may only be true of the hind wing, but even there it is closer to the margin than in the hind wing of *permianum*. I have not seen any fore wing that can be assigned to this species without question and the specimen mentioned above is the only one in the Harvard material which can be placed here with certainty, though some of the other wings with a more or less open type of venation may belong to this species.

Protereisma arcuatum (Sellards).

Figure 28.

Rekter arcuatus Sellards, 1907, Amer. Journ. Sci. (4) **23**: 349; fig. 6.
Protereisma arcuatum Tillyard, 1932, ibid. (5) **23**: 295; fig. 17.

Length of fore wing, 16–17 mm.; width, 4.7–5 mm.; hind wing a little smaller; costal margin straight for the proximal two-thirds, but curved downwards much sooner than in the foregoing species of the genus; apex pointed; cross-veins numerous; CuP distinctly waved. Tillyard states that the basal part of the anterior margin is strongly curved downward, but that assertion was apparently based upon Sellards' figure, which shows this part of the wing restored in dotted lines. Neither Sellards' specimen nor any specimens in the Yale collection show this part of the wing. In one wing of the Harvard collection (No. 3414ab), however, this region of the wing is preserved and is the same as in the other species of the genus.

The holotype was No. 598 in Dr. Sellards' collection, and was studied by me in 1927; since it is now lost, I designate No. 3414ab in the Museum of Comparative Zoology as the neotype. This fossil, which was collected in the lower layer, consists of the left hind wing, part of the right fore wing and most of the body, including the head, the prothorax, the abdomen and the genitalia. The latter are very well preserved and show that the insect was a male. On the same piece of limestone and only a few millimeters away is a specimen of *Protereisma permianum.* In addition to this specimen there is another representative of *arcuatum,* collected in the upper layer (No. 3413ab); this includes only the apical half of the wing, probably a fore one.

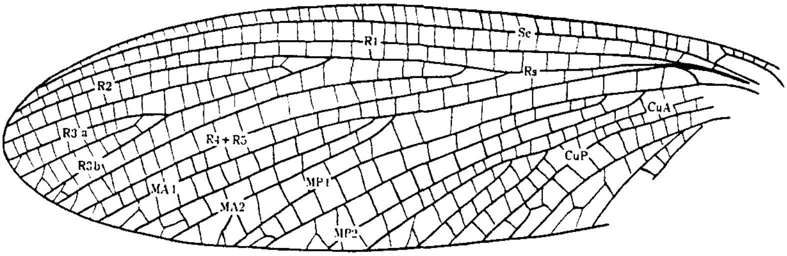


FIGURE 28. *Protereisma arcuatum* Sell., drawn from specimen No. 3414ab, Mus. Comp. Zoology.

Protereisma latum Sellards.

Figure 29.

Protereisma latum Sellards, 1907, Amer. Journ. Sci. (4) **23**: 349.
Protereisma latum Tillyard, 1932, ibid. (5) **23**: 256; fig. 7.

Length of fore wing, 22–25 mm.; width, 6.7–7.5 mm.; hind wing slightly smaller; costal margin straight for the basal two-thirds, distally curved as in *arcuatum;* cross-veins about as numerous as in *elongatum;* anal area of hind wing much broader than in *permianum.*

The holotype was originally No. 124 in the Sellards collection; since that is lost I designate No. 3419ab in the Harvard collection as the neotype; this was collected in the upper layer of limestone.

There are four good specimens of this insect in the Museum of Comparative Zoology: No. 3419ab, the neotype, is an excellent hind wing, complete and well preserved; No. 3392ab is a complete though somewhat wrinkled hind wing; No. 3391ab, is the basal two-thirds of a fore wing: and No. 3390ab is a fore wing lacking part of the anal area. All of these were collected by the writer in the lower layer, except for the neotype and No. 3391ab, which were taken in the upper layer. This fine species is easily recognized by its large size and relatively broad hind wing. Since the complete wing of this species has not previously been figured, I include here a drawing of the hind wing.

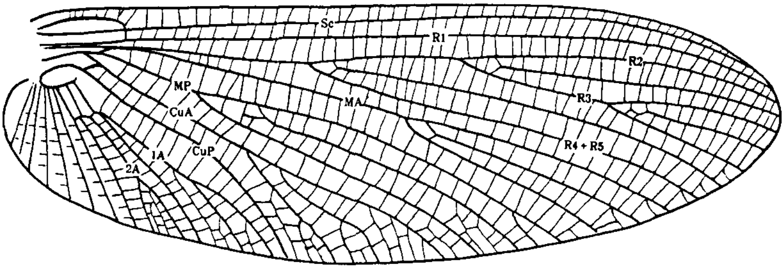


FIGURE 29. *Protereisma latum* Sell., hind wing, drawn from specimen No. 3419ab, Mus. Comp. Zoology.

Protereisma insigne Tillyard.

Protereisma insigne Tillyard, 1932, Amer. Journ. Sci. (5): 259; fig. 18.

Length of hind wing, about 27 mm.; width, 9.4 mm.; fore wing unknown; costal margin straight almost to the very apex of the wing, so that most of the apical curvature takes place in the hind margin, and the apex of the wing is anterior to the longitudinal axis of the wing; cross-veins numerous.

Holotype: No. 1112, Peabody Museum; a hind wing, lacking the distal part.

In the Museum of Comparative Zoology there is one specimen (No. 3436) which I consider to belong here; it consists only of the apical portion of the wing, but it has a width of 8 mm.; and could only fit this species or a new one. The chief feature of this fossil is that the apex is decidedly "off center," that is, above the longitudinal axis of the wing. A comparison of this fossil with the photograph of *insigne* given in Tillyard's paper shows that the outline of the two wings would make a perfect fit, the Harvard fossil including the portions missing in the type. The new specimen was collected in the upper layer of limestone.

Family MISTHODOTIDAE.

Fore wing: smaller and relatively broader than that of the Protereismatidae; costa submarginal, but not so strongly developed as in the previous family; Sc terminating on the margin before the apex of the wing; Rs arising from Rs near the base; MA and Rs fused basally, no free part of MA visible; Rs with 3 triads; MP with a single triad; CuA arising from the stem of Cu2 and joined to MP by a short strut, as in the Protereismatidae; CuA without a definite triad, at most with a terminal twig; anal area as in the Proteresimatidae; cross-veins relatively few.

Hind wing: broader than the fore wing, with a more rounded posterior margin; venational like that of the fore wing with the exception of the differences indicated in the Proteresimatidae.

These insects had a shorter abdomen and less elongate legs than the Protereismatidae; the structure of the antennae and tarsi is unknown. The family is much less common in the Elmo limestone than the Protereismatidae, and seems to be absent in the upper layer of the limestone, although additional material may show that it does occur there.

Genus *Misthodotes* Sellards.

Dromeus Sellards, 1907, Amer. Journ. Sci. (4) **23**: 351 (*nec Dromeus* Reiche).
Misthodotes Sellards, 1909, ibid. **27**: 151.
Misthodotes Tillyard, 1932, ibid. (5) **23**: 261.

This is the only genus in the family and consequently needs no further diagnosis. Three species have been described, *obtusus* Sellards (genotype), *biguttatus* Tillyard, and *ovalis* Tillyard. The wings of the latter species are distinguished by their large size, being 15 mm. long; those of *biguttatus* apparently differ from the others by the presence of a color pattern, consisting of two large blotches of brown. Neither of these two species is represented in the Museum of Comparative Zoology.

Misthodotes obtusus Sellards.

Dromeus obtusus Sellards, Amer. Journ. Sci. (4) **23**: 351.
Misthodotes obtusus Sellards, ibid., **27**: 151.
Misthodotes obtusus Tillyard, 1932, ibid. (5) **23**: 261; fig. 19, 20.

Length of fore wing, about 11 mm.; width, 4.3 mm.; length of hind wing, 10.5 mm.; width, 3.9 mm.; costal margin straight, except in the distal region; CuP slightly waved; imago wing without pigmentation.

The holotype specimen of this species was No. 1048 and its counter part 1385 in the Sellards collection; since these have been lost, I designate specimen No. 5470 in the Peabody Museum as the neotype. In the Harvard collection there are two specimens of this insect, No. 3425ab, consisting of a nearly complete hind wing, apparently from a subimago; and No. 3429, the apical part of a fore wing. The Yale specimen mentioned above consists of all four wings and many portions of the body, though details are not visible. Since this is a more complete specimen than either of the two in the Harvard material, I have designated it as the neotype. The various specimens of this species, especially the neotype, show that the fore and hind wings of the members of this family were like those of the Proteresimatidae in general features and differed from each other in the same manner.

Museum of Comparative Zoology,
Cambridge, Mass.

REFERENCES.

CARPENTER, F. M.

1930. The Lower Permian insects of Kansas. Part I, Introduction and the Order Mecoptera. Bull. Mus. Comp. Zool., **70** (2): 69–101.

1931a. The Lower Permian Insects of Kansas. Part 2, The Orders Palaeodictyoptera, Protodonata, and Odonata. Amer. Journ. Sci., (5) **21**: 97–139.

1931b. The Lower Permian Insects of Kansas. Part 3, The Protohymenoptera. Psyche, **37**: 343–374.

1931c. The Lower Permian Insects of Kansas. Part 4, The Order Hemiptera and Additions to the Palaeodictyoptera and Protohymenoptera. Amer. Journ. Sci. (5) **22**: 113–130.

1932. The Lower Permian Insects of Kansas. Part 5, Psocoptera and Additions to the Homoptera. Amer. Journ. Sci. (5) **24**: 1–22.

DUNBAR, C. O.

1924. Kansas Permian Insects. The Geologic Occurrence and the Environment of the Insects. Amer. Journ. Sci. (5) **7**: 171–209.

MARTYNOV, A. V.

1923. The Interpretation of the Wing-Venation and Tracheation of the Odonata and Agnatha. Rev. Russe Ent., **18**: 145–174. English translation in Psyche, **37**: 245–280, 1931.

1927. Über eine neue Ordnung fossilen Insekten, Miomoptera. Zool. Anz. **72**: 99–109.

1928. Permian Fossil Insects of N.-East Europe. Trav. Mus. Geol. Acad. Sci. U. R. S. S., **4**: 1–118.

1930. Permian Fossil Insects from Tikhje Gory, Order Miomoptera. Bull. Acad. Sci. U. R. S. S., **1930**: 951–975; 1115–1134.

1931a. On Some New Remarkable Odonata from the Permian of Archangelsk District. Bull. Acad. Sci. U. R. S. S., **1931**: 141–147.

1931b. Sur le subordre nouveau *Permanisoptera*, nom. nov. (Odonata) et sa position systematique. Comptes Rend. Acad. Sci. U. R. S. S., **1931**: 246–247.

1932a. New Permian Palaeoptera with a Discussion of Some Problems of their Evolution. Trav. Inst. Palaeont. Acad. Sci. U. R. S. S., **1**: 1–44.

1932b. On the Wing-Venation in the Fam. Meganeuridae (Meganisoptera). Comptes. Rend. Acad. Sci. U. R. S. S., **1932**: 42–44.

SCUDDER, S. H.

1885. New Genera and Species of Fossil Cockroaches, from Older American Rocks. Proc. Acad. Nat. Sci., Phila., **1885**: 34–49.

1890. The Insects of the Triassic Beds at Fairplay, Colo. Mem. Bost. Soc. Nat. Hist., **4**: 457–472.

1893. Insect Fauna of the Rhode Island Coal Field. Bull. U. S. Geol. Surv., **101**: 1–27.

SELLARDS, E. H.

1907. Types of Permian Insects. Part 2, Plectoptera. Amer. Journ. Sci. (4) **23**: 245–355.

1909. Types of Permian Insects. Part 3, Megasecoptera, Oryctoblattinidae, and Protorthoptera. Amer. Journ. Sci. (4) **27**: 151–173.

TILLYARD, R. J.

1926a. Upper Permian Insects of New South Wales. Part 1, Introduction and the Order Hemiptera. Proc. Linn. Soc. New S. Wales, **51**: 1–30.

1926b. Kansas Permian Insects. Part 8, The Order Copeognatha. Amer. Journ. Sci. (5) **11**: 315–349.

1926c. Kansas Permian Insects. Part 9, The Order Hemiptera. Amer. Journ. Sci. (5) **11**: 381; 395.

1928. Kansas Permian Insects. Part 12, The Family Delopteridae, with a Discussion of its Ordinal Position. Amer. Journ. Sci. (5) **16**: 469–484.

1931. Kansas Permian Insects. Part 13, The New Order Protelytroptera, with a Discussion of its Relationships. Amer. Journ. Sci., (5) **21**: 232–266.

1932a. Kansas Permian Insects. Part 14, The Order Neuroptera. Amer. Journ. Sci. (5) **23**: 1–30.

1932b. Kansas Permian Insects. Part 15, The Order Plectoptera. Amer. Journ. Sci. (5) **23**: 97–134; 237–272.

ZALESSKY, G.

1931a. Observations sur un Nouvel Insecte Libelluloïde du Permian du Bassin de la Kama. Bull. Acad. Sci. U. R. S. S., **1931**: 855–861.

1931b. Observations sur un Nouvel Insecte Libelloïde du Bassin du Fleuve Kama. Ann. Soc. Geol. Nord., **56**: 36–41.

1932. On the Wing Venation of Dragon-flies and Mayflies and their Phylogenetic Evolution. Bull. Acad. Sci. U. R. S. S., **1932**: 713–733.

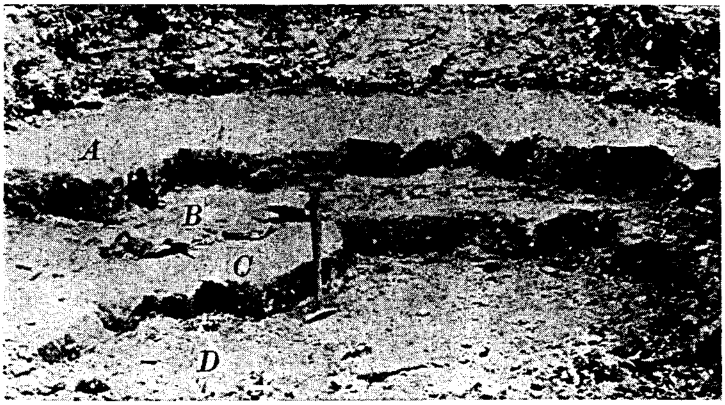
EXPLANATION OF PLATE 1.

FIGURE 1. Photograph of part of the 1932 quarry at Elmo, Kansas. A, upper layer of limestone; B, intervening shale; C, lower layer of limestone; D, top of stump layer.

FIGURE 2. Photograph of head of *Dichentomum tinctum* Till., showing eyes and palpi; specimen No. 3324ab, Mus. Comp. Zoology. Actual length of head, 1 mm. (The dark patches anterior to the eyes are crystals of gypsum.)

FIGURE 3. Photograph of fore tarsi and distal part of tibia of *Dichentomum tinctum* Till.; specimen No. 3348ab, Mus. Comp. Zoology. Actual length of tarsi, 1 mm.

FIGURE 4. Photograph of fore wing of *Delopterum minutum* Sell., specimen No. 5385ab, Peabody Museum, showing the forked Cu1 (× 20). Photograph by Professor C. O. Dunbar.



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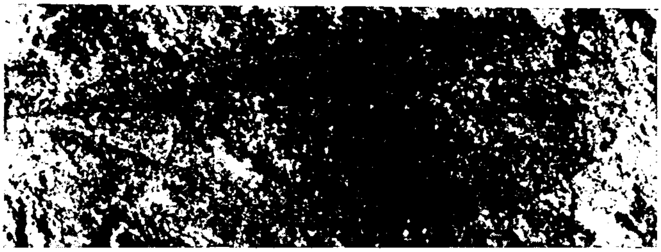

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METHODS AND PROCEDURES USED IN THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY PROGRAM OF INVESTIGATION OF THE PRESSURES AND VOLUMES OF WATER TO 460° C.

PART I.

By Frederick G. Keyes.

(Contribution from the Research Laboratory of Physical Chemistry,
Massachusetts Institute of Technology, No. 299.)

METHODS AND PROCEDURES USED IN THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY PROGRAM OF INVESTIGATION OF THE PRESSURES AND VOLUMES OF WATER TO 460° C.

PART I.

By Frederick G. Keyes.

(Read by title October 11, 1933.)

(1) Introduction.

Somewhat over a decade ago the extremes of the ranges in pressure and temperature of the experimentally determined properties of steam,[1] excepting the saturation pressure-temperature relation, did not exceed eleven atmospheres and 40° C of superheat. The urgent need of a more extended and highly exact knowledge of the properties of water, required for steam boiler and turbine design, resulted at the instance of Mr. George A. Orrok in the bringing together in June 1921 of a group of engineers and scientific men for the discussion of a comprehensive program of research. The group of fourteen, meeting at Harvard University, included important users of steam, several compilers of steam tables, a number of experimental experts and the chairmen of the three A. S. M. E. committees.[2] As a result of the extended discussion and exchange of views a program of investigation was planned to be carried out in three institutions. The U. S. Bureau of Standards was to undertake the measurement of quantities such as the heat function $H = U + pV$ for the liquid phase, and heats of evaporation. The pressure, volume, temperature relations of water over as extended a range of the variables as the most modern facilities would permit was to be carried out at the Massachusetts Institute of Technology. The measurement of the Joule-Thomson effect over wide ranges of pressure and temperature was to take place at the Harvard Engineering School. The final results of the investigations conducted under these three main divisions of the program were to supply a variety of data which, through the use of the two empirical principles of thermodynamics, could be interrelated and assurance provided that the consistency of the entire mass of experimental findings was satisfactory.

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[1] Amagat, Ann. de Chim. et de Phys. *29*, 545 (1893) had measured the volumes of liquid water over a limited range of temperature to 3000 atms. and a more extended temperature range to 1000 atms. The data had never been used for incorporation in steam tables, however, before Joseph H. Keenan's formulation of the "Thermal Properties of Compressed Liquid Water," Mech. Eng. *53*, 127 (1931).

[2] Mech. Eng. *43*, 557 (1921).

Reports of experimental findings have been presented in the Steam Tables Conferences at the yearly December meetings in New York City of the A. S. M. E. and the reports published in Mechanical Engineering.[3] These reports naturally constitute records of progress, provisional correlations, discussions of difficulties encountered and suggestions bearing on the means by which obstacles encountered were to be surmounted. The annual meetings have been in fact a sort of "clearing house" for the accumulating facts and information bearing on steam properties and have proved a most valuable feature of the comprehensive steam research program.

Interest in a more extended knowledge of the thermodynamic properties of water has also prompted investigations abroad, notably by H. L. Callendar, G. S. Callendar, and A. C. Edgerton in England, Jacob and Fritz, Knoblauch, Koch and others in Germany, and Havlīček and Zvonīček in Czechoslovakia. Committees were organized in these countries and in collaboration with the American Committee on Steam Research an international conference was held in London on July 8, 1929, to discuss the steam investigations. During the five days over which the Conference extended a "skeleton" table of thermodynamic properties was agreed upon. The latter table[4] consists of a set of selected fixed points in the saturated and superheated regions accompanied by an estimate of the reliability of each datum. It was also agreed in the conference that if at each of the points given in the Conference Table any existing table of steam properties lay within the estimate of reliability, then the latter was to be considered an "International Table" of steam properties. It was expected that the international fixed points tabulation would be revised in further conferences and a second international conference[5] was held in Berlin in June, 1930. The third conference is to be held in the United States at a time not yet definitely selected.

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[3] Mech. Eng. *45*, 165 (1923); *46*, 81 (1924); *47*, 103 (1925); *48*, 151 (1926); *49*, 161 (1927); *50*, 152 (1928); *51*, 109 (1929); *52*, 124 (1930); *53*, 132 (1931); *54*, 118 (1932); *55*, 113 (1933).

[4] Mech. Eng. *52*, 120 (1930).

[5] Mech. Eng. *53*, 289 (1931).

(2) PURPOSE OF THE PRESENT PAPER.

The present paper is the first of a series of papers and contains the detailed record of the experimental procedures and methods employed to obtain the data recorded in the succeeding papers. More than ten years of effort have been expended in experimentation and no resource has been neglected to develop apparatus and technique capable of yielding precise data on the pressure, temperature and volume properties of the liquid and vapor phases of water. The pressures have extended to more than 300 atmospheres. The temperature of experiment has been carried to 460°, and volume measurements have been made throughout the pressure and temperature ranges stated, including practically the whole of the saturation region. It is proposed in the papers to follow not only to present the experimental data obtained but to correlate the new data with the older body of water data and the results obtained by our American collaborators in the A. S. M. E. Steam Program,— N. S. Osborne, H. F. Stimson, and E. F. Fioch working at the U. S. Bureau of Standards; and H. N. Davis and R. V. Kleinschmidt who carried out the Joule-Thomson experiments at the Harvard Engineering School. The considerable body of important data obtained abroad is also to be included in the project of correlating all relevant steam data. Finally, certain new direct experimental determinations of $\left(\frac{\partial H}{\partial p}\right)_T$ at low pressures and temperatures are in progress and it is expected that the data will be available for consideration in completing our knowledge of the properties of steam in a region where reliable experimental results are obtained with difficulty by the usual methods.[6]

(3) EXPERIMENTAL BACKGROUND FOR THE WORK.

The project of measuring the pressures and volumes of liquid water and water vapor or steam was by no means a new project at the Research Laboratory of Physical Chemistry. William D. Coolidge[7] in 1903 had completed certain important investigations on the conductivity of aqueous solutions to 306° C. In the course of interpreting and correlating the data Coolidge had found the need of exact volume data for water at higher temperatures. The apparatus for new and extended measurements of volumes and pressures was accordingly designed and constructed. Actual measurements were carried out by R. D. Mailey, in 1910, using a Bourdon gauge for the pressure measurements. At about this time I undertook to determine the physical properties of Ammonia and adopted the Amagat "dead weight" or piston type of pressure gauge which has evolved under gradual development into the form which for purposes of distinction may be designated the Research Laboratory of Physical Chemistry type of absolute gauge.[8] Mr. Mailey redesigned the volume measuring portion of his apparatus and, using the piston gauge but retaining the platinum lined container previously used, made a series of measurements on water in 1912. These measurements were of a preliminary character and have never been published. The data and experience gained, however, have been of the greatest value and constitute a part of the background of the present methods and procedures herein described.

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[6] F. G. Keyes and S. C. Collins, Proc. Nat. Acad. Sci. *18*, 328 (1932).

[7] A. A. Noyes and W. D. Coolidge, Proc. Am. Acad. Arts and Sci. *39*, 163 (1903).

The measurements of the vapor pressure of water by Holborn and Baumann[9] as well as R. D. Mailey's work had indicated that water in contact with steel may under certain conditions give rise to permanent gas evolution at temperatures above 300° C. Such interaction of container material and water is a serious obstacle to accurate measurements of the pressure of water. Of course it has long been known that at a red heat iron becomes oxidized by steam yielding iron oxide and hydrogen, and Coolidge selected platinum as the container. A sphere of platinum from a single sheet was spun over a hard wood mould and the wood subsequently burned out. The sphere was then placed in a steel retaining shell consisting of two hemispheres so designed that they just enclosed the platinum hemispheres when screwed together. The neck of the platinum shell was spun back perpendicular to its axis and used as the seat for a gold gasket closure. Copper capillary tubing was used to connect the platinum sphere to the volume measuring device. The sphere was loaded with water by placing the end of the copper capillary below the surface of a quantity of boiling distilled water held in a glass flask and applying heat to the platinum lined container. Water was repeatedly driven into the container by heating and cooling the latter until it was judged to be loaded with air free water.

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[8] F. G. Keyes and Jane Dewey, Jour. Optical Soc. Am. *14*, 491 (1927). J. A. Beattie and W. L. Edel, Ann. Physik, *11*, 633 (1931). J. A. Beattie and O. C. Bridgeman, Ann. physik, *12*, 827 (1932).

[9] Holborn and Baumann, Ann Physik, *31*, 945 (1910).

The defect of a platinum lined or compound container is that its volume is a complicated and apparently erratic function of the pressure and temperature. The difficulty of obtaining a "perfect" fit of the platinum lining to the interior of the steel retaining shell is in fact quite serious. The coefficient of thermal expansion of the platinum, moreover, is smaller than that of steel so that a fit at one temperature could not insure a fit at another temperature. By means of pressure applied within the container the platinum lining can be forced into contact with the steel container but the inherent elastic properties of platinum requires that the relation between the volume of the container and the pressure involve a sort of discontinuity occurring at the point where the platinum shell yields and settles exactly into the steel retaining sphere. The pressure-volume properties of the container space can be made fairly reproducible at constant temperature since the elastic yielding of the lining will come to a fairly definite value as the application and release of pressure is repeated. On considerable change of temperature however the latter condition is upset for the platinum becomes annealed if the change is to higher temperatures and thus it is perceived that for measurements aiming at the highest precision the compound container is inferior to a container constructed of a single homogeneous piece of metal.

It is not difficult to predict what metals are most suitable from the point of view of chemical inertness toward water. The *equilibrium* ratio of the pressure of hydrogen to the pressure of steam at the temperature of measurement can in fact be readily computed. Practically it will suffice if this ratio is sufficiently small at the highest temperature of the steam measurements. To compute the ratio there is required the decomposition or equilibrium pressure of oxygen over the oxide of the container material at the temperature of interest and the equilibrium values for the decomposition of steam into its elements, hydrogen and oxygen. For silver interacting with water the oxide of silver is known to be Ag_2O and $2\ Ag + H_2O = Ag_2O + H_2$. The latter is however equivalent to the two equilibria:

$$2Ag + \tfrac{1}{2}O_2 = Ag_2O \tag{a}$$

$$H_2O = H_2 + \tfrac{1}{2}O_2 \tag{b}$$

The equilibrium values for reaction (a) and (b) are known and have been formulated.[10] The ratio pH_2/pH_2O is 7.0×10^{-28} at 150° C and 8.5×10^{-16} at 500° C. These values are small enough to qualify silver as a satisfactory metal from the chemical viewpoint. However, silver is a metal with unsatisfactory elastic properties in the temperature and pressure range over which it was desired to make volume measurements unless a compound or silver lined container was adopted. The early experience available with a platinum container indicated that the compound bomb should be used only as a last resource.

The equilibrium ratio of hydrogen to steam in the case of iron is not as simple as for silver due to the existence of several iron oxides.[11, 12] However for the oxide Fe_3O_4 equilibrium pressures are known and the ratio is computed to be $\frac{pH_2}{pH_2O} = 3.4$ at 500° C. This value and in fact similar values of the ratio for any metal gives the *equilibrium* value. In practice the equilibrium value may not be realized due to the effect of "passive resistance," the entire nature of which is still obscure. The fact is well known that smaller and apparently stationary values of the ratio are likely to be observed, particularly at lower temperatures.

One circumstance which has been frequently observed is that after an initial layer of oxide has formed on the surface of a metal the reaction to form more oxide proceeds much more slowly and may even come to an apparent stop. It must not be overlooked, moreover, that many or rather most metals absorb appreciable quantities of hydrogen and if this is the case the rate of formation of oxide may be comparable with the rate of absorption of hydrogen by the metal under the actual conditions of use of the container. In this event no permanent gas would be present but water would progressively disappear from the container, the oxygen of which it is composed combining with the metal to form oxide and the hydrogen diffusing into the metal.

Something should be stated regarding the preferred mechanical properties of the container material. A pure metal or as nearly pure as can be obtained commercially is desirable because pure metals are generally supposed to possess more regular and reproducible elastic and thermal expansivity characteristics; at least more nearly independent of previous history and treatment, than are most alloys. It is clear, in any event, that it is desirable to use as a container material a substance whose elastic properties have been thoroughly investigated. Unfortunately, elastic behavior has been investigated with any completeness for only a few metals. In the steam investigation these properties were desired over the range from room temperature to 500° C and it was found that the values of Young's modulus[13] and Poisson's ratio[14] are relatively fairly well known for nickel. Because of this and from such indications as could be adduced regarding its actual chemical behavior in the presence of steam it was decided to begin the measurements with a container of this substance. The dissociation pressure of oxygen over nickel oxide NiO is known[15] and the equilibrium ratio of hydrogen to steam is $\frac{pH_2}{pH_2O} = 4.2 \times 10^{-3}$ at 600° and 3.0×10^{-3} at 485°. The fact that nickel does not amalgamate with mercury at ordinary temperatures was also a point favoring the use of nickel.

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[10] Lewis and Randall "Thermodynamics" McGraw-Hill Book Co. 1923.

[11] Eastmen, J. Am. Chem. Soc. *44*, 975 (1922); Eastman and Evans, J. Am. Chem. Soc. *46*, 888 (1924).

[12] Lewis and Randall, loc. cit.

(4) THE STEAM CONTAINERS EMPLOYED.

Young's modulus for nickel in its relation to temperature is shown graphically in Fig. 1, and it is observed that above 425° C the change of the modulus with the temperature is quite abrupt. In fact some physical or quasi-chemical change in the nickel would seem to be indicated, the exact nature of which is still obscure. From the practical point of view it would of course be unwise to use a nickel container above 425° C and measurements were not extended beyond 400° C. The temperature dependance data for Young's modulus may be represented with sufficient fidelity by a linear function (except nickel above 200°) for most metals of the kind that would be considered for use as pressure vessels. Unfortunately however the data available are not impressive either as to quality, quantity, or temperature range. The modulus does however, and the rigidity likewise, diminish with temperature rise and the temperature coefficient of the latter is larger than the former. Thus for nickel below 200°,[16] the Young modulus decreases by 2.47 percent per 100° and the rigidity by 3.28 percent per

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[13] E. P. Harrison, Proc. Phys. Soc. London, *27*, 8 (1914).

[14] A. Bock, Wied. Ann., *52*, 609 (1894).

[15] Pease and Cook, J. Am. Chem. Soc., *48*, 1199 (1926).

[16] Schaefer, Ann. Physik, *5*, 220 (1901); *9*, 1124 (1902).

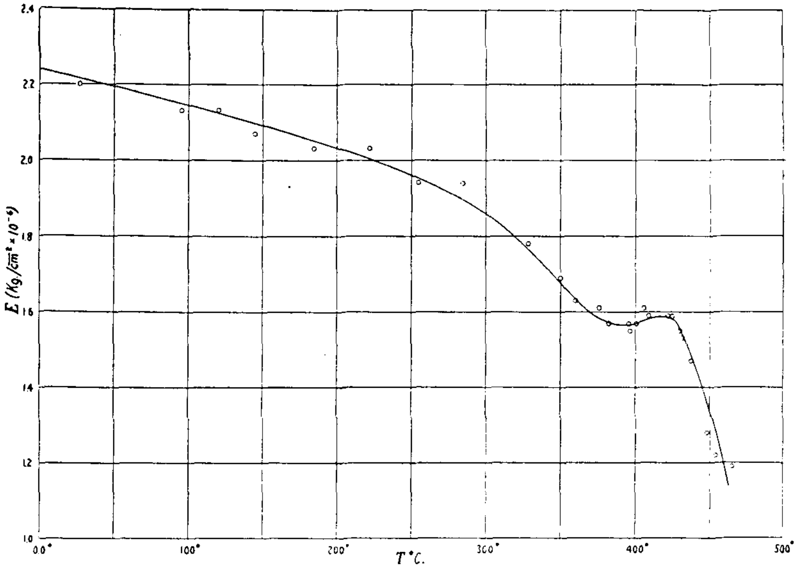

FIG. 1.

100°. From the classical elasticity theory we have, for the relation between Poisson's ratio (σ), Young's modulus (E) and the rigidity (μ):

$$1 + \sigma = \frac{E}{2\mu} = \frac{E_0(1 - \alpha t)}{2\mu_0(1 - \beta t)}$$

or

$$\sigma = \sigma_0 + (1 + \sigma_0)\left[(\beta - \alpha)t + \beta t^2(\beta - \alpha)\frac{1}{1 - \beta t} \right]$$

It is clear from the figures of Schaefer for nickel that part of the second term of the right hand member of the last equation may be neglected since $\beta\ (\beta - \alpha)$ is small relative to $(\beta - \alpha)$. This is the case with all metals so far as I can determine and the dependance of Poisson's ratio on temperature for nickel becomes on the basis of Schaefer's data:

$$\sigma = \sigma_0(1 + 8.1 \times 10^{-5}t) + 8.1 \times 10^{-5}t$$

It will be observed that the (0 — 200) data of Harrison for Young's modulus indicate a much higher (54 percent) temperature coefficient than that of Schaefer. The latter, however, measured both coefficients in the same investigations and the results may be relatively quite exact in which case the temperature dependance deduced for σ would be reliable. In any event the indication is that the variation of σ is smaller than the Young modulus and rigidity modulus, and always positive when the temperature coefficient of the latter exceeds the former. In the case of nickel, $\sigma = 0.33$ was employed in formula 1 and it is a property of the latter equation that it is quite insensitive to variation in σ. The extreme upper value of σ is 0.5 at the melting point and for temperatures of use far lower, as with nickel or steel in the present investigations, the assumption that σ is independent of temperature is often justified.

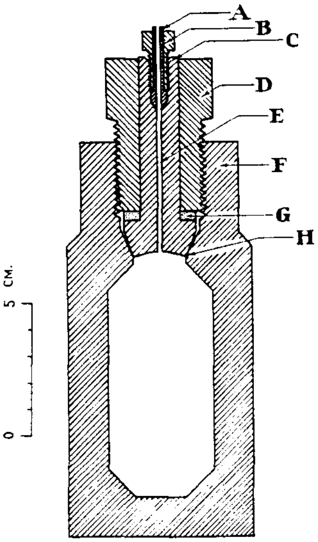

FIG. 2.

The form of the first or nickel container is shown in Fig. 2. The material, a special forging, was obtained from the International Nickel Company[17] and was the purest nickel obtainable commercially.

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[17] We take pleasure in acknowledging our indebtedness to Mr. H. T. Cole and Mr. H. G. Fales of the International Nickel Company for their interest and helpfulness not only in supplying specially forged pieces of nickel and monel metal but for much special information which proved of the greatest value.

The volume of the completed container was about 220 cc. Closure was effected by a device much used in this laboratory, that of using a cone closure piece *C*, Fig. 2, of nickel of 1½° less angle than the 45° angle cone of the seat machined at the bottom of the threaded neck, *F*. Contact is made at *H* and a bronze slip-ring lubricated with talc used at *G*. The member *D* drives the cone to a pressure tight seat and is of nickel. To connect the container with the measuring apparatus a nickel capillary[18] 0.62 mm. in bore and 2.25 mm. in outside diameter was employed.

The method of connecting with the container was to weld the capillary to the *Ni* cone *A* at the tip. The resulting slight irregularity produced at the tip of the cone was removed in the lathe and the end drilled to correspond to the bore of the capillary. The cone *A* was seated by means of the threaded driving piece *B*.

The constant temperature bath, to be described later, contained as bath fluid a molten mixture of sodium, potassium and lithium nitrates. Unless the threads of the container are properly lubricated it will be found impossible to disassemble the container after it has once been heated in the molten salt bath. The use of graphite is ineffective because it is completely oxidized during the long course of the heating required during measurements. The lubricant which has been found satisfactory is pure talc in finely powdered form mixed with heavy paraffin oil to a glycerine-like consistency. The mixture is brushed on the threads and the container locked in the usual manner. To open the container after long heating in the bath it is necessary to submerge the container in warm water to soften and dissolve the solidified salt which will be found to have penetrated into the threaded portion and oxidized the paraffin oil. The container is next warmed and oil allowed to enter the threads. After a short time it will be found that the threads may be "started" after which further warming and application of oil, if necessary, may be desirable. By proceeding with reasonable caution, avoiding hurry or forcing, it is possible to open and close the various types of containers repeatedly without harming the threads in the slighest degree. This statement is true not only for the case where the container is of nickel but where it is composed of ordinary low carbon or alloy steel. Cold worked steel, it may be noted, should always be annealed to remove strains before machining.

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[18] The nickel capillary we owe to the kindness of Mr. Philip Dorticos of the Edison Lamp Works at Harrison, New Jersey.

A point of no small importance in connection with a container formed from a solid ingot should be stressed. Experience over a period of some twenty years has shown that usually a bar of forged material has one or more fine capillary pores running parallel with the axis for considerable distances formed in the course of the elongation from small pockets or slag inclusions in the interior of the ingot during the forging or rolling operation. The first attempt to obtain a satisfactory nickel container for the steam work resulted in failure due to this kind of a defect. It would be most satisfactory of course to have the forging made from nickel which had been cast *in vacuo* but at the present time this does not appear to be commercially practicable. The successful nickel container of the form of Fig. 2 was finally made from a section of a long piece which had been forged a second time in a direction perpendicular to the original axis. Even this container, after several years use, developed a minute leak through the wall perpendicular to the axis.[19] This appeared however to be due to the progressive crystallization of the nickel to large grain size with attendant weakening of the container. At all events careful examination disclosed to Doctor Smith a fine crack running around the container for some distance. Independent experience has convinced the writer that large crystal grains do form in nickel on prolonged heating at high temperatures. The same phenomenon occurs in the case of tungsten.

The essentially cylindrical form of the nickel container, Fig. 2, was chosen because this form lends itself to simple treatment in determining the increase of the internal volume with pressure. The external form was actually hexagonal except for the neck *F*. The area of the hexagonal section was used to compute the equivalent radius of a circle of equal section and this radius was used in computing the dilation of the container volume with pressure by means of the formula:

$$\Delta V = V\left[\frac{3(1-2\sigma)}{E}\,\frac{p_i r_i^2 - p_0 r_0^2}{r_0^2 - r_i^2} + \frac{2(1+\sigma)}{E}\,\frac{(p_i - p_0) r_0^2}{r_0^2 - r_i^2}\right] \qquad (1)$$

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[19] Many years ago during the course of the commercial production of glass mercury arc rectifiers and mercury arc lamps, "epidemics" of leaky units would appear. The difficulty was finally traced to the same cause as has been described in the case of the forged billets used for the steam containers. The platinum wire then used for seals was drawn from ingots which had air bubble inclusions. The remedy in this case was to cut the platinum lead in two and weld the portions together in the oxy-hydrogen flame before using. This is still good practice in the laboratory where platinum wire is used for leads.

where ΔV is the volume displacement when the internal pressure is p_i and the outside pressure is p_0, r_i is the internal radius, E is Young's modulus, and σ Poisson's ratio. The contribution to the total increase in volume due to the ends of the cylinder was approximated by considering the ends to be equivalent hemispheres. The values of Young's modulus and Poisson's ratio used for the nickel container will be found in Table I.

The volume increase with pressure was also determined experimentally by evacuating the container and filling it completely with pure mercury. The apparent compressibility of the mercury was thus determined and from a knowledge of the true compressibility of mercury the volume increase of the container with pressure was calculated. The agreement of the volume increase computed and found experimentally is seen (Table I) to be in fair agreement. It was necessary, however, in the course of the measurements to go to termperatures where, owing to the lack of reliable compressibility data for mercury at high temperature, it was impractical to use mercury to determine experimentally the volume increase with pressure. The procedure adopted was therefore to compute the volume increase by the elasticity theory as described at all temperatures but the difference between the experimentally determined value of the volume increase and that computed at ordinary temperatures was applied throughout the temperature range of experiment. This amounts quite approximately to using the experimental value at ordinary temperatures and computing the increase of the quantity with temperature by elasticity theory using the elastic constants appropriate to the higher temperatures. The error introduced by entirely neglecting the variation of volume increase with pressure for No. 1 Ni container amounted as a matter of fact to 0.1% at the higher pressure range of measurements, 300 atmospheres.

The temperature expansivity of the container is a quantity which should be known with certainty. Reliance for the expansivity of nickel was placed upon the measurements reported in the Bureau of Standards Scientific Paper No. 426. The results were used by Doctor L. B. Smith to compute the constants of the empirical formula $l = l_0 + a_1t + a_2t^2 + a_3t^3$. The mean value of the quantity measured between t_1 and t_2 is given by

$$\frac{1}{t_1 - t_2}\int_{t_1}^{t_2}\left(\frac{\partial l}{\partial t}\right)dt = l = l_0 + \frac{a_1}{2}(t_1 + t_2) + \frac{a_2}{3}\,\frac{t_1^3 - t_2^3}{t_1 - t_2} + \frac{a_3}{4}\,\frac{t_1^4 - t_2^4}{t_1 - t_2}$$

It will be perceived by the use of the formula of Table II that the effect of an uncertainty of one percent in the expansion coefficient introduces an error of roughly 0.018 percent or one part in 5500 at 400°, in the determination of the volume of the container contents in passing from 0° to 400° C.

The interior surface of the container to a considerable depth is under high tension at the maximum pressures and since the coefficient of temperature expansion determined at one atmosphere was employed an error is involved in assuming that the latter applies under the pressures of the actual steam measurements. The temperature expansivity of liquid mercury is known[20] to diminish under pressure and undoubtedly the expansivity of nickel is slightly greater under the tensions which existed in our measurements. The effects appeared too complicated and available data too meagre, however, to attempt an evaluation of their magnitude and the one atmosphere expansivity was applied throughout in reducing the measurements.[21]

TABLE I.

Constants for Nickel Container No. 1.

Volume at 0°	219.642 ± 0.012
$r_0 = 4.228$ cm.	$r_i = 2.614$ cm.
Poisson's ratio	0.33

t° C.	Young's Modulus kg./cm.^{2} × 10^{-6}	Formula 1 ΔV	Stretch cm.^{3} per 100 atm. From Comp. of mercury.
30	2.183	.0526	.0535
100	2.138	.0541	.0559
200	2.082	.0568	.0541
250	1.984	.0585	Above 200° nickel amalgamated with the mercury.
300	1.884	.0616	
350	1.680	.0691	
400	1.600	.0725	
450	1.400	.0829	

The compressibility of mercury used in deducing the "stretch" of the container was:[22]

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[20] P. W. Bridgman, Proc. Amer. Acad. Arts and Sci. *47*, 345 (1911).

[21] K. T. Compton and D. B. Webster, Phys. Rev. *15*, 159 (1915), showed that the temperature dilation of piano wire increased 0.54 per cent when the load on a 0.0310 cm. wire was increased from 4.5 kg. to 9.5 kg.

[22] Carnazzi, Cim. (5) *5*, 180 (1903).

$\beta \times 10^6$	22.8°	52.8°	84.8°	110°	150.3°	191.8°
1 – 500 atm.	3.8	3.9	4.0	4.1	4.4	4.6

The measurement of the total volume of the containers used in the steam experimentation was carried out by evacuating the containers and filling with mercury and with water at the temperature of melting ice. The usual and well known precautions in determining volumes by this method were of course adopted. For weighing the containers a large Gurley balance capable of taking a load of 35 kilos (at this load sensitive to 5 milligrams) was employed. In Table II there is tabulated the volumes of the containers used. It is believed that the volumes are reliable to about one in ten thousand relative to the accepted densities of mercury and water. The values of the elastic constants and thermal expansion are also given in the table.

A detail of some importance is worth mentioning in connection with the calibration of the containers using mercury. The normal vapor pressure of mercury at 360° is one atmosphere and evidently no trace of mercury must be allowed to remain in the containers during measurements of steam pressures. On the other hand if the closure, piece *E*, for example, of the nickel container was removed for removing traces of mercury, it would be difficult or impossible to reestablish the closure without disturbing the volume of the container. Of course the determination of volumes could be carried out after having made a series of steam measurements, but in the event of an accident to the container in the course of a "run" it might be found impossible to reduce the measurements taken prior to the accident. As a matter of fact experience showed that the mercury could be run out very completely leaving only a matter of milligrams of mercury behind. The procedure of passing dry nitrogen into the warmed container by means of a fine capillary passed through the closure piece *E* proved to be satisfactory as a means of evaporating the mercury.

The container of nickel, Fig. 1, was replaced early in 1926 by a pair of containers designed especially for volumes of steam greater than correspond to 20 cc. per gram. At about this time the development, production and knowledge of the physical characteristics of certain alloy steels had progressed to a point where tests of the resistance of the alloys to slight chemical action by water warranted investigation. The nickel container had not on first use proven as satisfactory with respect to its resistance to attack as had been hoped, although after an initial oxide skin had formed, the behavior of this container appeared to be satisfactory. It was found, however, in the case of machinery steel that a slow and very regular apparent loss of water from the container took place and this was definitely traced to the formation of an oxide film which reduced the rate of decomposition of the water to the rate at which the hydrogen was absorbed by the steel. The observed properties of the steam were then quite approximately a regular and calculable function of time and allowance for the water loss was made. Needless to state, however, many hours of time consumed in making measurements were lost because of the reaction of the water with the container.

While the oxide film in the interior of the No. 1 nickel container, was sufficient to retard water decomposition it should be stated that the mechanical properties of pure nickel are far from satisfactory for pressure work. Upon long heating at high temperatures the metal tends to crystallize and it flows or "creeps" excessively under stress. This is not serious as far as the constancy of the working space of the container is concerned at the upper pressures of the steam investigation, but the threads at the closure tend to yield and the locking nut requires taking up frequently.

The choice of the spherical form for the second container was dictated partly by the fact that the dilation by pressure is small and easily computed by the formula:

$$\Delta V = V \frac{3(1 - 2\sigma)}{E} \frac{p_i r_i^3 - p_0 r_0^3}{r_0^3 - r_i^3} \tag{2}$$

The method of construction will be clear from Fig. 3 which shows two hemispheres C and E machined to fit together, D, leaving a V shaped groove to facilitate the welding together of the two portions. The piece at F was cut off subsequently and used to determine the temperature dilation of the alloy. The method of closure is the same in principle as described, A and B Fig. 2, for the case of the nickel container. The possession of duplicate containers made it possible to save much time since one sphere could be prepared and loaded with a water sample while the second was in use for making measurements.

The thoroughly annealed machinery steel used for the second container was not satisfactory from the point of view of resistance to chemical reaction. At higher temperatures (200°) the hydrogen attained a pressure of 0.2 atms. It was found, however, on subsequent examination that in exhausting the No. 2 container at 350° preparatory to loading with water the protective oxide film which had been purposely formed prior to use as a protective coating had been entirely removed or decomposed during the course of the heating and pumping. This was somewhat astonishing since preliminary calculation had seemed to indicate that the vapor pressure of the oxygen over the oxide was low enough at 360° to insure a sufficiently low rate of oxide decomposition. However the spongy surface of iron resulting from this decomposition proved, as might in fact be anticipated, to be extremely reactive and enough hydrogen was collected to make identification positive. By lowering the temperature at which the exhaustion of the container was carried out it was found possible to preserve the oxide film intact and hydrogen evolution during steam measurements ceased. Contrary to the experience with the nickel container, however, it was found that decomposition of water had not really stopped entirely, but that the hydrogen was absorbed by the steel as rapidly as generated. By keeping careful account of the time, it was nevertheless possible to apply corrections for the regular loss of water due to gradual oxide formation.

By this time (1930) so-called Nirosta steel (18 percent Chromium, 8 percent Nickel) in large homogeneous ingots became available and after some initial difficulties incident to inexperience in machining the alloy, a pair of containers similar in form and content to the No 2 machinery steel containers was completed. The welding of the hemispheres proved impossible using oxygen-acetylene, but thanks to the General Electric Company their atomic-hydrogen welding process proved entirely successful and two very perfect containers became available. The coefficient of expansion with temperature of the alloy as determined on samples cut from the *F* portion (Fig. 3) are given in Table II. With Nirosta steel the range of temperature in the measurements was extended to 460° C.

During the course of the time the Nirosta containers were under construction a chrome-vanadium steel container similar in every way to the nickel container was constructed, Container No. 1A. This was used to make repeat measurements on liquid water compressibilities up to the critical region, 374° C. Finally a third cylindrical type of container of Nirosta steel, No. 1B, was constructed and used for repeat measurements on water at high densities.

By way of résumé, there were three forms of container used in the steam investigation, the cylindrical No. 1 (Fig. 1) of nickel, No. 1A of chrome-vanadium alloy steel (about 216.7 cc. capacity), and No. 1B of Nirosta steel (about 286.2 cc. capacity); No. 2A, spherical type of

Nirosta alloy steel similar to No. 2 machinery steel container. Finally a small Nirosta steel cylinder was used in vapor pressure measurements. The cylindrical form was used for the measurement of volumes not greater than corresponds to a specific volume of 20 cc. per gram. The spherical type, roughly 550 cc. total volume, was used for the range 20 cc. per gram to 150 cc. per gram. The latter is the saturation volume of steam at about 191.9° C, a little higher temperature than the limit of the explored region prior to 1920. The Nirosta or 18 percent chromium—8 percent nickel alloy, in use for many months continuously at temperatures as high as 460° C, showed very slight or practically no signs of chemical interaction with water. The coefficient of expansion is considerably greater than either nickel or machinery steel, but on the other hand the expansivity is regular and reproducible. Table II summarizes the data pertaining to the containers.

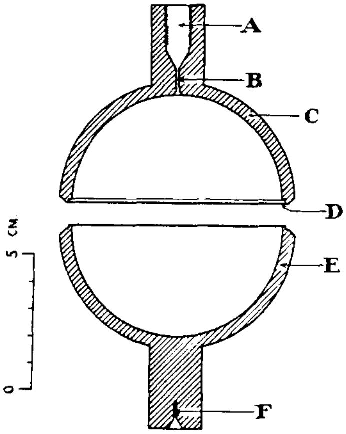

FIG. 3.

The heat treatment employed with the chrome-vanadium steel was not extended beyond the point where it could be readily machined. In the case of the Nirosta steel it was heated to about 1200° C and quenched thereby leaving it in a condition for machining without too great difficulty.

The capillary connecting tube between the container and the volume measuring portion of the steam in the earlier series of measurements was of nickel. The nickel capillary proved, however, to be

TABLE II.

DATA PERTAINING TO CONTAINERS USED.

Container	$E \times 10^{-6} \frac{\text{Kg}}{\text{cm.}^2}$	σ	V_0(Hg)	V_0(H_2O)	At 200° Stretch 100 atm. Eq. 1 or 2
No. 1 Nickel Cyl.	Table 1	0.33	219.642 ± 0.012		.0568
No. 1A Chrome-Vanadium Cyl.	2.17	0.33		216.661 ± 0.005	.0469
No. 1B Nirosta 18/8 Cyl.	2.00	0.28		286.165 ± 0.013	.0649
No. 2 Mach. St. Sphere	2.17	0.30	546.708 ± 0.006		.0966
No. 2′ Mach. St. Sphere	2.17	0.30	550.028 ± 0.008		.0966
No. 2B Nirosta 18/8 Sphere	2.00	0.28	542.881 ± 0.006		.0857
No. 2B′ Nirosta 18/8 Sphere	2.00	0.28	539.023 ± 0.004		.0857
No. 3B Nirosta small Cyl.				7–8 cc.	

V_t (Nickel $= V_0\,(1.0000 + 3.95 \times 10^{-6}t + 1.252 \times 10^{-8}t^2)$.[23]

V_t (Mach. steel) $= V_0\,(1.0000 + 3.342 \times 10^{-6}t + 3.123 \times 10^{-8}t^2 - 2.109 \times 10^{-11}t^3)$.[24]

V_t (Chrome-vanadium $= V_0\,(1.0000 + 3.666 \times 10^{-6}t + 1.338 \times 10^{-8}t^2 - 1.215 \times 10^{-11}t^3)$.[24]

V_t (Nirosta 18/8) $= V_0\,(1.0000 + 4.815 \times 10^{-6}t + 1.818 \times 10^{-8}t^2 - 1.335 \times 10^{-11}t^3)$.[24]

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[23] Dr. Leighton B. Smith obtained this equation from a formulation of the nickel data contained in Paper No. 426, U. S. Bur. of Standards, 1922.

[24] The expansion coefficients were measured by Doctor Smith using the Pulfrich interferometer apparatus operating on specimens cut from the container and the data formulated by the least square procedure.

Compositions of Steels. The nickel was analyzed by Professor L. F. Hamilton who found as follows: cobalt 0.46, iron 0.31, carbon 0.13, copper 0.12, manganese 0.081, sulfur 0.009, silicon 0.098. The Nirosta steel contained chromium 18.0, nickel 8.00, carbon 0.15. The Chrome-vanadium steel contains carbon 0.34, manganese 0.73, phosphorus and sulphur each 0.019, chromium 0.92, vanadium 0.21.

subject to longitudinal imperfections resulting in such a large number of failures that mild seamless steel was substituted until finally Nirosta capillary tubing was produced by the Summerill Tube Company of Bridgeport, Pa. The steel capillary proved to be troublesome for the bore 0.6 mm. was so small that in time it became stopped due to gradual oxidation of the inner walls at the portion submerged in the high temperature bath. The Nirosta capillary on the other hand was highly satisfactory in every way.

(5) THE THERMOMETERS.

The scale of temperature on which it would be desirable to report the steam data is of course the absolute, or thermodynamic scale (Kelvin Scale). Unfortunately the relation of the latter scale to the indications of the several practical or secondary temperature measuring devices is not yet known to the degree of exactness that is commensurate with the reproducibility and sensitiveness of the platinum resistance thermometer, 0.001° C. It is, in fact, not difficult now for investigators working in different laboratories to construct platinum resistance thermometers which will indicate, for example, the normal boiling temperature of the same sample of benzophenone (employing of course perfected designs of boiling point apparatus) in agreement within at least 0.01° or 0.02° C.

The perfection in the construction and use of platinum resistance thermometers sensitive to 0.001° C has proceeded parallel with the improvements in the construction and mode of operation of the apparatus used for the fixed points until it is relatively easy to maintain and reproduce the temperature of boiling water and sulfur to about ± 0.001° C and ± 0.013 respectively. On the other hand the reproducibility of the normal sulfur boiling temperature on the hydrogen or nitrogen gas scale as determined by various independent investigators shows a lack of accord which amounts in some instances to tenths of a degree.

At present the standard practice is to determine the electrical resistance of the platinum-resistance thermometer at three temperatures, namely: the equilibrium temperature of water, saturated with air, and ice at one atmosphere; the international-atmosphere boiling point of water; and the normal boiling point of sulfur. These three temperatures are by international convention and agreement of 1927[25] assigned the values 0° C, 100° C, and 444.6° C and employed to compute the value of δ in the Callendar interpolation formula:

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[25] Burgess, U. S. Bur. of Stand. Jour. of Research, Re. Paper No. 22, p. 635, Oct. 1928.

$$t - (pt) = \delta\left(\frac{t}{100} - 1\right)\frac{t}{100} \tag{3}$$

where t is the absolute or thermodynamic temperature on the centigrade scale and (pt) the "temperature" defined in terms of the resistance of the platinum thermometer by the formula:

$$(pt) = \frac{100(R - R_0)}{R_{100} - R_0}$$

R, R_0 and R_{100} representing the resistance at temperature t, the ice freezing point and normal boiling point of water. The Callendar formula is equivalent to the assumption that the relation between thermodynamic temperature and the electrical resistance of pure platinum is given by the quadratic expression:

$$R = R_0(1 + At + Bt^2)$$

It is to be emphasized however, that no sufficient knowledge is yet available for establishing the validity of the quadratic relationship since in practice t is not known on the thermodynamic scale.

Clearly, two major difficulties arise when the relation between this particular scale of the platinum resistance thermometer and the true thermodynamic scale is sought. The first difficulty is the defect in precision with which the normal sulfur boiling temperature is known, admittedly uncertain by perhaps as much as 0.1° C. The second has to do with the sufficiency of the empirical formula (3) as an interpolation formula. Little can be stated to inspire complete confidence[26] in the use of formula (3) for it is well known that it fails below 0° C where gas thermometry has reached a degree of refinement surpassing that attained above zero.

The scale of temperature uniformly adopted for the steam investigations is that of the platinum-resistance thermometer constructed of pure platinum having a mean temperature coefficient between 0° and 100° greater than 0.00391. The Callendar formula (3) is used for interpolation and the value of δ determined in accordance with the afore-mentioned 1927 International scale specifications.

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[26] Burgess, Le Chatlier, "Measurement of High Termperature" John Wiley and Sons, 1912, p. 199.

Because of the unsatisfactory state of knowledge with respect to the relation between the highly perfected platinum-resistance thermometer and the thermodynamic scale of temperature an extensively planned investigation of the subject was undertaken in 1920 in this laboratory. The investigations initiated under the program are beginning to provide very consistant data encouraging the belief that a body of information will become available pertaining to important details bearing on the behavior of the platinum-resistance thermometer and its relation to the several gas scales of temperature. The data required to reduce the gas scale readings to the thermodynamic scale of temperature are also being obtained by three independent methods.

It is hoped that similar gas thermometric investigations may soon be undertaken in other laboratories in order that the present lack of positive knowledge regarding the relation of the platinum resistance scale to the absolute temperature scale may be reduced. This is greatly to be desired for a final statement regarding the consistency of the great body of steam measurements which is nearing completion cannot be written until the present clouds over the temperature scale situation are cleared.

An example of the nature of the difficulty of comparing data from different experimental sources by means of the two empirical principles of thermodynamics may suffice to illustrate the importance of the temperature scale problem. N. S. Osborne, H. F. Stimson and E. F. Fioch[27] have completed extremely precise measurements of the latent heat of evaporation of liquid water from 50° to 270° C. The measurements at the Massachusetts Institute of Technology on the other hand have provided values for the relation between saturation pressure, temperature and specific volumes which may be used to compute latent heats by means of the thermodynamic formula of Clapeyron $L_e = T\frac{dp}{dT}(v_1 - v_2)$ where T is the thermodynamic temperature and p, v_1 and v_2 the pressure and volumes of unit mass of the vapor and liquid determined at T on the thermodynamic scale of temperature. Now to obtain $\frac{dp}{dT}$ the operation of differentiation, or its equivalent, must be carried out on the expression relating the saturation pressure to the thermodynamic temperature. If, however, in place of the true thermodynamic temperature T an approximation to the thermodynamic scale of temperature, say Θ scale is used to relate pressure to "temperature" the quantity $\frac{dp}{dT}$ cannot be evaluated without exact knowledge of the relation between T and Θ. The values of v_1 and v_2 will also correspond to the Θ temperature scale. The Clapeyron equation "latent heat" computed is therefore not a true thermodynamic quantity. Clearly then, lack of certainty of the relation of the temperature scale of the thermometers used in the steam investigation to the thermodynamic scale of termperature leads to serious difficulties in comparing directly measured and thermodynamically derived quantities. The nature of the existing temperature scale situation is of a kind that can only be dealt with by the most refined and painstaking investigations conducted preferably by several groups independently.

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[27] Trans. of Am. Soc. Mech. Eng. *52*, 191 (1930).

One other example of the fundamental importance of an improved knowledge of the relation of the accepted scale of temperature to the thermodynamic scale is of immediate interest. The heat capacity of steam in the vapor or gaseous phase has been determined experimentally by Knoblauch and Koch.[28] The pressure-volume-temperature properties of the same phase have also been measured in the course of the M. I. T. investigation. From the latter data the values of $\left(\frac{\partial^2 v}{\partial T^2}\right)_p$ can be evaluated and if the temperature scale used to relate v to temperature at constant pressure is on the thermodynamic scale of temperature then a comparison of the experimentally determined heat capacity data may be made with similar values computed from the expression $c_p = c_{p_0} - T\int\left(\frac{\partial^2 v}{\partial T^2}\right)_p dp$. In the latter expression c_{p_0} represents the heat capacity for the gas at very low (zero) pressure, a pure temperature function. A cursory survey of the Koch c_p values shows that for important regions of the c_p, T plane the value of the second term of the right hand member of the above equation is several times as large as c_{p_0}. It is therefore clear that an important test of the consistency of the Koch c_p values and the present pressure-volume-temperature data becomes possible only when v can be related to the thermodynamic temperature because the evaluation of a second derivative may be brought into grave error by what may appear to be a slight trend in the relation of volume to temperature in the primary data. In other words the formula

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[28] Knoblauch and Koch, Zeit. d. Ver. Deutch Ing. *72*, 1733 (1928).

$$c_p = c_{p_0} - T \int \left(\frac{\partial^2 v}{\partial T^2} \right)_p dp$$

has precise meaning only when T is true thermodynamic or Kelvin scale temperature.

There were five platinum resistance thermometers used in the M. I. T. steam investigation since experimental work started early in 1922. Owing to the world wide disturbance of industry by the war satisfactory pure platinum wire for the thermometers was practically impossible to obtain for some years subsequent to 1919. Samples obtained from 1919 to 1922 from England, Germany and the United States proved of very mediocre quality but the best of that obtainable had to be employed. Thermometer No. 1 was of the flat calorimetric four-lead type with copper leads. The steel thermometer case was filled with dry hydrogen and sealed. The method of aging or conditioning thermometers employed at this laboratory at the time was to heat them to about 500° C and to cool in liquid air repeatedly until the ice point remained unchanged. The calibration points employed (1922) were ice, boiling water, naphthalene and benzophenone, the values for the two latter fixed points being taken from determinations made by the U. S. Bureau of Standards. These fixed points were not normal boiling temperatures given by gas thermometer measurement. No. 1 thermometer was used in all measurements made prior to June 15th, 1925. On the latter date a leak in the iron case was discovered which had evidently occurred at a time sufficiently in advance of June 15th to admit air and oxidize one of the copper leads. By this time, however, the deficiency in quality of platinum had become increasingly exasperating not alone in the steam investigation, but in several other investigations in progress in the laboratory. The Bureau of Standards had earlier undertaken the problem of platinum purification and a supply sufficient for a number of thermometers was finally secured. The quality of this platinum has proved to be everything that could be desired. Over forty thermometers of different types have been constructed for the various investigations of the laboratory and subjected to careful test and control. The experience gained induces the belief that no better quality of platinum has ever been employed in constructing platinum resistance thermometers.

Thermometer No. 2 B. S. was of the flat type made of the Bureau of Standards platinum and having pure gold leads. Aging and calibration was carried out as in the case of No. 1 thermometer except that the sulfur boiling point (444.6° N. P.) was used in determining the δ of the formula (3). In order to interpret the temperature readings of the No. 1 thermometer in terms of the No. 2 B. S. instrument constructed of the superior Bureau of Standards platinum wire, comparison was made with a duplicate of the No. 1 thermometer constructed and used by Mr. Charles K. Lawrence in his measurement of some properties of amonia.

About January 1926, a leak developed in the case of thermometer No. 2 B. S. admitting water. No. 3 B. S. thermometer was of the strain free type inclosed in a steel sheath. During the aging and calibration process the indications of the thermometer became somewhat erratic and it was discarded.

Two new strain-free thermometers were next constructed, No. 4 B. S and No. 5 B. S., both of the strain-free type with pure gold leads and encased in pyrex glass. Both thermometers were aged and calibrated using sulfur N. B. P. 444.6° [29] as the high temperature fixed point substance. The thermometers were calibrated and used in close-fitting thin-walled nickel protecting sheaths. Each was compared directly and at numerous points with the Lawrence duplicate of thermometer No. 1. The No. 5 thermometer has been held as a reserve and No. 4 used in the course of observations of steam properties. The constants of the thermometers are given in Table III.

TABLE III.

Platinum Resistance Thermometers.

	R_0	R_{100}	$(R_{100} - R_0)/100R_0 \times 10^3$	δ
No. 1	17.0889	23.7164	3.878	1.504
No. 2 B S	16.8063	23.4030	3.925	1.491
No. 3 B S strain-free	20.1310	27.9968	3.907	1.52*
No. 4 B S " "	18.2320	25.3831	3.92228	1.4965
No. 5 B S " "	17.6679	24.5917	3.91886	1.5002

* Contaminated wire.

The resistance of the thermometers was measured with a Mueller[30] bridge manufactured by the Leeds and Northrup Company in 1920. The bridge was built into the laboratory type of bridge thermostat and during more than ten years the bridge coils have not at any time differed in temperature more than a few degrees from 30° C. The coils were calibrated in terms of the 10 ohm coil in January 1922, January 1926, and September 1931 by means of a Carey Foster bridge modified for convenience to facilitate bridge calibrations. The change in the bridge coils with time between the first and second calibrations was 0.04 percent and between the second and third 0.01 percent.

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[29] The form of sulfur bath in use in this laboratory has not been described. The sulfur is contained in pyrex glass and electrically heated. The thermometers are of course shielded.

[30] Mueller, Bureau of Standards Scientific Paper No. 228.

The temperatures reported in the M. I. T. investigation may be regarded as all on the scale of the No. 4 platinum resistance thermometer using for interpolation the Callendar formula 3 and the fixed points and recommendations of 1927 defining the international scale of temperature. The readings of the thermometer No. 1 were corrected as has been described by 10° comparisons between a duplicate designated the Lawrence thermometer, and the No. 4 and No. 5 thermometers.

(6) THE PRESSURE GAUGE.

The most reliable and exact measurements of pressure can be made with the floating piston or "dead-weight" type of gauge devised by Amagat.[31] The type in use in this laboratory[32] has been under development since about 1910. The gauge is represented in Fig. 4, which consists of two parts, one an isometric drawing representing the gauge assembly, the second showing the piston block with the pressure shut-off and method of mounting the piston-cylinder in the block.

The principle upon which the gauge is based is that exemplified in determining the weight, applied on a scale pan suspended from the piston, required to equilibrate the force tending to move the piston out of its cylinder. To annul frictional effects the piston must be floating in oil and the details of the scheme by which true equilibrium is determined between the weights and the pressure exerted by the fluid under measurement, is of vital importance. Evidently the force of gravity at the place where the gauge is used must also be known if comparisons between pressures measured with a piston gauge by different observers in different localities is of importance. The deformation of the piston-cylinder by pressure, by temperature, and due to aging of the material of construction are likewise of essential importance in measurements of precision.

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[31] Meyer, Bur. of Standards Jour. of Research, *6*, 1061 (1931) has written out a brief but interesting history of this type of pressure gauge. The paper gives in addition much interesting information pertaining to a quite different design of precision pressure gauge.

[32] F. G. Keyes and R. B. Brownlee, Thermodynamic Properties of Ammonia, John Wiley and Sons, 1916, p. 11. F. G. Keyes and Jane Dewey, Jour. Opt. Soc. of Am., *14*, 491 (1927). J. A. Beattie and W. L. Edel, Ann. Physik, *11*, 633 (1931). J. A. Beattie and O. C. Bridgeman, Ann. Physik, *12*, 827 (1932).

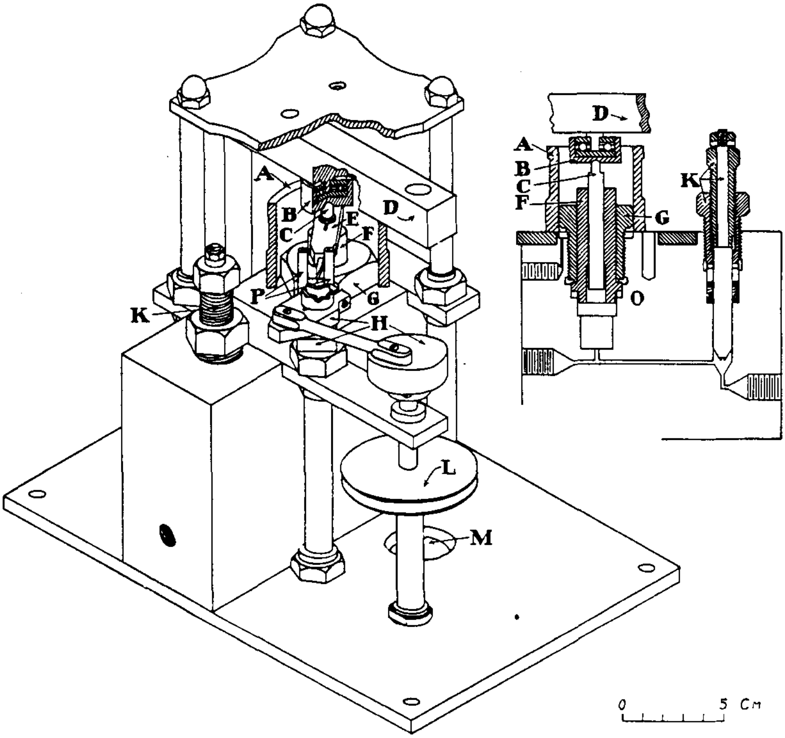

FIG. 4.

The pressure range which the standard four piston R. L. P. C. gauge has been designed to cover is from several to about 1200 atmospheres. If the attempt was made to cover the range with a single piston-cylinder the sensitivity would be less than desirable at low pressures or an inconveniently large amount of weights required at the highest pressures. The design therefore includes four separate piston-cylinder combinations but each cylinder has the same outside dimensions fitting the gauge block interchangeably. The series of piston-cylinders are designed to have constants of 0.5 mm./g., 1 mm./g., 2 mm./g., and 4 mm./g. The constant is related to the effective piston diameter by the formula:

$$c = \frac{40g}{\pi \rho g_s d_e^2}{}^{*} \qquad (4)$$

where g, g_s are the values of the gravitational constant at the place of use, g_s the standard gravitational constant (980.665), ρ the density of mercury in cc./g. at zero degrees, and d_e the effective diameter in cm. The load on the scale pans for the maximum pressure is 250 kilos.

In Fig. 4 the piston of hardened steel is indicated by *C*, its cylinder by *F* and the nut locking the cylinder into the block by *G*. An annealed aluminum gasket is used to seal the cylinder into *O* its block. The piston supports, through the intermediary of a ball bearing *B* at its upper end, a cross bar *D* for the scale pan. The brass collar *A* is merely an oil catch for the oil accumulating as a consequence of the continuous and necessary leak of oil from the space between the cylinder and its piston. The shut-off *K* is based on the old standard R. L. P. C. design[33] in use for some twenty years in all pressure work. The design makes it possible for the stem of the valve to move forward and backward through the packing without twist; a point of advantage if the shut-off is to be used frequently at higher pressures. The packing consists of fine woven cotton sheeting material cut on the bias in strips about 2 to 3 cm. in width. It is passed repeatedly through purified ceresin wax held at 110° C until no further air bubbles appear on the material. The strips are then hung to drain and when the wax has solidified the required amount, cut to the desired width (2 to 3 cm.), is wrapped tightly around the piece to be packed. A hard-rubber piece fitting the piston tightly is used at the bottom of the packing gland, the piston with its ceresin packing next and finally a second hard-rubber piece. A steel driving member compresses the packing by means of the nut as depicted. It has always been found that the driving of the packing should proceed by intervals of successive tightenings of the driving nut rather than by attempting to settle the packing into its final state by excessive tightening at once. This is essentially the packing gland scheme that has always

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[*] The value of g at M. I. T. is 980.397.

[33] Keyes, Townsend and Young, Jour. Math. and Phys. M. I. T. *1*, 278, fig. 9, (1922).

been used in this laboratory and which has proven serviceable to pressures approaching 2000 atms.

The isometric drawing shows the parts labeled corresponding to the piston-cylinder detail. The letter *E* designates a member clamped to the piston below *B* and serving as a means to oscillate the piston to and fro (30 per minute) by means of the hardened steel rollers *P* actuated by the mechanism *H*, and belt drive to pulley *L*. The holes *M* in the base plate permit the passage of the suspension rods of the scale pan. A hole is provided in the block for a thermometer well and shown in the smaller drawing below the letter *G*.

The first piston-cylinder No. 1 T used in the steam investigation was made by using an automobile motor valve tappet. It was in the form of a cylinder surmounted with a hexagon head having a threaded hole but accurately ground and selected for uniformity of diameter over its entire length. The cylinder was bored and lapped to receive the tappet or piston. The gauge No. 1 T was first calibrated in July 1923 against an open glass mercury manometer 8 meters in length and the constant obtained was 1.01443 mm. per gram at 30° C.

A second calibration in October, 1926, using a jacketed constant temperature column gave a constant 1.015024 mm. per gram at 30° C. Due to grit in the oil the piston of gauge No. 1 was scratched in March 1931. No. 1 T gauge was then replaced by gauge No. 28 constructed in 1923 and having a hardened piston-cylinder made by West and Dodge of Cambridge. Dr. O. C. Bridgeman's calibration of this piston in May, 1927, gave the value 1.99672 mm. per gram at 30° C. By this time, however, data pertaining to a number of gauges had indicated unmistakably that secular changes were taking place in the steel of all the piston-cylinders possessed by the laboratory. Gauge No. 28 on June 16, 1931, was calibrated at the pressure of liquid CO_2[34] at 0° and the constant found to be 1.99770 mm. per gram at 30° C. On June 24, 1931, the constant of gauge No. 1 T was determined at the CO_2 point and the constant found to be 1.015815 mm. at 30° C. The rate of change of the gauges[35] with time is 0.0182 percent per year for No. 1 T in the first period of 3.21 years and 0.0165 percent in the second period of 4.72 years while No. 28 changed on an average by 0.0122 percent from May 1927 to June 1931. Gauge No. 52 having the largest diameter 1.37 cm. has the least yearly change (0.009).

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[34] O. C. Bridgeman, J. Am. Chem. Soc. *49*, 1174 (1927).

[35] J. A. Beattie and O. C. Bridgeman, Ann. Physik, *12*, 827 (1932).

The precision of the calibration of the gauges is not inferior to one per twenty thousand and the details of the calibration process are fully described in the references cited in this section.

The device used for detecting the equilibrium is in principle identical with that devised by the writer some twenty years ago.[36] In this scheme the oil of the gauge is separated from the fluid whose pressure is to be measured by a mercury *U* provided with electrically insulated needles in each arm (*J*, *F* Fig. 18). The mercury level in the *U* is adjusted by means of a small mercury injector piston, *H*, so that it just makes contact with the needles at zero pressure. Any displacement of the mercury meniscus will evidently cause one or the other of the contacts to break. Thus if a weight is applied to the scale pan in excess of the pressure being measured the gauge oil-mercury meniscus contact will open. If, on the other hand, the weight is too small the other mercury contact will open. The rapidity of operation and sensitivity of the equilibrium detecting device is all that could be desired. The equilibrium obtained therewith is moreover independent of the rate of leak of oil from the crevice between the piston and cylinders. The latest electrical circuit in use for the pressure equilibrium detecting device has been described by Beattie and Edel.[37]

TABLE IV.

Date when Calibrated	Type piston-cylinder	Constant C, mm./gr.	Percent change per year
July 18, 1923	No. 1 T (auto. tappet)	1.014430	
Oct. 4, 1926	No. 1 T (auto. tappet)	1.015024	0.0182
June 24, 1931	No. 1 T (auto. tappet)	1.015815	0.0165
May 1927	No. 28 (made 1923)	1.99672	
June 16, 1931	No. 28 (made 1923)	1.99770	0.0122
May 1927	No. 52	0.499446	
May 1931	No. 52	0.499628	0.009
July 1932	No. 52	0.499681	

Temperature correction of gauge constant $C_t = C_{30}(1 + 2.2 \times 10^{-5}(30 - t))$. Corrections to all pressure observations were applied for the "aging effect" of the piston-cylinders.

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[36] See for example Keyes and Brownlee, Thermodynamic Properties of Ammonia, John Wiley and Sons, 1916; also F. G. Keyes, Am. Soc. Refrig. Eng. *1*, 9 (1914).

[37] J. A. Beattie and W. L. Edel, Ann. Physik, *11*, 633 (1932). See Fig. 3, p. 636.

The Table IV summarises the data pertaining to the gauges used in the investigation. All pressures are expressed in international atmospheres or the force per unit area due to a column of mercury at 0° C, 76 cm. in length, the internationally accepted value of the gravitational constant being 980.665 dynes. At this laboratory the gravitational constant is 980.397.

A knowledge of all liquid levels within the apparatus is evidently required before the pressure actually exerted by the fluid of interest under investigation can be stated. The design of the apparatus was executed in such a way that the required liquid levels could be easily and exactly determined.

The oil injector which makes it possible to compensate for the oil escaping from the crevice between the piston and its cylinder has a capacity of about 100 cc. It is represented at *B* in Fig. 5 and the most important detail is the oil tight sliding piston represented in Fig. 5*A*. The screw (18 threads per inch) which advances the oil-pump piston is depicted at *A*; the oil cylinder at *B*, a swivel type joint is provided at *C* and the piece *D* of mild steel should be a fair sliding fit in the uniform lapped hole of the cylinder (clearance about one or two thousandths inch). The end of *D* opposite the joint, *C*, is turned down to 0.65 cm. diameter, leaving it one inch in length. The end is threaded to receive a nut *G* and carefully turned washer *F*. Discs of machinery belt leather 0.3 cm. thick are cut 0.1 cm. larger in diameter than the base of the cylinder and placed on the stem of *D*, represented at *E*, and compressed by means of the nut *G* and washer *F*. The piston *D* is now placed in a lathe and ground true with a tool post grinder until, well oiled, it can be drawn into the cylinder *B* by means of the screw. The torque at the end of the 30 cm. spokes required to draw *D* into the cylinder need not exceed twenty to thirty pounds.

The assembly drawing of the steam apparatus, Fig. 9, illustrates the application of the equilibrium detecting device as it was actually applied in part of the steam investigation. The water sample was here confined by liquid water or by mercury as circumstances required and the two risers were connected to a manifold into which the volumnometer *I* connected. The mercury in riser *H* made connection with the oil line of the gauge, *W*, and was held at the needle point, represented within *H*, by means of an oil injector pump not shown in this drawing (see *B*, Fig. 5).

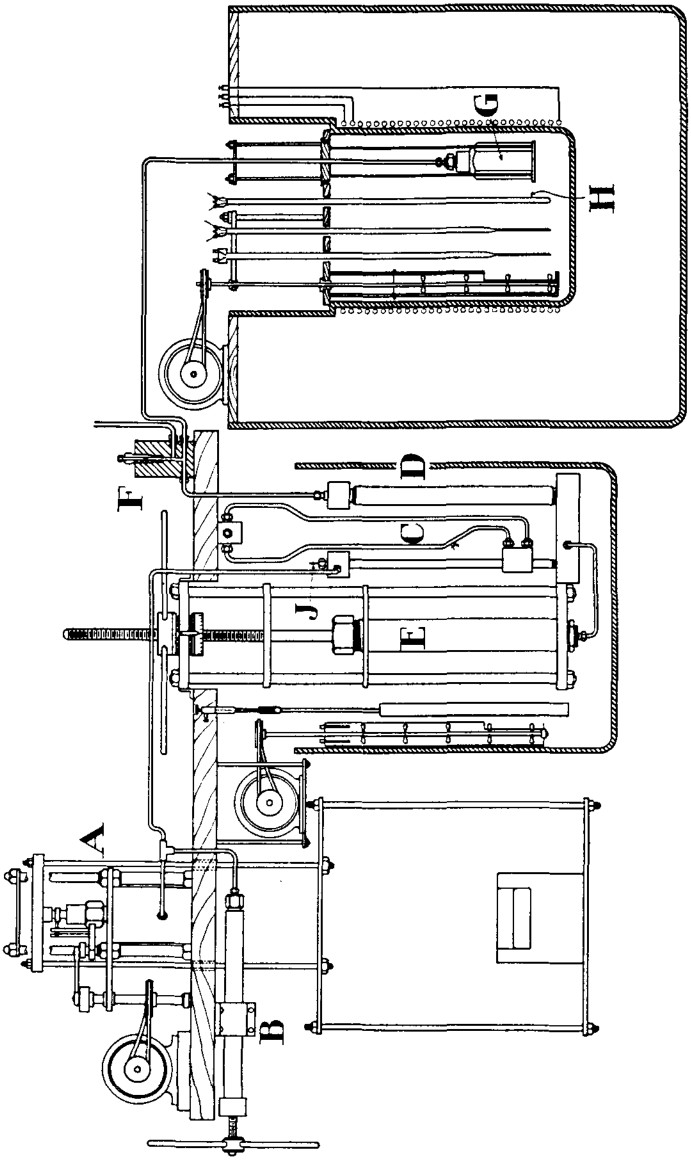

FIG. 5.

(7) MEASUREMENT OF VOLUME CHANGES.

The details of the volumnometer are represented in Figures 5, 6, and 7. Figure 5 represents the apparatus assembly used where mercury and liquid water confine the water sample and *E* shows the volumnometer assembled. It consists of a cylinder of steel which receives a uniformly cylindrical piston of selected drill rod. The packing of the piston to enable it to be displaced without leak or wear is similar in principle to that already described for the stems of pressure valves. Figure 6 shows the detail of the packing gland for the piston of the volumnometer, *F* representing the cylinder. The ceresin packing is represented at *E*, the hard rubber rings at *D*, the driving ring at *C* and the threaded nut used to compress the packing by *B*. The piston is designed to allow a clearance of about 0.05 mm. to 0.1 mm. in the cylinder and the hole in the packing nut *B* may be as much as 1.0 mm. larger in diameter than the piston in order to avoid scoring of the piston as it moves to and fro through the nut. In fact it will be evident that the piston should not come in contact with steel except in the case of the driving ring. The latter may have an internal diameter 0.05 to 0.1 mm. greater than the piston diameter and should be made from annealed steel.

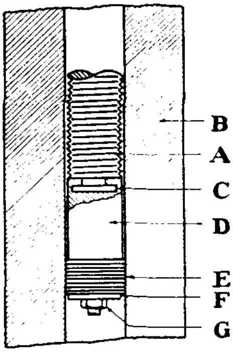

FIG. 5*A*.

In the case of a small volumnometer such as is used in the steam measurements, total displacement 50 cc., a length of drill rod may be selected for the piston, care being exercised to choose a length uniform in external diameter and having a smooth surface. The thread is cut directly at one end of the piston. The type of thread used is of the so-called trapizoid type represented at *G* in Fig. 7, which causes the thrust to be taken by thread surfaces perpendicular to the axis of the volumnometer piston. A *V* thread tends to wedge and brings about increased friction at the higher pressures. Thus at pressures of 350 atm. for which the steam volumnometer was designed the total thrust on the piston is 710 kilos (1570 lbs.) and the trapizoid thread insures that the wear on the piston threads and driving nut *F* is a minimum. The thread is lubricated with Dixon's graphite grease.

Larger volumnometers (250 cc.) for higher pressures (1500 atm.) have been constructed and used, while one volumnometer of 1000 cc. displacement has been used to 100 atm. It is best with larger diameter

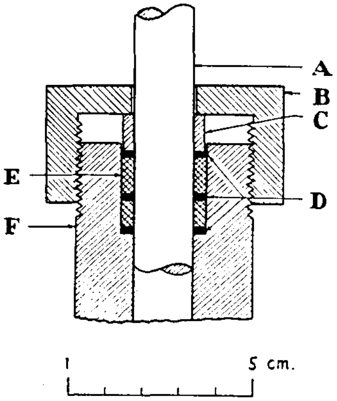

FIG. 6.

pistons to use a large diameter driving thread. The standard practice is to cut the thread on a 5.08 cm. (2 inch) piece of selected drill rod and in the case of a 250 cc. volumnometer to use a 2.22 cm. (7/8″) or 2.54 cm. (1″) piston. One very good method of joining the piston and threaded member is to turn a one degree taper for about 5 cm. on one end of the piston and to bore accurately the end of the threaded portion to correspond, but somewhat deeper than 5 cm. The taper end of the piston is then cooled in liquid air and dropped into the tapered hole. It is well to provide an excised cavity at the bottom of the tapered hole to make certain that the taper end of the piston "clears" and does not come in contact with the bottom of the tapered hole. The contraction of the piston taper in cooling from liquid air temperatures is of the order of 0.05 mm. If the tapers are accurately prepared and smooth a joint will result which has all the properties that would result were the whole assembly machined from a single piece of stock. It is not practical to turn a volumnometer piston and its thread out of a single bar in the case of the larger pistons because a lathe would be required having an exceptionally long bed. Even if the lathe were available the precautions and time involved to avoid

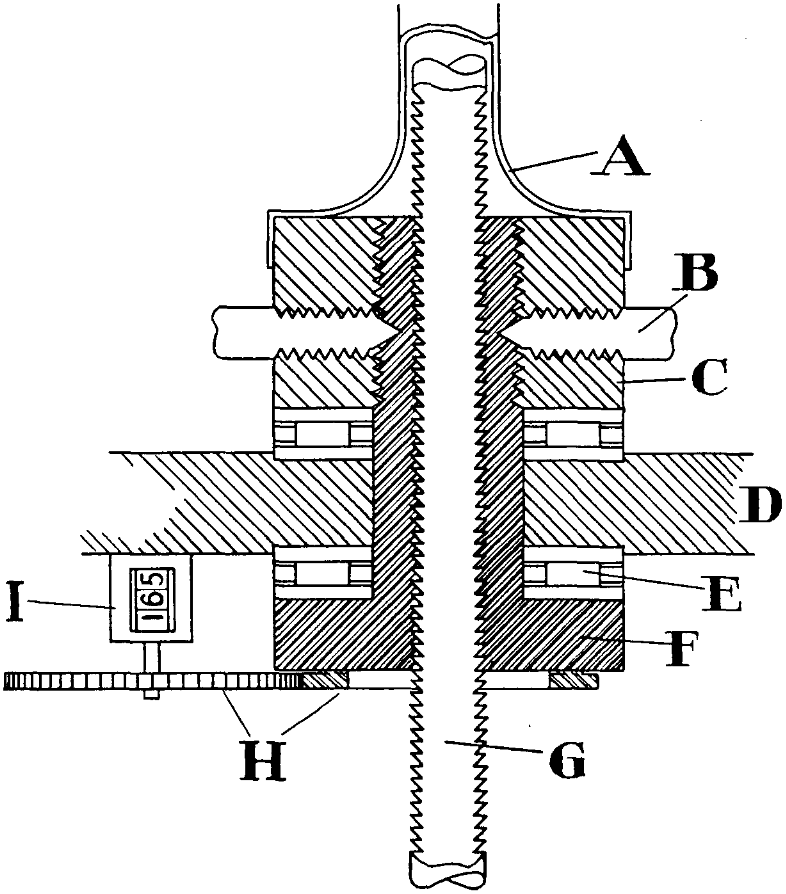

Fig. 7.

distortion and deflection of the five foot length of steel during the machining operations become quite formidable. Other methods of joining the two pieces of drill rod stock will occur to the reader, such as threading and pinning; but the method suggested has been used, is satisfactory, and not difficult to execute.

The material of the nut *F* to receive the thread is a matter of considerable importance. Tobin bronze has been found, after some experience, to be the best material. The nut is machined from a solid piece of rolled stock and the operation of thread cutting proceeds quite easily because of the favorable cutting properties of Tobin bronze. Sometimes to insure as perfect a fit as possible I have left the nut thread just short of a final fit and milled the end of the piston screw to form a tap. The finishing cut may then be made by carefully working or tapping with the piston screw through the nut and the result is as perfect a fit as could be desired. A skilled mechanician will, however, have no difficulty in machining the nut with a threading tool to a satisfactory fit. The length of the nut is 5 to 6 times the thread diameter in the case of screws to 2.22 cm. diameter and a somewhat smaller ratio for larger diameter screws.

The bearings for the nut are likewise of great importance and it has been found that the roller thrust bearings manufactured by the Gwilliam Company of Brooklyn, New York, are entirely satisfactory. They consist of two hardened and ground flat steel rings with a bronze cage between, containing hardened steel rollers. The bearings are represented at *E* and one set is used above and one below the frame cross-support member *D*. It will be noted that the hardened rings are set into a recess turned into the member *D* and of course the rings also make a good fit with the barrel of the Tobin bronze nut. The surface of the bronze nut external to the bearing and parallel to the axis is divided into 100 parts and numbered at each 10 marks for use in estimating fractions of a turn. Whole turns are counted by means of a bicycle "cyclometer," *I*, actuated by means of gears, *H*, one of which is secured by screws to the face of *F* perpendicular to the nut axis. A gear having the same number of teeth is of course fitted to the cyclometer shaft.

The locking member *C* is made of steel. After having been adjusted to give the proper bearing adjustment it is bored and threaded on the lateral face to receive the four spokes *B* used both to lock and to operate the nut. In recently executed designs a bevel gear has been fitted to *C* and a motor used to operate the nut.

(8) CALIBRATION OF THE VOLUMNOMETER.

It will be quite obvious that in spite of all care in the choice of a uniform piston the irregularities of the volumnometer piston diameter joined with the inevitable errors of the actuating screw will cause non-uniformity in the volume displacement of the piston. The delivery or volume displacement per turn of the screw will therefore vary as the screw of the volumnometer is moved over its range. The diameter of the piston for the steam work was 1.587 cm. (5/8 inch) and possessed 27 cm. of travel. The number of turns per cm. was 7.0866 (18 per inch) and the displacement per turn about 0.28 cm.[3] The nut periphery divided into 100 parts permitted estimations of 0.001 turn or 0.00028 cm.,[3] the nominal volume-sensitivity of the volumnometer.

The calibration is carried out with the volumnometer mounted in its position for actual use, exhausted and filled completely with mercury through a 0.06 cm. capillary. A very fine glass capillary is prepared and joined to the end of the steel capillary projecting from the constant temperature bath surrounding the volumnometer. The mercury within the latter is now expelled in five turn portions and weighed. Each portion weighs approximately 19 g. but since the nominal error corresponding to five turns is not inferior to one in five thousand the weight on repeated five turn expulsions of mercury may not be expected to be consistent to better than about four milligrams.

A convenient way of using the data is to divide the total volume by the total number of turns and construct a graph consisting of the differences of the mean volume from the actual volumes as ordinates and turn numbers as absissae. From such a graph the volume displacement from zero to the nth turn may be rapidly computed or the difference in volume between the nth and the $(n + j)$th turn. A convenient way to verify the calibration is to weigh the successive ejections for each ten turns. Fig. 8 is the graph of the calibration for the steam volumnometer. During the calibration it is important that the kerosene bath surrounding the volumnometer should be maintained at constant temperature (30° C is convenient). The degree of regulation required is easily computed for when the volumnometer contains 50 cc. of mercury, 1° C change in temperature will cause the mercury to change volume by 9×10^{-3} cc. The volume displacement for 1/1000th turn is however 0.0003 and accordingly the temperature of the bath must not change by more than 1/30th degree. It is of course quite easy, using steel inclosed mercury regulators, to hold the temperature constant to within ± 0.005 at 30° C.

A glass tip at the end of the steel capillary leading from the volumnometer must be of fine bore, 0.01 to 0.02 cm., and cut off squarely at the exit. It is well also to arrange for a minimum of exposed or unthermostated capillary, otherwise variations in room temperature will mar the consistency of the weighings. Thus a change of temperature of one degree with one cubic centimeter of unthermostated mercury would cause a volume displacement approaching 1/1000th turn of a ⅝ inch piston such as used in the steam work.

The mercury must also be ejected very slowly during the calibration, otherwise a rise in pressure will take place within the volumnometer. The isothermal compressibility of mercury is about 3.9×10^{-6} per $cm.^3$ or for 50 cc. roughly 2×10^{-4}, about the order of sensitivity of the volumnometer. Moreover the conductivity for heat of the body of the latter is finite and on rapid pressure rise there will be an imperfectly adiabatic increase in temperature if the mercury is quickly ejected and then weighed. This effect may, in fact, be important since at 300 K. the temperature rise on adiabatic compression is 0.0032° C per atmosphere. When due care is exercised with respect to the indicated technique of ejecting the mercury and the usual procedures followed in accurate weighing practice, consistent calibrations of ten turn portions to within one in five thousand are easily attained.

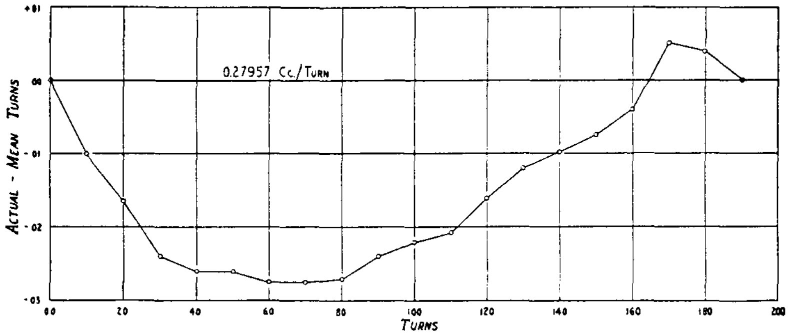

Fig. 8.

In the case of the steam investigation the container volumes are believed to be known to the order of one part in possibly twenty thousand; certainly to one in ten thousand. The total volume of the container, No. 1, Fig. 2 for example, was 219.642 cc. at 0° so that if this is assumed known to ± 0.01 cc. and the error in the volumnometer for the worst case (50 cc.) contributed an uncertainty of ± 0.01 cc. it would be expected that the error in the total volume measured (220 + 50 cc.) should not exceed ± 0.02 cc. The objective was to measure volumes with a precision of one in ten thousand and this degree of precision appears to have been attained if the consistency of the duplicate measurements for liquid water under pressure are used as a basis for judgment. Of course mere consistency using the same apparatus may be very misleading, but in the case of the volumes of liquid water, duplicate series of measurements using containers of two different metals and different gauges provided data which showed about the agreement anticipated.

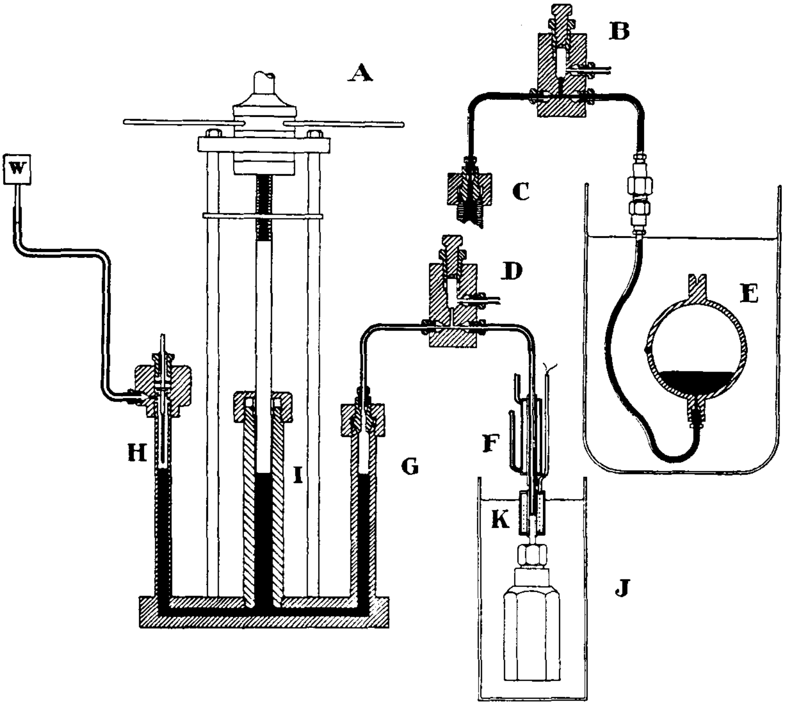

FIG. 9.

Some of the measurements of the volume of the vapor phase of water were made in the 550 cc. spherical containers using mercury, Fig. 9, *E*, to confine the vapor. *B* is the loading block, *C* the upper part of *G*, while *I*, *H*, and *A* correspond as in Fig. 5. In this case special conditions enter which diminish the precision with which vapor volumes may be determined. In the first place the total mass of water in the container may, at larger specific volumes, exist in relatively appreciable amount on the container walls in the adsorbed state. The amount of water in the latter state will depend of course on the temperature and the pressure, increasing markedly as the temperature is lowered and augmenting with pressure. Because of the adsorptive property of water, volumes corresponding to specific volumes greater than 150 cc. per gram (saturation temperature 191.93° C) were not measured. Nothing, in fact, short of a new method which by its nature is capable of furnishing data independent of the adsorption phenomenon will suffice to give precise values of the volumes of steam at low pressures and temperatures.[38]

A further obstacle to securing exact volume measurements on water vapor at low pressures where mercury confines the fluid resides in the difficulty of correcting the observed volumes or if it is preferred, the observed pressures for the effect of mercury vapor. The effect of the presence of mercury vapor may not be considered allowed for by subtracting from the total pressure the normal equilibrium pressure of mercury vapor at the temperature of experiment. Gibbs[39] proved, and Poynting[40] independently pointed out that the normal equilibrium pressure, vapor pressure, of a subtance was altered when pressure was applied to the condensed phase. Thus if subscripts one and two denote respectively the vapor phase and condensed phase the thermodynamic equation governing the state is:

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[38] F. G. Keyes and S. C. Collins, Proc. Nat. Acad. Sci. *18*, 328 (1932).

The method described in this paper has since been greatly improved and tested further using gaseous ammonia and carbon dioxide. The indications are that $\left(\frac{\partial H}{\partial p}\right)_T$, $(H = U + pV)$ can be very exactly determined at low temperatures and pressures. The coefficient is equivalent thermodynamically to $\left(\frac{\partial v\tau}{\partial \tau}\right)_p$ where v is volume and τ reciprocal Kelvin temperature. From the empirical but accurate representation of the latter coefficient as a function of temperature and pressure complete information regarding pressure-volume-temperature properties can be obtained. The results obtained by the use of the method are entirely unaffected by adsorption effects. Moreover, while the method is particularly valuable for low pressures, it is not necessarily restricted to this region. However, the low pressure and low temperature range is particularly interesting in the case of steam and measurements are now being made.

[39] J. W. Gibbs, Scientific Papers, Longmans Green and Co., 1906, p. 154, Eq. 272.

$$(v_1 dp_1 = v_2 dp_2)_T \tag{5}$$

Using the idealized expression $pv = RT$ for the gas phase and assuming the condensed phase to vary linearly with pressure the following expression for the equilibrium pressure is obtained:

$$p_1 = p_{\text{sat}} \exp\left[\frac{V_0}{RT}(p - p_{\text{sat}}) - \frac{\alpha}{2RT}(p_2 - p_{2_{\text{sat}}})\right] \tag{6}$$

where the volume of the condensed phase is assumed given by $V_0 - \alpha p$. It is evident that the equilibrium pressure, p_1, will always increase since V_0 is necessarily positive, and α small.

The kind of correction here referred to is commonly termed the "partial-pressure-of-mercury" correction, but partial pressure loses meaning as consideration turns from the conception of idealized gas mixtures with constituents behaving as though each produced its pressure effect independently. The equation (6) will therefore be expected, as the temperature rises, to become inexact as a means of correcting for the effect of mercury in an apparatus where the vapor of a substance is confined with mercury as in the case under discussion.

A better approximation to allow for the presence of mercury vapor can be secured only when a knowledge of the properties of mixtures of steam and mercury are available. If the pressure-volume-temperature properties of mercury vapor were known it would be possible to compute the required corrections with some confidence by making use of the body of information now available in connection with the equation of state of pure substances and their mixtures.[41]

The data obtained for the volumes of steam where mercury confined the water (150 cc. to 20 cc. per g.) extend to 330° C. Considerations to be discussed in detail in the paper dealing with superheated steam will indicate that the correction for the mercury-vapor-effect given by Equation (6) suffices to possibly 300° but that significant error results if Dalton's partial pressure rule is used for correcting the measurements taken above this temperature.

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[40] Poynting, Phil. Mag., *12*, 32 (1881).

[41] F. G. Keyes and H. G. Burks, J. Am. Chem. Soc., *50*, 1100 (1928). L. J. Gillespie, Phys. Rev., *36*, 121 (1930). H. T. Gerry and L. J. Gillespie, Phys. Rev., *40*, 269 (1932).

The volume measurements made in the vapor phase corresponding to 50 cc. per gram and less were made by what may be referred to as the "constant-volume-variable-mass" method. Here, Fig. 9, the vapor phase in the container, *J*, was confined by liquid water in the capillary above the level of the constant temperature bath in which the container was supported. The amount of water in the latter is of course determined, with a given total sample of water originally introduced into the system, by the setting of the volumnometer piston. In this way measurements over a range of steam densities can be made. A major difficulty with this method lies however in the definiteness with which the water-liquid-vapor phase meniscus can be located. After considerable experience a capillary heater was designed and constructed by my collaborator, Doctor Leighton B. Smith. The heater represented in Fig. 9 at *K* is adjusted in a position which leaves its lower three fourths submerged below the surface of the bath fluid.

The detailed construction of the heater is made evident in Fig. 10 where *F* represents the capillary connecting the container to the loading block *D* of Fig. 9. The cross section representation shows the capillary at *F* over which is drawn a silver tube *E*. A mica frame perforated with holes *C* to receive wire is constructed as indicated at *D*. The leads from the electrical resistance wire are threaded through a mica strip *B* and insulated from the metal of the case by two mica strips shown in the section drawing. The case *G* is of steel and all joints are silver soldered. The device is important, indeed indispensable where, as quite generally is the case, the fluid under investigation must be confined by the same fluid.

By introducing electrical energy the contents of the capillary can be heated a few degrees above the temperature of the main body of vapor in the container. A cooler, *F*, Fig. 9 through which water is circulated is next placed just above the heater. It will be readily perceived that by this means the meniscus or liquid-vapor-junction in the capillary will be sharpened and the limits of its position fixed, particularly when the vapor in the container is near saturation. The 0.06 cm. capillary had a volume of 0.0029 cm.^3 per cm. length and if the liquid-vapor junction, as appears to be the case, is localized within ± 0.5 cm., the error in determining the mass of water in the vapor phase can be stated. At specific volume 20 cc. per g. there would be about 11 gr. of water in the container No. 1. Using the figure 0.003 cm.^{3} for the uncertainty in locating the capillary liquid-vapor junction it turns out that the vapor density of the container content would be uncertain by about one part in thirty-seven hundred. At higher vapor densities the error in computing the mass of vapor would of course be correspondingly less; at smaller densities on the other hand greater.

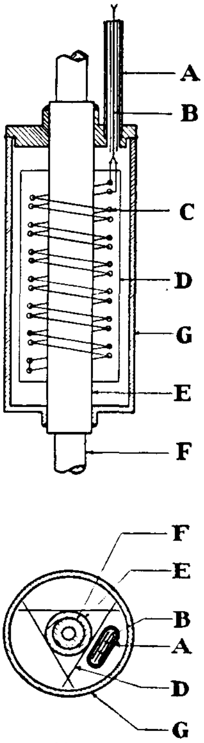

FIG. 10.

The earlier measurements were made without the capillary heater and at temperatures not far from saturation. Long periods were required before equilibrium was attained at the water-vapor junction in the capillary. Even with the exercise of every care it was difficult to be sure that equilibrium existed and it was only after installing the heater that the desired measurements close to saturation could be made.

(9) DILATION OF VOLUMNOMETER AND OTHER APPARATUS PARTS DUE TO PRESSURE.

The volumnometer and all accessory apparatus parts containing pressure transmitting fluids are kept at constant temperature 30° C during measurements. It is practically impossible to compute the total pressure dilation of the system in the bath but it is quite simple to determine the net pressure-stretch effects experimentally. To do so it is necessary to close off the capillary at the point where the container is attached. The shift in reading of the volumnometer is then determined at various pressures and for pressure increasing and decreasing. The readings should coincide but it is good practice especially after the apparatus has been newly assembled to increase the pressure to a value considerably in excess of the highest pressure to be used subsequently in measurements on the fluid of interest and to maintain the high pressure for a number of hours. This procedure may be described as a "seasoning" operation and it is usually found that on releasing the pressure the packing drive-nut of the volumnometer (*B*), Fig. 6, can be tightened.

After thorough seasoning a series of volumnometer displacements may be taken along the course of rising pressure and followed by similar readings as the pressure is relieved. The two series should be identical to within one or two thousandths of a turn of the volumnometer piston nut. If not, the seasoning should be repeated until accord is obtained but as a rule reproducibility is attained after a thorough seasoning of several hours at twice the upper limit of the pressure it is expected to use in actual measurements on substances.

The simplest manner of using the apparatus dilation measurements is in the form of a graph where the volumnometer readings in turns are recorded as ordinates and the pressures as absissae. Should mercury be added or withdrawn from the system prior to connecting with the container and making measurements on a fluid, a new graph may be constructed, or corrections computed, by making use of the data on the compressibility of mercury by Bridgman[42] $(4.2 \times 10^{-6} - 1.54 \times 10^{-10}p)_{30°}$ (or Carnazzi—Table I) and computing the adjustments required.

The compressibility of the oil (53.1×10^{-6} cc./cc. at 25° C) is not a factor since contact of the mercury is always maintained with the insulated needle used to maintain the location of the mercury-oil junction at the needle point *H*, Fig. 9. A slight but usually negligible effect enters in allowing for the oil pressure in computing total pressures since the oil density (0.9083 g/cc. $+ 5.3 \times 10^{-4}t$) increases with pressure.

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[42] P. W. Bridgman, Int. Crit. Tables, Vol. II; see also Cim. (5) *5*, 180 (1903).

Another minor factor deserving mention is connected with the finite compressibility of the volumnometer piston.[43] The piston is calibrated at substantially atmospheric pressure but in actual use it is under pressure. The compressibility of steel is however of the order 0.6×10^{-6} and in the case of the steam volumnometer each 100 atmospheres with fifty cubic centimeters of piston exposed to pressure would cause a contraction in piston volume equivalent to about 0.010 turn. In applying this correction it is best to note the piston volume exposed to pressure at the time of the pressure dilation determinations and add the number of turns corresponding to the piston compression for each reading. The equation ($6.0 \times 10^{-7} \cdot \Delta \cdot n \cdot p$) where Δ is the piston volume per turn in cm.^{3}, n the number of turns reckoned from the zero used in the dilation experiments and p the pressure, can then be used to compute the compression effect for use in reducing actual measurements.

Reference should also be made to the fact that in the measurements using liquid water as the confining fluid for the container contents an exact knowledge is required for the compressibility of water at the temperature 30° C of the volumnometer bath. This is however easily secured by bringing the container completely filled with liquid water to 30° C and measuring the decrease in the number of turns of the volumnometer over the entire range of pressure. After these readings have been corrected for pressure "stretch" of the container, volumnometer and accessory parts, and volumnometer piston compressibility, it is a simple matter to correlate the data and obtain an equation for the volume of water at 30° C as a function of pressure.[44]

During actual measurements the container capillary or rather portions of it will in general be at temperatures up to 10° different from 30°, and accordingly the fluid in the capillary connecting tube will be subject to a temperature gradient. The amount of water in this state is, however, small and it is best to proceed with the reduction of the observations neglecting provisionally the corrections for the compressibility of the connecting tube fluid. The resulting data, while slightly inexact, may than be used to obtain the required true mass of the capillary fluid in each measurement and a final value computed for the amount of substance in the container. As a matter of fact, in the case of water, the compressibility in the range 20° to 50° is almost constant and in the measurements of liquid water volumes under pressure the average value of the compressibility between 20° and 40° suffices for computing the mass of water in the capillary.

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[43] The frame rods holding the volumnometer are evidently stretched when the contents are under pressure. It is also evident, however, that this effect is included in the determinations of the "apparatus stretch" due to pressure. Of course a certain displacement of the volumnometer readings under pressure relative to the practically zero-pressure mercury calibration readings enters, but the piston is sufficiently regular to render the volume error of this displacement negligible.

[44] At 30° C between 100 and 200 atm., we obtain a mean diminution of volume for 1 cc. amounting to 43.62×10^{-6} cc. Further data will be given in the paper dealing with the volumes of liquid water.

(10) CASE WHERE MERCURY CONFINES THE WATER SAMPLE.

The apparatus pressure-stretch data are of course only valid for the quantity of mercury present in the apparatus at the time the data were taken. If mercury is used to confine the water sample *F*, Fig. 9, a part of the mercury will be at the same temperature as the water sample which is of course in the case of water, in general higher than that of the volumnometer bath (30° C). Proper allowance must therefore be made for the temperature dilation of the mercury transferred from the volumnometer to the container and its compressibility. The temperature dilation of mercuty at atmospheric pressure is well known but the compressibility as a function of pressure and temperature is not known over a wide range of temperature. We have used the data given by Carnazzi[45] for the reason that Bridgman's more recent data does not extend to high enough temperatures. There is very great need of a determination of the *p-v-t* properties of liquid mercury over a long temperature and pressure range.

In spite of the fact that the data for liquid mercury are deficient no great error is introduced in measurements where the volume of fluid under measurement is large relative to the volume of mercury in the container. Thus for example, at 150 cc. per gram the 550 cc. container would have about 525 cc. of steam and 25 cc. of mercury contained in it. For a temperature of 300° the difference in compressibility of the mercury from that at the volumnometer temperature would be 1.3×10^{-6} cc. per cc. of mercury. The total volume change of the mercury for a pressure difference of 15 atm. is therefore 5.0×10^{-4} cc. or about 0.0018 piston turns. It is evident from this example that an error of fifty percent in the temperature coefficient of compressibility is not too serious.

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[45] Carnazzi, Cim., (5) *5*, 180 (1903).

The temperature dilation is more important for it is nearly fifty times the compressibility and nothing is known regarding its change with pressure at higher temperatures. The latter is however undoubtedly small. In all work at this laboratory the one atmosphere temperature dilation of mercury has been used.

(11) PREPARATION OF GAS-FREE WATER.

Mere boiling of water in a flask at atmospheric pressure is not sufficient to remove the last traces of dissolved air. Nor will exposing a sample of water to a vacuum at or just above the freezing point (vapor pressure 4.8 mm.) even for considerable time suffice. Prolonged shaking in a discontinuously applied water aspirator pump vacuum will pretty completely remove dissolved air[46] but in the preparation of water for the steam measurements it was more satisfactory to repeatedly distill in vacuum. To follow the reduction in dissolved gas content a gauge of the McLeod type was employed wherein a trapped sample of the gas phase could be compressed to small volume and to pressures when desired several times the vapor pressure of the water. In this way the degree of failure of the vapor sample to collapse completely is a measure of the ratio of gas to water. By the repeated distillation of water in vacuum it was found possible to reduce the volume ratio of permanent gas to water vapor at 20° to much less than 5×10^{-9}. The latter number corresponds to roughly 0.00004 percent of air by weight or roughly one in two and a half million.

The final apparatus used for distilling, testing and loading the larger water samples is represented in Fig. 11. Communication of the entire system with a large capacity oil pump was possible by opening the mercury seal represented at *D*. About one liter of water was contained in the bulbs at *B* where the water could be slowly distilled with intermittent pumping from one bulb to the other as many times as was desired. It will be noted that under certain conditions a value for the gas content determined by the McLeod gauge, *E*, may be a true value for the gauge but not representative of the gas content in the bulbs, *B*. This is particularly likely to be the case during active distillation. However, by allowing the system to stand over night the readings at *E* become independent of time and fairly representative of the essential gas content of the water sample.

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[46] F. G. Keyes and L. J. Gillespie, Jour. Biol. Chem. *13*, 244 (1912).

(12) METHOD OF LOADING WATER SAMPLES INTO STEEL CONTAINERS.

The loading of the water sample into glass container *A* Fig. 11 is quite simple. It proceeds by distillation on cooling *A* with ice water.

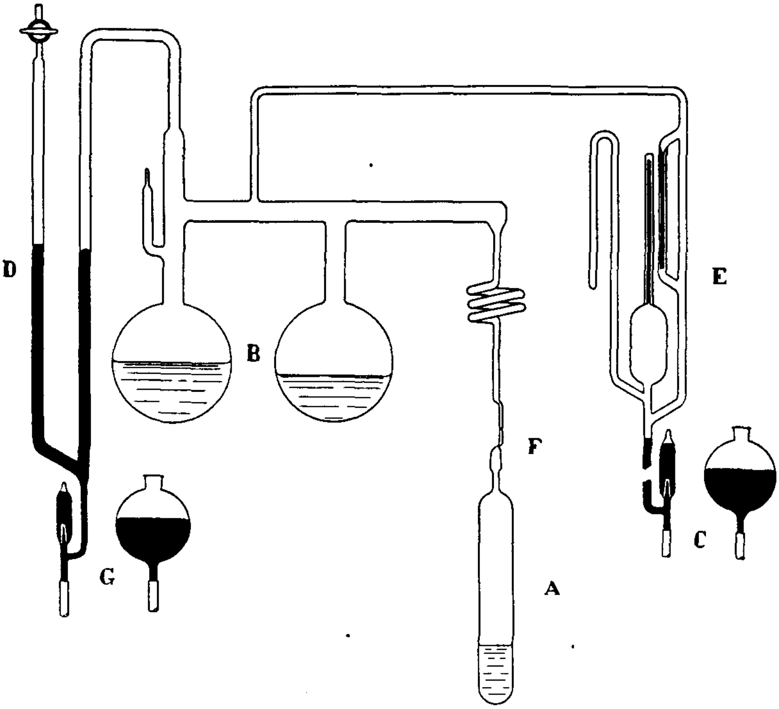

FIG. 11.

Obviously it is necessary to heat *A* prior to loading in order to free the walls from adsorbed carbon dioxide, air, or other gases. When the desired volume of water (the volume of *A* as a function of the length being known) is collected, the flask is sealed from the apparatus at *F*. The weight of water within *A* can then be exactly determined from the known total weight of the water plus container *A* after the water has been transferred. The method of loading the substance sample into the intermediate container has already been described[47] as well as the principle employed to transfer the contents of the latter to the metal container used for the measurements. The method of transfer will be clear from Fig. 12 where *F* the steel capillary leading to the metal container is joined to the glass member of the loading apparatus with de Khotinski cement.[48] Vacuum is applied through cock *E* while *G* is a supplementary tube used in unloading definite amounts of water from the metal container in the course of measurements. After a suitable vacuum has been created in all communicating parts of the loading apparatus the glass container *A* is rotated slightly in the ground joint *B*. By this means the sealed off tip *D* (previously marked with a super-hardened steel edge) is brought into contact with the internal boss *C* and snaps off without fragmentation. The water contents then flow into the metal container and at the proper time the steel shut-off *D* of Fig. 9 on the steel capillary line is closed. The water left clinging to the walls of *A* and in the capillary or other parts of the loading apparatus is removed through cock *E* by means of a pump, the water vapor being held in a weighed trap *H* cooled with liquid air. The method described for transferring substances without contact with air has been used with success in this laboratory for a variety of purposes.[49]

The loading process for smaller quantities of water, 2 grams to 50 grams, and particularly where the material was to be confined with mercury, was carried out with a modification of the apparatus of Fig. 12. The design of the smaller water sample container will be evident from *B*, Fig. 13; a large number of such containers were sealed to a manifold joined to the distillation apparatus, Fig. 11. In this way a series of water samples of proper amount to cover the range of specific volumes from 150 cc. per g. to 20 cc. per g. could be prepared. The glass vessel and its contained water, after weighing, was placed in *A* and the ground glass stopper inserted. The ground joint at *C* is for

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[47] F. G. Keyes and W. A. Felsing, J. Am. Chem. Soc., *41*, 599, 604 (1918).

[48] F. G. Keyes, R. S. Taylor and L. B. Smith, Jour. Math. and Phys. M. I. T. *1*, 217 (1922).

A word of caution is worth stating relative to the care in handling water in conatiners where no permanent gas is present. The liquid water behaves in all respects like a piece of steel and will shatter the glass if the position of the container is too quickly reversed.

[49] F. G. Keyes, J. Am. Chem. Soc., *34*, 779 (1912).

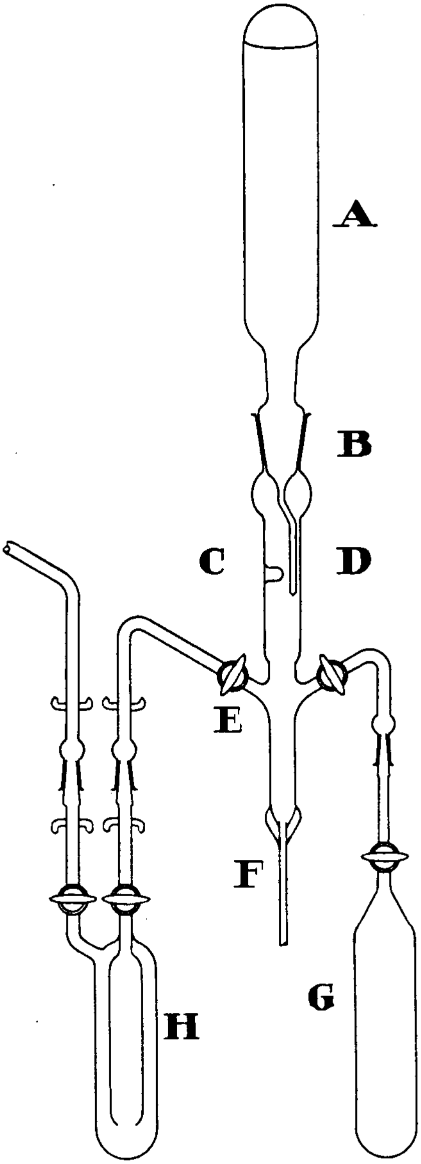

Fig. 12.

rotating the sample holder *A*, the latter being off-set relative to the ground end. The internal boss *D* causes the scratched tip of the container *B* to snap off when *A* is rotated. The system was pumped out through cock *E* prior to unloading *B*.

The Fig. 13 also illustrates the special device used when the water sample subjected to measurement was to be confined with mercury. The disposition and general design of the parts will be evident from the figure.[50] Referring to the figure, *G* amounts to a screw driver whose shank passes through a packing gland, *H*. Immediately below the end of the screw driver is a threaded plug bored axially to about 0.15 cm. diameter. The lower end of the hollow plug is prepared for locking against a gold closure disc, *K*, 0.025 cm. in thickness. When the apparatus is set up for loading *J* is left screwed off the seat about 0.2 cm. leaving ample opportunity for the water to pass, at the time *B* is unloaded, along *F*, past the gold disc and into the container attached to *L*. The container boss (*E*, Fig. 9) on the spherical container in the present method of loading is cooled with liquid air, and in this way all the water originally in the container *B* is transferred to the metal container. The screw driver is finally used to lock the contents of the container by forcing *J* against the gold closure disc *K*. The portion of the apparatus carrying the screw driver is now removed and replaced with *M*; the latter being the connecting means to the volumnometer part of the apparatus.

The volumnometer and connecting tubes when mercury is used to confine the water sample must be filled with a quantity of mercury appropriate to the amount of water in the steel container. After joining the latter to the volumnometer system by means of the cone joint *M*, the air in the capillary connecting tube is exhausted and the steel cock (*B*, Fig. 9) connecting therewith closed. Mercury can now be injected into the exhausted capillary by means of the volumnometer and volumnometer readings made at a series of pressures with the mercury in contact with the gold closure disc. These readings are called the "zero-set" readings and by making use of the apparatus pressure-stretch data, the amount of mercury injected by the volumnometer at any pressure and temperature is determined. Communication of the mercury with the interior of the container is readily effected by heating the steel container in its thermostat bath to 200° under a pressure on the mercury side of 50 to 100 atm. In

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[50] F. G. Keyes, L. B. Smith and D. B. Joubert, Jour. Math. and Phys. M. I. T., *1*, 191 (1922)—Fig. 3.

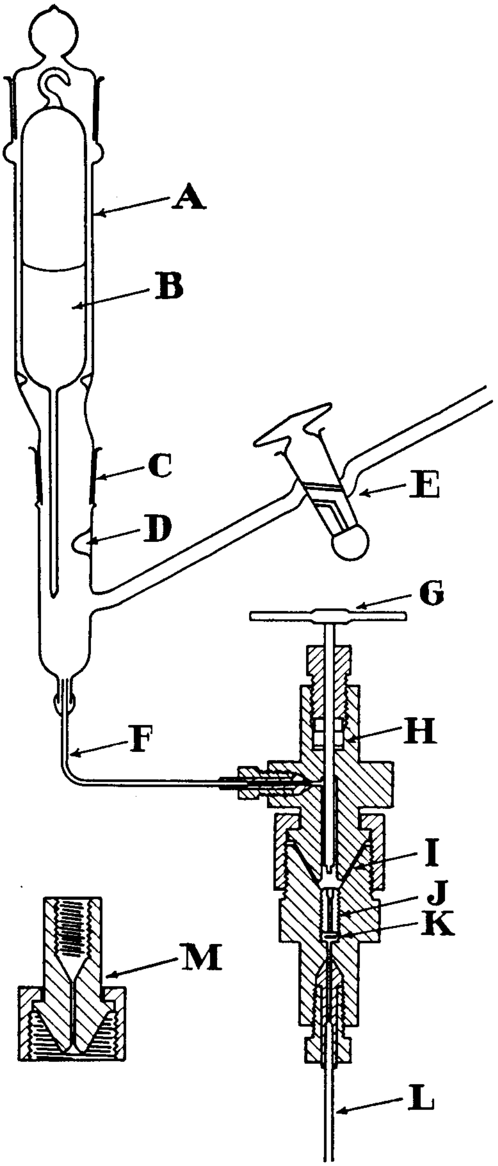
A
B
C
D
E
F
G
H
I
J
K
L
M

FIG. 13.

this way the gold disc is weakened by amalgamation with mercury and punctured. The disposition of the container in the bath support is shown in Fig. 14, wherein may be seen the stirrer tube and spherical level-indicating steel float for the molten salt bath.

(13) The Constant Temperature Bath.

Constancy of the high temperature bath during measurements is an important matter, and it is not difficult to determine the degree of regulation desirable in measuring, for example, the vapor pressure of water. Thus at 370° C the change of pressure for 1° C is 2.4 atm. and the vapor pressure itself is 207.8 atm. The gauge sensitivity is, say, one in twenty thousand or 0.01 atm. on the vapor pressure. The temperature must therefore not deviate more than 0.004° C or ± 0.002° C.

The construction of automatically regulated baths operating over the range 100° C to 500° C have not so far as known been described until recently although work toward the goal of a bath constant to 0.002° C had been started in 1919. The bath used throughout the steam investigation was built in 1922 and consists of a wrought iron pot wound with swaged iron-clad insulated nichrome wire electrical resistance heaters 15 meters in length. The pot consisted of a 25.4 cm. Shelby seamless tube 45 cm. long with a wrought iron welded bottom. The pot is finally mounted in the center of a large galvanized can leaving about 20 cm. of annular space for asbestos-magnesia heat insulating material. A galvanized iron collar fits the outside top of the pot and makes a driving fit with a wooden annular ring forming the closure for the iron can. The wooden ring at the same time serves as a support for electrical connections, switches, and stirring motor.

Figure 5 shows a cross section of the bath at the right of the figures in which *G* is the nickel No 1 container and *H* the thermostat actuating member. The objects in the bath are all supported from a machined steel cover. The container support is however mounted on a machine disc forming a close fit with the hole bored for its reception in the bath cover. In this way the container may be easily withdrawn without disturbing the heavy bath cover.

The bath fluid consisted of the eutectic mixture of sodium, potassium and lithium nitrates (27.3 percent $LiNO_3$, 18.2 percent $NaNO_3$, and 54.4 percent KNO_3 by weight) fluid at 135° C and entirely serviceable up to 500° C. The material is very clean and highly fluid. When a fresh batch is put into use it is a good plan to place scraps of iron, steel wool or the like in the molten mass. In this way

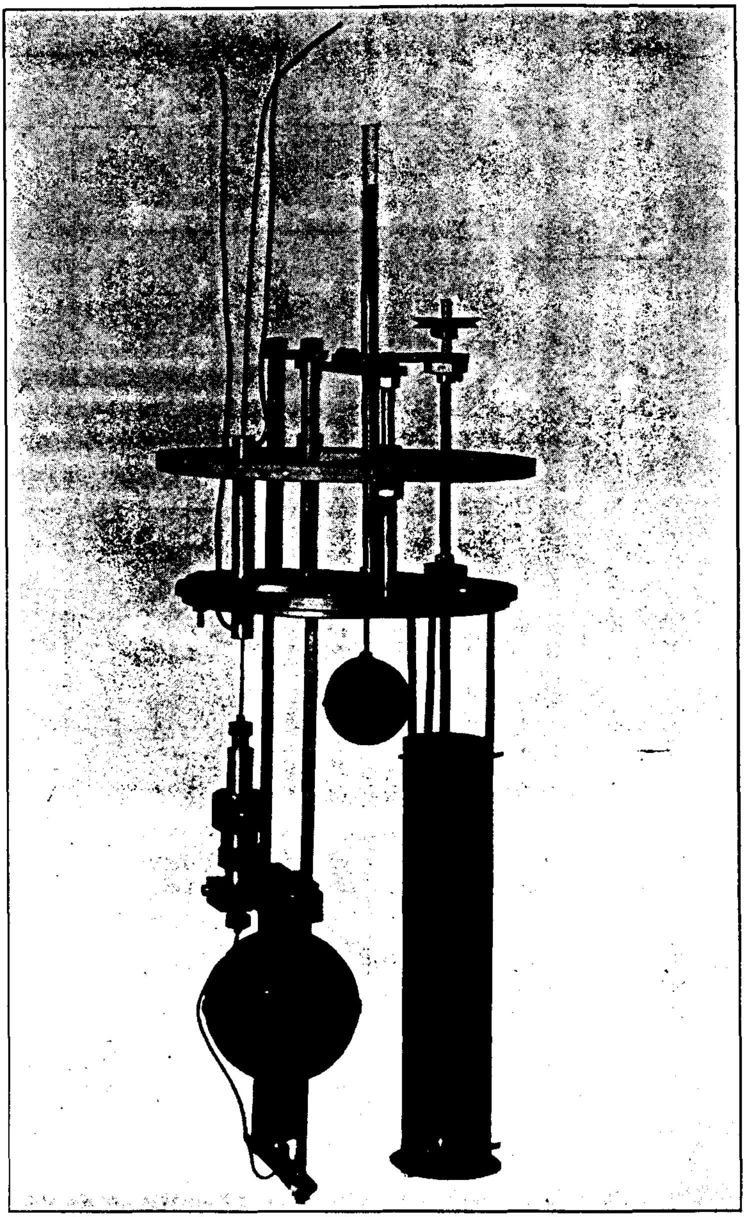

FIG. 14.

it has been found that the mixture loses its power to oxidize steel. The mixture must not be allowed to solidify until all apparatus objects contained are removed because the mixture contracts to such an extent on cooling that it will, for example, bend or even pull the propellers from the shaft.

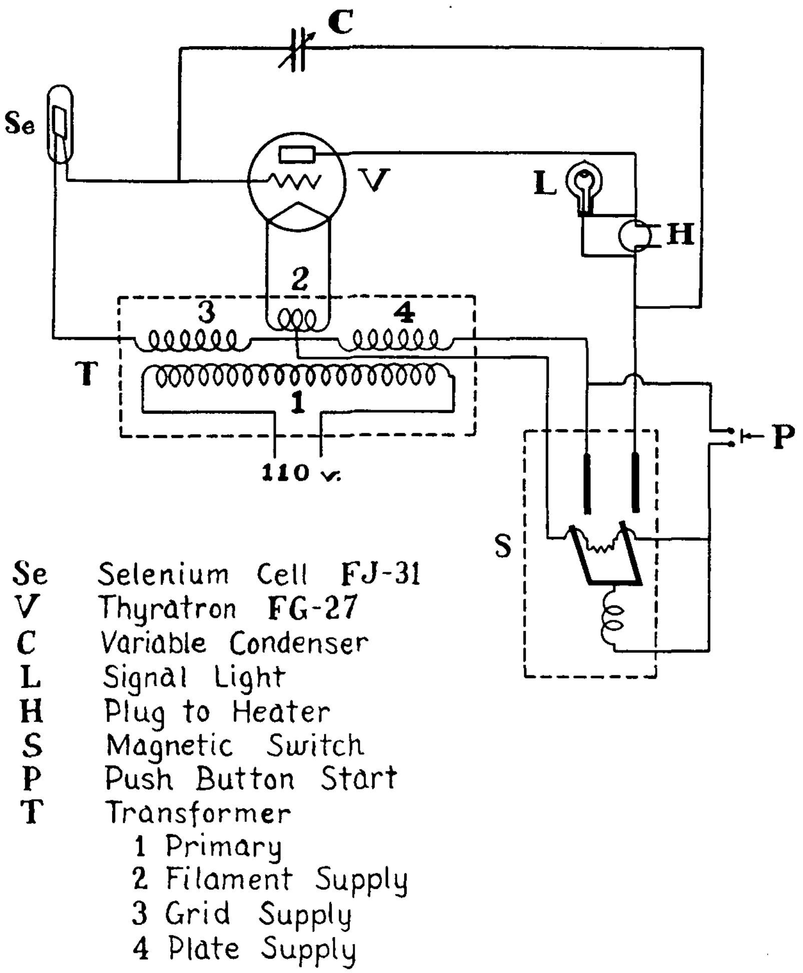
C
Se
V
L
H
2
3
4
T
1
110 v.
S
P
Se Selenium Cell FJ-31
V Thyratron FG-27
C Variable Condenser
L Signal Light
H Plug to Heater
S Magnetic Switch
P Push Button Start
T Transformer
1 Primary
2 Filament Supply
3 Grid Supply
4 Plate Supply

FIG. 15.

The temperature regulation is entirely automatic and the bath element consists essentially of a flat type platinum resistance thermometer of 50 ohms 0° C electrical resistance. The latter or "regulator" is balanced on a Wheatstone bridge and when unbalanced, due to a change in the resistance of the regulator brought about by a change in the bath temperature, a beam of light reflected from the galvanometer mirror falls on a selenium cell. The light falling on the cell causes a relay to act introducing or ceasing to introduce energy into the bath through a small heater placed near the stirrer case. The earlier system has been already described by Doctor L. B. Smith in the annual report printed in Mechanical Engineering, Vol. 48, page 155, with figures 8 and 9. A later and improved design of the bath with full details of the temperature regulation scheme has recently been published by J. A. Beattie.[51] Since the latter publication additional improvements have been made by Doctor Collins and myself enabling automatic regulation to be maintained to ± 0.0005° C. The automatically controlled bath described and used in the steam investigation was continuously improved until toward the beginning of 1926 when bath constancy of ± 0.002° was realized. This degree of constancy is in fact sufficient for the most exacting portions of the steam measurements. The details of the most recent scheme of the automatic regulating circuit will be seen from Fig. 15 and with the notations on the diagram comment is scarcely required.[52]

The thirty degree volumnometer oil bath was automatically controlled by a conventional type of mercury-in-steel regulator. A Beckmann thermometer indicated that extreme fluctuations did not exceed a few thousandths of a degree.

(14) APPARATUS ASSEMBLY.

The apparatus described in the preceding pages for measuring *p-v-T* properties is of quite general applicability and is in principle the same as has been used for taking observations extending to about 1500 atm. for some twenty years.[53] There are however various ways in which the component parts may be used depending upon the substance and temperature range to be explored. It has already been indicated that to temperatures of 300° mercury may be used to confine the fluid under measurement. At this temperature the vapor pressure of mercury is about 0.324 atm. and because of uncertainty in the corrections to be applied to allow for the "partial-pressure" effect of mercury it is not safe to greatly exceed 300°. For steam at specific volume 150 cc. per gram and 300° the pressure is about 16.61 atm. and accordingly the ratio of the normal equilibrium pressure of mercury to the steam pressure is about one in fifty-one. Of course allowance must be made for the increase in pressure of the mercury due to the pressure of the steam (Poynting effect) but this effect is only one in four thousand on the total pressure in the preceding example. There must be considered also the failure of Dalton's partial pressure rule already referred to under the section "Calibration of the Volumnometer." The use of Gallium as the confining liquid instead of mercury would be a major improvement for this metal has a negligible vapor pressure at 500°.

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[51] James A. Beattie, Rev. Sci. Instr. *2*, 458 (1931).

[52] In this connection see J. A. Beattie and D. D. Jacobus, J. Phys. Chem., *34*, 1254 (1930).

[53] F. G. Keyes, Am. Soc. Refrig. Eng., *1*, 9 (1914). F. G. Keyes, L. B. Smith and D. B. Joubert, Jour. Math. and Phys. M. I. T., *1*, 191 (1922). F. G. Keyes, R. S. Taylor and L. B. Smith, Jour. Math. and Phys. M. I. T., *1*, 211 (1922). F. G. Keyes and R. B. Brownlee, "Thermodynamic Properties of Ammonia," John Wiley and Sons, 1916.

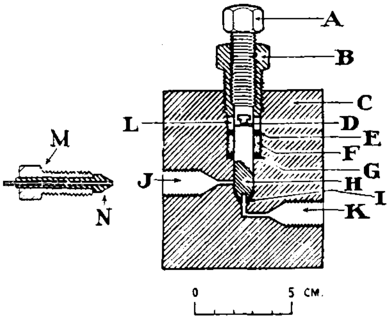

Fig. 16.

At temperatures greater than 300° it is necessary to confine the water with itself. Until the critical temperature is reached however, pressures less than saturation will cause vapor to appear in the container and a two phase junction will make its appearance in the connecting capillary. The exact location of this miniscus junction is a matter of great importance and the heater developed to facilitate its localization has already been described in section eight. The Fig. 5 is a diagram of the assembly as used in the steam investigation but

Fig. 9 indicates the apparatus scheme where mercury confines the water sample.

The shut-off used in the steam investigation* is shown in Fig. 16 where the valve member *H* is made with a cone having a slightly less taper than the seat. A hole is bored in the lower end, *I*, and the edges at the cone are sharp. The design facilitates closure without excessive pressure exerted by means of the member *A*. It is important that the valve part, *H*, should not turn in the packing *E*, *F*, *G*, and to avoid turning a species of universal joint is provided at *D*. The packing is of ceresin impregnated cotton already described. It is compressed by means of the mild steel ring, *L*, driven by the member *B*. The method of making the joints with capillary tubes has already been referred to but at the left of Fig. 16 a detail is given. The cone ended piece, *N*, is welded over with the acetylene flame and subsequently drilled out to the bore of the capillary. The cone is then dressed to a size slightly sharper than the seat *J*, *K* and driven tight by means of the member *M*. This type of joint is serviceable to pressures of 1000 atm. and has been used with success to about 2000 atm.

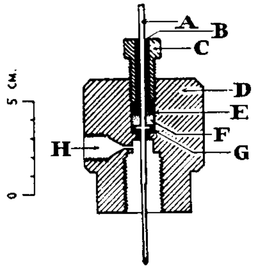

FIG. 17.

The electrical insulating joint is represented in its construction by Fig. 17. For higher pressures, 1000 to 2000 atm., it is best to turn the needle *A* from a single piece of stock. Otherwise the ring on the needle may be silver soldered to the needle. The piece upon which the needle rests, *G*, is of porcelain although hard rubber may be used. *F* is cut from ordinary laboratory rubber stopper. A few discs of mica are provided at *E*. Insulation between the needle stem and the driving screw, *C*, is provided by using hard rubber.

---

* A later design is shown at K, Fig. 4.

The capillary into which the needle projects is about 0.3 cm. in diameter and *D* may be locked onto the capillary portion by using an annealed aluminum gasket. Capillary tube connections to the gauge or elsewhere are made by the usual cone joint, *M*, *N*, Fig. 16, at *H*. This type of insulating needle is very practical and lasts indefinitely.

(15) SPECIAL APPARATUS FOR MOST RECENT VAPOR PRESSURE MEASUREMENTS.

Many measurements of the vapor pressure of water were made in the apparatus assembly represented in Figs. 5 and 9. However after continued experience had shown the superiority of Nirosta chrome-nickel steel as a container, a simplified apparatus was assembled during 1932 and used exclusively for vapor pressure measurements. The Fig. 18 indicates the disposition of parts, *E* being the Nirosta cylinder containing the water sample loaded through *C*, and attached to the shut-off block *B*. Varying amounts of water could be introduced at will by operating the volumnometer *I* and forcing water from *G* into the container. In this manner it was possible to wholly collapse the vapor phase in *E* and demonstrate the presence or absence of permanent gas. Moreover when desired water vapor could be removed from *E* by opening the cock at *B* and pumping at *C*.

Liquid water communicated directly with the mercury of the *U* in block *L* and the mercury-water miniscus maintained in position by means of pressure exerted by the oil injector attached to *W*. *F* and *J* are insulating needles and the position of exact contact of the mercury with both needle points could be made before measurements began by means of the mercury injector *H*. The shut-off *K* served to interrupt pressure communication between the water sample and the gauge connected to *W*.

The function of the capillary *U* block *L* is two fold. It serves to separate the oil of the gauge from the water sample and also to indicate when pressure equilibrium has been established in the same way that the mercury-oil miniscus does at *H*, Fig. 9. Thus too great a weight on the gauge scale pan will cause the vapor in *E* to collapse thereby displacing the mercury-oil miniscus in *J*, and making contact at the mercury-water miniscus at *F*. Too little weight on the gauge scale pan on the other hand causes vapor in *E* to be evolved from the liquid phase and the mercury in the *U* block *L* is displaced breaking contact in *F* and making contact in *J*. In this way the *U* acts as a very sensitive detector of pressure equilibrium. It was found in practice by Doctor Smith that the sharpness of the water-needle contact was affected by the finite electrical conductivity of the water. At a convenient time the water side of the *U* was disconnected and replaced with oil. The shift of the mercury level in the *U* with pressure was than determined by means of the mercury injector *H*.

The sensitivity of the equilibrium detector is readily established as follows. The bore of the capillary containing the insulating needles *F*, *J*, Fig. 18, is 0.3 cm. The diameter of the piston of the pressure gauge is 0.682 cm. It has been found by independent trial that ± 0.001 cm. displacement suffices to make or break contact. Therefore a piston movement of ± 0.0002 cm. will suffice to produce ± 0.001 cm. motion of the oil-mercury miniscus. With larger diameter pistons the detectable piston movement is less and for piston diameters less than the diameter of the insulated needle capillary the device is less sensitive. It will be observed that the functioning of the detector is independent of oil leak.

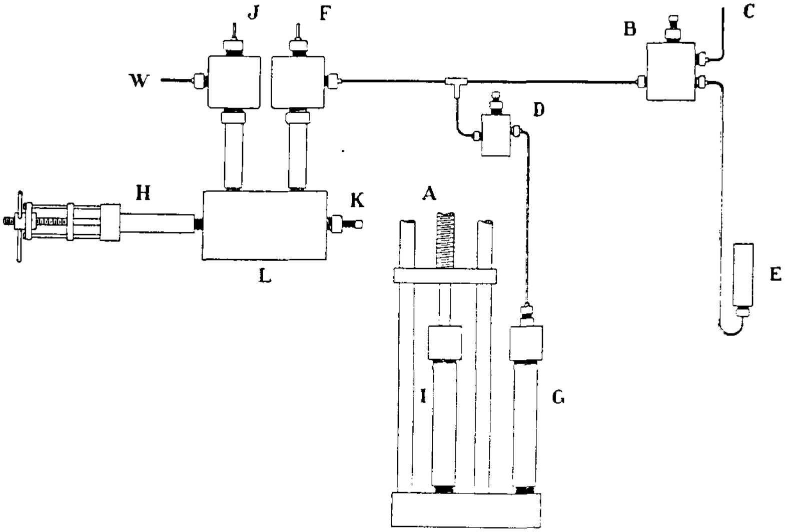

FIG. 18.

The electrical circuit used for the equilibrium detector has been greatly modified during the many years it has been used. The simplest system is the thyratron FG-27 circuit which permits a current of only 1×10^{-6} amperes to flow from the mercury to the needle. It is essential in any electrical circuit that is used to have a very small current pass the junction for it has been found that too great a current causes the formation of an oil-mercury emulsion at the miniscus which not only diminishes the liquid mercury in the system but causes an uncertain contact.

CAMBRIDGE, MASS.

Proceedings of the American Academy of Arts and Sciences.

Vol. 68. No. 13.—December, 1933.

RECORDS OF MEETINGS, 1932–1933.

OFFICERS AND COMMITTEES FOR 1933–1934.

LIST OF THE FELLOWS AND FOREIGN HONORARY MEMBERS.

STATUTES AND STANDING VOTES.

RUMFORD PREMIUM.

INDEX.

(Title Page and Table of Contents.)

RECORDS OF MEETINGS.

One thousand one hundred and ninety ninth meeting.

OCTOBER 19, 1932.—STATED MEETING.

The Academy met at its House at 8.20 P. M.

The PRESIDENT in the Chair.

There were present seventy-five Fellows and eight guests.

The records of the meeting of May 11 were read and approved.

The Corresponding Secretary reported the receipt of letters accepting Fellowship from C. R. Adams, M. J. Ahern, J. R. Angell, J. C. Aub, C. F. Batchelder, A. A. Bennett, Welles Bosworth, W. B. Cabot, Leonard Carmichael, W. R. Castle, Jr., G. C. Cell, J. P. Chamberlain, C. D. Clifton, J. C. Cobb, K. J. Conant, J. A. Cousens, R. S. Crane, M. W. Croll, W. J. Cunningham, E. C. Cutler, H. S. Dennison, A. S. Dewing, W. B. Donham, G. H. Edgell, Felix Frankfurter, Glennon Gilboy, J. C. Grew, Albert Haertlein, W. P. Hall, William Healy, Einar Hille, H. W. Holmes, E. R. James, William James, P. C. Jessup, Hayward Keniston, T. W. Lamont, E. F. Langley, K. S. Lashley, R. R. Lawrence, W. G. Leland, J. A. Lowell, F. B. Lund, W. W. McLaren, Sayre Macneil, Calvert Magruder, K. L. Mark, L. C. Marshall, C. E. Merriam, F. C. Mills, O. L. Mills, Everett Morss, H. G. Moulton, A. D. Nock, G. R. Noyes, William Cardinal O'Connell, W. F. Ogburn, Milman Parry, A. S. Pease, A. J. Peters, David Rapport, A. L. Ripley, G. B. Roorbach, L. S. Rowe, F. B. Sayre, W. C. Schumb, H. R. Shepley, H. E. Smith, Payson Smith, Carl Snyder, C. Henry Taylor, C. Holt Taylor, William Thomson, E. S. Thurston, E. S. Webster, R. H. Wetmore, D. V. Widder, J. H. Williams, W. F. Willoughby, B. L. Young; and of letters accepting Foreign Honorary Membership from Mineichiro Adatci, F. Orpen Bower, A. L. Bowley, R. G. Hawtrey, Hu Shih, Paul Mazon, Kingo Miyabe, Charles Rist, A. C. Seward, F. W. Thomas.

He also announced that Dr. Herbert J. Spinden had represented the Academy at the inauguration of William A. Boylan as President of Brooklyn College, on June 21, 1932; and that the President of the

Academy would act as its delegate at the conference on the "Obligation of Universities to the Social Order," at New York University, November 15–17, 1932.

The Corresponding Secretary announced that the Council had made the following grants from the Permanent Science Fund:

1. To George F. Beck, Washington State Normal School, Ellensburg, $200, for the purchase of apparatus and for field expenses and assistance in prosecuting his investigations in paleobotany.

2. To Willi M. Cohn, Cambridge, $300, to aid him to go to the Pacific coast, where he can obtain the use of a Hartmann microphotometer for the purpose of evaluating the records made at the recent solar eclipse on the polarization of the solar corona.

3. To Gleason F. Kenrick, Tufts College, $300, for the purchase of oscillographic equipment for use in his investigations of the Kennelly-Heaviside layer.

4. To Bret Ratner, New York University, $800, to aid in the support of his investigations on anaphylactogenic properties of split proteins and amino acids of milk, etc.

5. To John F. Sly, West Virginia University, Morgantown, $500, to aid in his investigations of town and county expenditures in England.

The President announced the death of ten Fellows:—Nathaniel Allison (Class II, Section 4), John Ripley Freeman (Class I, Section 4), Elijah Clarence Hills (Class IV, Section 3), Edward Washburn Hopkins (Class IV, Section 3), William White Jacques (Class I, Section 2), Charles Willison Johnson (Class II, Section 3), William Williams Keen (Class II, Section 4), Oliver Dimon Kellogg (Class I, Section 1), George Augustus Sanderson (Class III, Section 1), Edward Wyllys Taylor (Class II, Section 4); and of one Foreign Honorary Member:—Jean Adrien Antoine Jules Jusserand (Class III, Section 2).

The following communications were presented:

Mr. Joshua Whatmough: "Prunae Vatillum (Hor. *Sat.* i. 5. 36)."

Motion Pictures of Harvard Botanical Garden in Cuba and of the Barro Colorado Island Laboratory in the Panama Canal Zone, with Special Reference to the Wild Life. Incidental Remarks by Mr. William H. Weston, Jr.

Three papers were presented by title: "The Saltatorial Rodent Dipodomys. The Functional and Comparative Anatomy of its

Muscular and Osseous Systems, by A. Brazier Howell, presented by H. V. Neal; "The Compressibility of Eighteen Cubic Compounds," by P. W. Bridgman; "The Pressure Coefficient of Resistance of Fifteen Metals down to Liquid Oxygen Temperatures," by P. W. Bridgman.

The meeting was dissolved at 10.10 P. M.

One thousand two hundredth meeting.

NOVEMBER 9, 1932.—STATED MEETING.

The Academy met at its House at 8.20 P. M.

The PRESIDENT in the Chair.

There were present forty-two Fellows and fifty-two guests chiefly members of the American Statistical Association, who had been invited to attend by the President.

The records of the meeting of October 19 were read and approved.

The Corresponding Secretary reported the receipt of a letter from Frederick K. Morris, accepting his election as a Fellow.

The President announced the death of two Fellows:—William Patten (Class II, Section 3) and William Lambert Richardson (Class II, Section 4).

The following communication was presented:

Mr. Carl Snyder: "Recent Progress in Monetary Theory."

Two papers were read by title: "Note on Stellar Perturbations of nearly Parabolic Orbits," by Ernst J. Öpik, presented by Harlow Shapley; "The Eclipsing Binaries RW Arae and SW Ophiuchi," by Martha B. Shapley, presented by Harlow Shapley.

The meeting was dissolved at 10.20 P. M.

One thousand two hundred and first meeting.

DECEMBER 14, 1932.—STATED MEETING.

The Academy met at its House at 8.20 P. M.

The PRESIDENT in the Chair.

There were present fifty-two Fellows and fifteen guests.

The records of the meeting of November 9 were read and approved.

The Corresponding Secretary reported the receipt of letters from Paul Claudel and Pierre Janet accepting Foreign Honorary Membership; and from B. P. Clark, Howard Coonley, C. K. Drinker, E. S. Drown, E. C. Jeffrey, and R. C. Sturgis, resigning Fellowship.

He also announced that the President had appointed Wallace W. Atwood as delegate of the Academy at the First Congress of the Pan American Institute of Geography and History, to be held at Rio de Janeiro, beginning December 26, 1932.

The President announced the death of two Fellows:—William Sydney Thayer (Class II, Section 4) and Henry Pickering Walcott (Class II, Section 4).

The following communication was presented:

Mr. James B. Conant: "The Structure of Chlorophyll and the Problem of Photosynthesis."

The following papers were read by title: "Compressibilities and Pressure Coefficients of Resistance of Elements, Compounds, and Alloys, Many of them Anomalous," by P. W. Bridgman; "The Effect of Pressure on the Electrical Resistance of Single Metal Crystals at Low Temperatures," by P. W. Bridgman; "The Volume-Pressure-Temperature Relations of Fifteen Liquids," by P. W. Bridgman; "Radioactivity Measurements. 1. The Radium Content of the Keweenawan Basalts and Some Accessory Minerals. 2. The Occurence of Radium, Uranium, and Potassium in the Earth," by William D. Urry, presented by A. C. Lane.

The meeting was dissolved at 9.20 P. M.

One thousand two hundred and second meeting.

JANUARY 11, 1933.—STATED MEETING.

The Academy met at its House at 8.20 P. M.

The PRESIDENT in the Chair.

There were present sixty-four Fellows and three guests.

In the absence of the Recording Secretary, Vice-President Gulick was asked to assume his duties.

The records of the meeting of December 14 were read and approved.

The Corresponding Secretary reported that Mr. Edwin B. Wilson had represented the Academy at the conference on the obligation of universities to the social order, held at New York University, November 15–17, 1932.

He also reported that the Council had reclassified Foreign Honorary Member Wolfgang Köhler in Class IV, Section 1, instead of Class II, Section 3, as formerly.

The Corresponding Secretary announced that the Council had made the following grants from the Permanent Science Fund:

1. To P. W. Bridgman, Harvard, $200, to be applied as part salary of an assistant in his researches on the effects of high pressures.

2. To C. F. Brooks, Blue Hill Observatory, $75, for designing, constructing, and standardizing a cup anemometer having the heating element enclosed, for use on Mt. Washington.

3. To L. R. Cleveland, Harvard, $750, for assistance, apparatus, and materials in the prosecution of his anatomical and cytological researches on the protozoa of the wood feeding roaches, *Cryptocercus punctulatus*.

4. To W. C. Rose, University of Illinois, $250, to be used in part payment of the salary of his assistant in an investigation seeking to isolate a hitherto unknown component of casein essential for life, and related problems of nutritional chemistry.

The President announced the death of three Fellows:—John Joseph Carty (Class I, Section 4), Eliakim Hastings Moore (Class I, Section 1), and James Hardy Ropes (Class IV, Section 1).

Dr. Elihu Thomson then told of his personal observations of the great Leonid Meteor Shower of November 14, 1867, in Philadelphia. He also read a thrilling description of the experiences of Mr. Van Gestel at Anjer, Java, which town was overwhelmed at the great explosion of the volcano Krakatoa in the Sunda Straits in August 1883.

The following papers were read by title:

"The Purification and Physical Properties of Organic Compounds. 1. The Interpretation of Time-Temperature Curves in Freezing Point Determinations and as a Criterion of Purity," by Evald L. Skau, presented by F. G. Keyes.

"Studies in the *Bromeliaceæ.* IV," by L. B. Smith; and "The *Bromeliaceæ* of Trinidad," by W. E. Broadway and L. B. Smith, presented by B. L. Robinson.

The meeting was dissolved at 9.30 P. M.

One thousand two hundred and third meeting.

FEBRUARY 8, 1933.—STATED MEETING.

The Academy met at its House at 8.35 P. M.

The PRESIDENT in the Chair.

There were present forty-one Fellows.

The records of the meeting of January 11 were read and approved.

The Corresponding Secretary reported the receipt of a letter from Dr. James H. Means, resigning Fellowship.

The President announced the death of two Foreign Honorary Members: John Galsworthy (Class IV, Section 4) and Sir Robert Jones (Class II, Section 4).

The following communication was presented:

Mr. Arthur E. Kennelly: "Approximate Laws of Fatigue in Racing Animals, as Deduced from World's Racing Records," illustrated by lantern slides.

The meeting was dissolved at 9.55 P. M.

One thousand two hundred and fourth meeting.

MARCH 8, 1933.—STATED MEETING.

The Academy met at its House at 8.40 P. M.

The PRESIDENT in the Chair.

There were present twenty-eight Fellows and seven guests.

The records of the meeting of February 8 were read and approved.

The Corresponding Secretary announced that Professor Charles A. Kofoid had been appointed the delegate of the Academy at the Fifth Pacific Science Congress to be held in Victoria and Vancouver, B. C. in June 1933.

On the recommendation of the Council the following appropriations were made for the ensuing year:

From the income of the General Fund, $7,725, to be used as follows:

for General and Meeting expenses	$ 800
for Library expenses	2,000
for Books, Periodicals, and Binding	1,400
for House expenses	2,500
for Treasurer's expenses	1,025
	$7,725

From the income of the Publication Funds, $2,223, to be used for publication.

From the income of the Rumford Fund, $3,230, to be used as follows:

for Research	$2,000
for Books, Periodicals, and Binding	350
for use at the discretion of the Committee	880
	$3,230

From the income of the C. M. Warren Fund, $959.45, to be used at the discretion of the Committee.

On the recommendation of the Council an additional appropriation of $75 from the income of the General Fund for the current year was voted for the use of the Treasurer.

The President announced the appointment of the Nominating Committee:

Frederick A. Saunders, of Class I
Charles T. Brues, of Class II
Arthur N. Holcombe, of Class III
Irving Babbitt, of Class IV

Mr. Alfred C. Lane proposed an amendment to Article 7 of Chapter X of the Statutes, changing the word "prepare" to "provide."

He also proposed the appointment of a standing committee to provide biographical notices of deceased members, to consist of six members.

It was

Voted, To refer both matters to a committee consisting of Mr. Lane and both Secretaries, for report to the Academy.

The Corresponding Secretary read a short statement concerning Joseph Priestley.

The following communication was presented:

Mr. Hervey W. Shimer: "The Problem of the Changes in the Forms of Life from the Mesozoic to the Cenozoic Era."

The meeting was dissolved at 9.45 P. M.

One thousand two hundred and fifth meeting.

April 12, 1933.—Stated Meeting.

The Academy met at its House at 8.35 P. M.

The President in the Chair.

There were present twenty-eight Fellows and six guests.

The records of the meeting of March 8 were read and approved.

The Corresponding Secretary announced the appointment of Mr. Arthur E. Kennelly as delegate of the Academy at the dedication of the George Eastman Research Laboratories of the Massachusetts Institute of Technology on May first.

He also reported the receipt of a letter from Mr. Thomas Reed Powell resigning Fellowship.

The President announced the death of two Fellows:—Arthur Messinger Comey (Class I, Section 3), Arthur Bliss Seymour (Class II, Section 2); and of one Foreign Honorary Member, William Cawthorne Unwin (Class I, Section 4).

The President reported that he had attended the dinner given by the Massachusetts Institute of Technology in honor of Dr. Elihu Thomson and had presented to him a testimonial from the Academy.

On the recommendation of the Committee appointed at the March meeting, it was voted to amend the Statutes as follows:

Add to Chapter XI, Article 2 a new section, ix, as follows: "The Committee on Biographical Notices to consist of six Fellows, two to be elected each year, six in 1933, one of them to be a Secretary of the Academy, to see that biographical notices of the Fellows are provided."

Second, to change Article 7 of Chapter X by replacing the word "it" with the words "the Committee on Biographical Notices" and to change the word "prepare" to "provide."

It was also voted to further amend the Statutes by striking out the words "which may be needed to complete their sets, but" in Chapter XI, Article 2, Section iv, paragraph 2.

The following communication was presented: Mr. Reuel W. Beach: "The Beginnings of Printing in the Americas."

The following paper was read by title: "Least Squares and Laws of Population Growth," by Edwin B. Wilson and Ruth R. Puffer.

The meeting was dissolved at 10.10 P. M.

One thousand two hundred and sixth meeting.

MAY 10, 1933.—ANNUAL MEETING.

The Academy met at its House at 8.25 P. M.

The PRESIDENT in the Chair.

There were present forty-three Fellows and six guests.

The records of the meeting of April 12 were read and approved.

The Corresponding Secretary announced that the Council had made the following grants from the Permanent Science Fund:

1. To Prof. Howard H. Beard, Louisiana State University Medical Center, New Orleans, $400, to aid in carrying on his studies in nutritional anemia of the rat.

2. To Dr. L. R. Cleveland, Harvard Medical School, $500 in the year 1933 and $500 in the year 1934, to aid in the publication of a monograph on the morphology of certain cockroaches and their parasites.

3. To Dr. Charles V. Green, Research Associate, Roscoe B. Jackson Memorial Laboratory, Bar Harbor, Maine, $400, for the purpose of aiding an investigation into the inheritance of the susceptibility to spontaneous mammary tumors in a mouse interspecific cross.

4. To Prof. Hudson Hoagland, Clark University, Worcester, $475, to aid in providing him with an amplifier, photographic supplies, tubes, batteries, etc., these being certain necessary pieces of equipment for a quantitative analysis of the frequencies of nerve impulses from spontaneously discharging neuromasts in fishes, which will show the receptive function of these cells.

5. To Dr. R. J. Van de Graaff, Research Associate, Dept. of Physics, Massachusetts Institute of Technology, $500, for the purpose of aiding in the construction of a new source of X-rays for treating cancer and other diseases.

The President reported the death of three Fellows:—Eugene Watson Burlingame (Class IV, Section 3), William Henry Holmes (Class IV, Section 4) and George Herbert Palmer (Class IV, Section 1).

He also made remarks on the recent expansion of the Academy to eight hundred Fellows and suggested that some of the meetings might be held on the third floor. He expressed the hope that money might be found for increased publications in the humanities.

The following report of the Council was presented:

REPORT OF THE COUNCIL.

Since the last report of the Council there have been reported the deaths of twenty-two Fellows:—Nathaniel Allison, Eugene Watson Burlingame, John Joseph Carty, Arthur Messinger Comey, John Ripley Freeman, Elijah Clarence Hills, William Henry Holmes, Edward Washburn Hopkins, William White Jacques, Charles Willison Johnson, William Williams Keen, Oliver Dimon Kellogg, Eliakim Hastings Moore, George Herbert Palmer, William Patten, William Lambert Richardson, James Hardy Ropes, George Augustus Sanderson, Arthur Bliss Seymour, Edward Wyllys Taylor, William Sydney Thayer, Henry Pickering Walcott; and four Foreign Honorary Members:—John Galsworthy, Sir Robert Jones, Jean Adrien Jules Jusserand, William Cawthorne Unwin.

Ninety-nine Fellows and thirteen Foreign Honorary Members were elected by the Council and announced to the Academy in May 1932.

The roll now includes 675 Fellows and 105 Foreign Honorary Members (not including those elected in May 1933).

The annual report of the Treasurer, Ingersoll Bowditch, was read, of which the following is an abstract:

GENERAL FUND.

Receipts.

Income on hand April 1, 1932		$ 1,838.66	
From Investments	$2,802.90		
From Assessments	5,640.00		
From Admissions.	420.00		
From Interest on Deposits	139.35		
From Rumford Fund Income	200.00	9,202.25	$11,040.91

Expenditures.

Expenses of Library	$2,024.00		
Treasurer's Expenses	1,152.05		
Books and Binding	1,460.14		
General Expenses	856.26		
House Expenses	2,693.78		
President's Expenses	45.00		
Interest on Bonds bought	165.01	$ 8,396.24	
Income transferred to Principal		364.75	$ 8,760.99

RUMFORD FUND.

Receipts.

Income on hand April 1, 1932	$ 1,221.28	
From Investments	3,737.65	$ 4,958.93

Expenditures.

Purchase and Binding of Books	$ 404.14		
Research	2,450.00		
Transferred to General Fund Income for care of books	200.00		
Discretion of Committee—bal. 1931 transferred to Capital	129.17	$ 3,183.31	
Interest on Bonds bought	12.44		
Income transferred to Principal	194.53	206.97	$ 3,390.28

PUBLICATION ACCOUNT.

Receipts.

Income on hand April 1, 1932		$11,343.34	
From Income Appleton Fund	$ 855.49		
From Income Centennial Fund	2,122.74		
From Authors' Reprints	66.50		
From Sale of Publications	436.99	3,481.72	$14,825.06

Expenditures.

Publications	$6,090.22		
Vault Rent—part	10.00	$ 6,100.22	
Interest on Bonds bought	—.11		
Income transferred to Principal	172.75	172.86	$ 6,273.08

C. M. Warren Fund.

Receipts.

Income on hand April 1, 1932	$ 960.80	
From Investments	1,141.12	$ 2,101.92

Expenditures.

Research	$2,060.00		
Vault Rent—part	3.00	$2,063.00	
Income transferred to Principal		61.00	$ 2,124.00

Francis Amory Fund.

Receipts.

From Investments	$ 2,433.00

Expenditures.

Publishing Statement	$ 55.50	
Income transferred to Principal	1,676.00	
Interest on Bonds bought	62.22	$ 1,793.72

Permanent Science Fund.

Receipts.

Received for above fund	$ 5,557.25

Expenditures.

Grants from above fund	$ 5,725.00	
Application blanks for grants	57.25	$ 5,782.25

The following reports were also presented:

REPORT OF THE LIBRARY COMMITTEE.

During the year 184 volumes and thirty-six unbound numbers of periodicals have been borrowed by 20 different Fellows and 21 different libraries, and many more have been consulted at the Academy. All books taken out have been returned or satisfactorily accounted for. It may be of interest to note that while the number of Fellows borrowing books has not greatly increased in the past ten years, the number of libraries has quadrupled.

The number of volumes on the shelves at the time of the last report was 43,029. During the year 488 volumes were added, largely by binding serials, making the number now 43,517. This includes 65 purchased from the General Fund, 30 from the Rumford Fund, and 392 received by gift or exchange. The number of pamphlets added was 76. Several publications of the American Institute of Physics have been added to the periodical subscription list:—Physics, Journal of Chemical Physics, Journal of the Acoustical Society of America, and Journal of the Optical Society of America. The Academy had previously been receiving their other publications:—Physical Review, Review of Physical Instruments, and Reviews of Modern Physics, which it will continue.

At a recent meeting of the Library Committee the question of the number of copies of the Academy's Proceedings and Memoirs which should be kept in reserve was discussed, and it was decided that no copies of any volume or number should be distributed when the number of copies in stock has been reduced to one hundred, without consulting the Library Committee, and that no copies should be sent out when the number is below fifty, without consulting the Council.

On making a survey of the present stock of the publications, it was found that some volumes and numbers are entirely exhausted, while in other cases the number of copies is much below one hundred.

Volumes 1 through 31 of the Proceedings were issued not in separate numbers as at present, but in whole or half volumes. Of these there is a reasonably liberal supply of volumes 1 through 26, and 28, but volumes 27, 29, 30, and 31 are entirely out of print.

Of the more recent volumes which were issued in separate numbers, the supply of the numbers in volumes 32 through 47 (with a few exceptions) is decidedly small, while some are exhausted. Beginning with volume 48 and continuing to date, the stock is adequate, with a few exceptions noted below:

Vol. 48, no. 2
" 57, " 7
" 60, " 14
" 61, " 3, 4, 5, 12
" 62, " 2, 3, 9

With some exceptions the stock of the Memoirs is fair, though of some of the earlier volumes there are less than a hundred copies. Of volume 3 of the Old Series, published in 1809, there are only a few copies, while the numbers of volumes 12 and 13 of the New Series are in some cases completely, in others almost wholly, exhausted.

If any Fellows have copies of back numbers of the Memoirs or Proceedings for which they have no use, especially of volumes 32–47, 1896–1912, the Committee would greatly appreciate it, if these copies could be returned to the Academy to add to the stock.

Mrs. Ball, who is the real librarian, has kept everything in order, has done everything desired and has been most obliging and efficient.

The following appropriations were placed at the disposal of the Librarian during the past year:

Balance from General Fund	$ 390.09
Balance from Rumford Fund	135.61
Appropriation from General Fund	3,400.00
Appropriation from Rumford Fund	350.00
Total	$4,275.70

The expenses charged to the library during the financial year ending March 31, 1933, were:

Salaries	$2,000.00
Binding:	
General Fund	$ 719.75
Rumford Fund	92.40
	$ 812.15

Purchase of Books and Periodicals:

General Fund	$ 740.39
Rumford Fund	311.74
	$1,052.13
Miscellaneous	24.00
Total	$3,888.28

There remained an unexpended balance of $387.42, as follows:

General Fund	$ 305.95
Rumford Fund	81.47
Total	$ 387.42

As usual this balance will be carried over to next year.

Respectfully submitted,

ALFRED C. LANE, *Librarian*.

May 10, 1933.

REPORT OF THE RUMFORD COMMITTEE.

The Rumford Committee held six meetings during the Academic year April 1, 1932–March 31, 1933, on April 13, May 19, Oct. 9, Nov. 14, Feb. 1, and March 8.

The Committee made the following grants, during the year, from the Rumford Fund, in aid of researches in light and heat.

1932		*Amount*
May 11.	No. 298. Prof. Kenrick, Tufts College, for equipment in research on the Effect of Solar Radiation on the Kennelly-Heaviside Layer	$ 400
May 27.	No. 299. Dr. Willi Cohn, c/o A. Little & Co., Cambridge, for equipment in a research on the Polarisation of Light from the Solar Corona	250
Oct. 19.	No. 300. Dr. Cecilia Payne, Harvard Observatory, for equipment in research in Stellar Photometry and Colorimetry	200

1932		*Amount*
Oct. 19.	No. 301. Prof. R. H. Frazier, Mass. Institute of Technology, for equipment in researches on the Thermal Properties of Metals.	100
Dec. 14.	No. 302. Prof. D. C. Stockbarger, Mass. Institute of Technology, for equipment in research on the Production of Artificial Crystals in Optical Apparatus	500
Dec. 14.	No. 303. Prof. A. C. Hardy, Mass. Institute of Technology, for equipment in Colorimetric Research.	500
Dec. 14.	No. 304. Prof. J. R. Roebuck, University of Wisconsin, for equipment in Joule-Thomson Measurements	100
	Total	$2,050

The Committee has voted twice unanimously, and recommended the award of the Rumford Medal by the Academy, at the meeting of May 10, 1933, to Prof. Harlow Shapley, Director of the Harvard Observatory, "for Researches on the Luminosity of Stars and Galaxies." The Committee also recommends that this medal be delivered to Prof. Shapley at a special Boston or Cambridge meeting, next December, to be arranged in association with the American Association for the Advancement of Science, and provisionally assigned to Wednesday, December 27, 1933.

Reports of progress in researches aided by Rumford grants have been recently received from W. Cohn, F. H. Crawford, G. H. Dieke, R. H. Frazier, W. R. Fredrickson, J. W. McBain, C. E. Mendenhall, H. H. Plaskett, J. R. Roebuck, F. E. Ross, H. Shapley, J. C. Stearns, H. T. Stetson, and D. C. Stockbarger.

The following recent publications concern researches that have been aided by grants from the Rumford Fund.

Nos.

284. By Prof. T. C. Poulter, Iowa Wesleyan College, "Lens Effect of Pressure Windows," and "Permeability of Glass and Fused Quartz to Ether, Alcohol, and Water," *Physical Review*, June 1, 1932.

Nos.

304. By Prof. J. R. Roebuck, "The Joule-Thomson Effect in Helium," *Physical Review*, Jan. 1932.

298. By Prof. G. W. Kenrick and G. W. Pickard, "Observations of the Effective Height of the Kennelly-Heaviside Layer and Field Intensity during the Solar Eclipse of August 31, 1932," *Proc. Inst. Radio Engrs.*, April 1933, Vol. 21, No. 4, pp. 545–566.

294. By Prof. Harlan T. Stetson, Ohio Wesleyan University, "Sun Spots and Radio Reception," Smithsonian Report, 1931, pp. 215–228.

293. By Prof. P. W. Bridgman, Harvard University, "The Compressibility of Eighteen Cubic Compounds," *Proc. Am. Ac. of Arts & Sc.*, Vol. 67, No. 9, Dec. 1932, pp. 345–375.

288. By Prof. J. C. Stearns, Univ. of Denver, *Physical Review*, 1932.
    (a) "Diurnal Variation of Cosmic Rays," with others, Vol. 41, No. 2, pp. 119–126.
    (b) "Factors Influencing Ionisation Produced by Cosmic and Gamma-Rays," Vol. 40, No. 4, May 15, pp. 636–637.
    (c) "Solar Component of Cosmic Rays," Sept. 1932, Vol. 42, No. 2, pp. 317–318.

Respectfully submitted,
ARTHUR E. KENNELLY, *Chairman*.

REPORT OF THE C. M. WARREN COMMITTEE.

The Committee had at its disposal at the beginning of the fiscal year 1932–1933, $2,116.80, of which $2,060 has been appropriated during the past year.

Since the last annual report grants have been made as follows:

May 13, 1932: To Professor Marston T. Bogert, Columbia University, $150, to pay for analytical work needed for the publication of certain researches.

May 13, 1932: To Professor Henry Gilman, Iowa State College, $300, for the purchase of chemicals to be used in connection with his study of organometallic compounds.

May 13, 1932: To Professor Harold Hibbert, McGill University, $400, to be used in connection with $1,000, obtained elsewhere, for the purchase of apparatus needed in his research on polysaccharides.

May 13, 1932: To Professor Carl L. A. Schmidt, University of California, $60, for the purchase of a galvanometer to be used in his study of the physical chemistry of certain proteins.

May 28, 1932: To Professor W. E. Bradt, State College of Washington, $300, to be used to purchase apparatus needed in the study of the electrochemical properties of the higher melting intermetallic compounds.

May 28, 1932: To Professor Edward Kremers, The University of Wisconsin, $250, to study the derivatives of the cymene, thymol and carvacrol, hydrothymoquinone, thymoquinone, etc. group to be followed up by bacteriological and pharmacological studies.

December 23, 1932: To Dr. C. R. Johnson, Princeton University, $100, to be used in the purchase of platinum apparatus required in the determination of certain atomic weights.

December 23, 1932: To Professor E. L. Skau, Trinity College, $100, to be used in the purchase of liquid air and other materials necessary in his investigation on the purification of organic compounds and the determination of their physical properties.

December 23, 1932: To Dr. Hermann Schmid, Technische Hochschule, Vienna, $100, to be used for apparatus needed in his work on the study of intermediate substances in chemical reactions.

December 23, 1932: To Professor Wm. Lloyd Evans, The Ohio State University, $300, to pay for the services of a skilled assistant to continue his work on the action of an aqueous solution of potassium hydroxide on l-Rhamnose and the behavior of aqueous solutions of potassium hydroxide on Wohl's methyl triose.

Fifteen applications are before the Committee for consideration at its next meeting.

Reports of progress, during the current year, have been made by Professors Allen, Bogert, Gilman, Johnson, Schmid, Schmidt, Skau.

The papers listed below, which have been published since the last report of the Committee, describe the results of investigations aided by the Warren Fund. In each case there is an acknowledgment by the author of the assistance received.

C. F. H. Allen and E. W. Spanagel:

(A Communication to the Editor)

"The Structure of the Diketone Obtained from Anhydracetonebenzil."

C. F. H. Allen and E. W. Spanagel:

"The Reactions of Anhydracetonebenzil with Certain Halogen Compounds." J. Am. Chem. Soc., *54*, 4338 (1932).

C. F. H. Allen, G. F. Frame, J. B. Normington and C. V. Wilson:

"The Condensation of Certain -Ketonic Esters with Aromatic Aldehydes. II." Canadian Journal of Research, Vol. *8*, February, 1933.

Henry Gilman and R. H. Kirby:

"The Relative Reactivities of Organolithium and Organomagnesium Compounds." J. Am. Chem. Soc., *55*, 1265 (1933).

Willis D. Gallup and Carl L. A. Schmidt:

"The Action of Nitrous Acid on Salmin, Guanidine, Urea, and Certain Related Compounds." University of California Publications in Physiology, Vol. *7*, No. 12, pp. 201–210, August, 1930.

Evald L. Skau:

"The Purification and Physical Properties of Organic Compounds. I. The Interpretation of Time-Temperature Curves in Freezing Point Determinations and as a Criterion of Purity." Proceedings of the American Academy of Arts and Sciences, Vol. *67*, No. 12, January, 1933.

Evald L. Skau and Blair Saxton:

"Some Errors Inherent in the Usual Determination of the Binary Freezing Point Diagram." Journal of Physical Chemistry, Vol. XXXVII, No. 2, February, 1933.

Evald L. Skau and Blair Saxton:

"The Freezing Point-Solubility Relations of Geometrical Isomers. II. The Dynamic Isomerism of the Anisaldoximes." Journal of Physical Chemistry, Vol. XXXVII, No. 2, February, 1933.

David Davidson and Marston T. Bogert:

"Isovioluric Acid (Alloxan-6-Oxime)." Proceedings of the National Academy of Sciences, Vol. *18*, No. 7, pp. 490–496, July, 1932.

Respectfully submitted,

James F. Norris, *Chairman.*

May 10, 1933.

REPORT OF THE COMMITTEE OF PUBLICATION.

The Committee of Publication has held one meeting during the year with all members present. Most of the work of the committee has been carried on by correspondence.

During the year 1932–33 one memoir and fourteen numbers of the Proceedings have been issued. These include numbers 5 to 13 of volume 67 and numbers 1 to 5 of volume 68. Three papers are now in press. A memoir by Dr. L. R. Cleveland has been accepted and the sum of $3,000 set aside for its publication. The Committee on the Permanent Science Fund will contribute an additional $1,000 for this purpose. The Committee of Publication has expressed its willingness to spend $500 annually towards the publication of Professor Lake's collection of Greek minuscule manuscripts. As usual it has been found necessary to decline publication of a number of manuscripts submitted by non-members of the Academy.

The Treasurer renders the following financial statement for the year April 1, 1932 to April 1, 1933.

Receipts.			
Balance April 1, 1932	$10,338.47		
Appropriation 1932–33.	3,282.25		
Rumford Fund Balance	1,004.87		
Receipts from sales.	503.49		
Available for publication 1932–33		$15,129.08	
Expenses.			
Publication Fund Expenditures	$ 5,345.24		
Rumford Fund Expenditures	754.98		
Total		6,100.22	
Unspent Balance April 1, 1933			$ 9,028.86
Appropriation 1933–34.			2,223.00
Available for publication 1933–34.			$11,251.86

The committee desires to express its appreciation of the generous financial assistance given by the Rumford Committee, by the Committee on the Permanent Science Fund, and by the American Council of Learned Societies. Mr. Marshall, representing the latter organization, is helping the committee to secure a suitable publisher of the Greek minuscule manuscripts collected by Professor Kirsopp Lake.

In accordance with recommendations made by the special committee appointed by President Ford it is planned to distribute annually to all Fellows of the Academy and to all Foreign Honorary Members a copy of the records number of the Proceedings. With the assistance of the Secretaries a special brochure giving a general account of the nature of the Academy and the scope of its activities is to be prepared for distribution.

Respectfully submitted,
H. V. NEAL, *Chairman.*

May 10, 1933.

REPORT OF THE HOUSE COMMITTEE.

The House Committee has had funds at its disposal amounting to $3,369.26, made up as follows:

Balance from previous year	$ 400.35
Appropriations for 1932–33	2,700.00
Received for use of rooms and telephone	268.91
	$3,369.26

Of this amount the sum of $2,259.73 has been spent for the routine expenses, janitor, light, power, heat, telephone, etc., and $702.96 has been spent for upkeep and equipment, making a total of $2,962.69, and leaving an unexpended balance of $406.57. The chief items of upkeep this year have been the re-caning of the lecture hall chairs and extensive repairs on the metal and glass front doors of the Academy's house. Quite a saving has been effected by using coke instead of anthracite coal for heating, with satisfactory results.

Meetings have been held as follows:

The Academy	8
American Antiquarian Society	1
American Chemical Society, Northeastern Section	6
Archaeological Institute of America, Boston Society	2
Harvard-Technology Chemical Club	1
Japan Society of Boston	3
Massachusetts Girl Scouts	2
Mediaeval Academy of America	1
New England Botanical Club	9
Total	33

The Council Chamber has been used for Academy Council and Committee meetings, and also by the Trustees of the Children's Museum, the New England Farm and Garden Association, the Thursday Evening Club, etc.

A detailed list of expenditures follows:

Janitor		$ 976.00
Electricity	Light	233.10
	Power	108.12
Fuel		563.95
Elevator		88.25
Gas		65.10
Telephone		108.84
Water		77.28
Ash tickets		9.90
Upkeep		691.96
Furnishing and equipment		11.00
Janitor's supplies and sundries		29.19
Total		$2,962.69

Respectfully submitted,
S. BURT WOLBACH, *Chairman.*

May 10, 1933.

It was *Voted,* To award the Rumford Premium to Dr. Harlow Shapley for researches on the luminosity of stars and galaxies.

On the recommendation of the Treasurer, it was *Voted*, That the annual assessment for the ensuing year be $15.

On the recommendation of the Treasurer, it was *Voted*, That the Statutes be amended so as to allow the appointment of a special committee to administer the Amory Fund.

The annual election resulted in the choice of the following officers and committees:

George H. Parker, *President*
Harry M. Goodwin, *Vice-President for Class I*
Walter B. Cannon, *Vice-President for Class II*
Edwin F. Gay, *Vice-President for Class III*
Arthur S. Pease, *Vice-President for Class IV*
Tenney L. Davis, *Corresponding Secretary*
Walter E. Clark, *Recording Secretary*
Ingersoll Bowditch, *Treasurer*
Alfred C. Lane, *Librarian*
Herbert V. Neal, *Editor*

Councillors for Four Years.

John C. Slater, of Class I
John W. M. Bunker, of Class II
John H. Williams, of Class III
William C. Greene, of Class IV

Finance Committee:

Thomas Barbour
Paul J. Sachs
Alfred L. Ripley

Rumford Committee.

Arthur E. Kennelly
Elihu Thomson
Percy W. Bridgman
Harry M. Goodwin
Charles L. Norton
Harlow Shapley
Norton A. Kent

C. M. Warren Committee.

James F. Norris
Gregory P. Baxter
Walter L. Jennings
Arthur D. Little
Frederick G. Keyes
Reid Hunt
Charles A. Kraus

Committee of Publication.

HERBERT V. NEAL, *Chairman*

EDWIN C. KEMBLE, of Class I	JOSEPH H. BEALE, of Class III
FREDERIC T. LEWIS, of Class II	ROBERT P. BLAKE, of Class IV

Committee on the Library.

ALFRED C. LANE, *Chairman*

RAYMOND C. ARCHIBALD, of Class I	WILLIAM Y. ELLIOTT, of Class III
THOMAS BARBOUR, of Class II	HENRY B. WASHBURN, of Class IV

Auditing Committee.

GEORGE R. AGASSIZ
JOHN E. THAYER

House Committee.

S. BURT WOLBACH, *Chairman*

WILLIAM H. LAWRENCE
ROBERT P. BIGELOW
DAVIS R. DEWEY

Committee on Meetings.

THE PRESIDENT
THE RECORDING SECRETARY
DANIEL F. COMSTOCK
LEIGH HOADLEY
CHARLES J. BULLOCK
JOSHUA WHATMOUGH

The Corresponding Secretary announced that the following had been elected members of the Academy:

FELLOWS.

CLASS I.

Section 1.	Jesse Douglas, Cambridge
	Marshall Harvey Stone, New Haven, Conn.
Section 3.	Louis Frederick Fieser, Waltham
	George Bogdan Kistiakowsky, Cambridge
	Henry Weeden Underwood, Jr., Cambridge
Section 4.	Harold Kilbrith Barrows, Winchester
	Arthur Edwin Norton, Cambridge
	Frank Edward Winsor, West Newton

CLASS II.

Section 1. Charles Franklin Brooks, Milton
Frank Morton Carpenter, Cambridge
Section 3. Philip Bard, Cambridge
Alden Benjamin Dawson, Cambridge
Frank Rattray Lillie, Chicago, Ill.
Jeffries Wyman, Jr., Chestnut Hill
Section 4. Reginald Fitz, Brookline
Edgar Erskine Hume, Washington, D. C.
George Burgess Magrath, Boston
George Cheever Shattuck, Brookline
Soma Weiss, Cambridge

CLASS III.

Section 1. Harry Augustus Bigelow, Chicago, Ill.
Henry Wolf Biklé, Philadelphia, Pa.
Benjamin Nathan Cardozo, Washington, D. C.
John Dickinson, Philadelphia, Pa.
Theodore Francis Green, Providence, R. I.
Learned Hand, New York, N. Y.
Harlan Fiske Stone, Washington, D. C.
Section 2. Sir Herbert Brown Ames, Brookline
George Hubbard Blakeslee, Worcester
Edwin Montefiore Borchard, New Haven, Conn.
Robert Treat Crane, New York, N. Y.
Bruce Campbell Hopper, Cambridge
Henry Aaron Yeomans, Cambridge
Section 3. Harold Hitchings Burbank, Cambridge
Sheldon Glueck, Cambridge
Edward Sagendorph Mason, Cambridge
Robert Ezra Park, Chicago, Ill.
Josef Alois Schumpeter, Cambridge
William Isaac Thomas, New York, N. Y.
Abbott Payson Usher, Belmont
Section 4. George Hoyt Bigelow, Milton
James Vance May, Boston
Albert Warren Stearns, Billerica

Class IV.

Section 1. Gordon Willard Allport, Cambridge
John Gilbert Beebe-Center, Swampscott
James McKeen Cattell, New York, N. Y.
Curt John Ducasse, Providence, R. I.
Walter Samuel Hunter, Worcester
Lee Sullivan McCollester, Tufts College
Carroll Cornelius Pratt, Cambridge
Howard Crosby Warren, Princeton, N. J.
Frederic Lyman Wells, Newton Highlands
Section 2. Henry Thatcher Fowler, Providence, R. I.
Halford Lancaster Hoskins, Tufts College
Edward Sapir, New Haven, Conn.
Section 3. William Nickerson Bates, Philadelphia, Pa.
Campbell Bonner, Ann Arbor, Mich.
Carleton Brown, Upper Montclair, N. J.
Hans Kurath, Providence R. I.
Ivan Mortimer Linforth, Berkeley, Cal.
Howard Rollin Patch, Northampton
George Benson Weston, Cambridge
Harry Austryn Wolfson, Cambridge
Karl Young, New Haven, Conn.
Section 4. John Alden Carpenter, Chicago, Ill.
Charles Collens, Newton Center
Cyrus Edwin Dallin, Arlington
Carl Engel, Washington, D. C.
Leo Rich Lewis, Tufts College
David Stanley Smith, New Haven, Conn.
Charles Herbert Woodbury, Boston

FOREIGN HONORARY MEMBERS.

Class I.

Section 3. Jaroslav Heyrovsky, Prague, Czechoslovakia
Fritz Paneth, Königsberg, Germany

Class II.

Section 3. Hans Spemann, Freiburg i. B., Germany
Section 4. Sir Arthur Keith, London, England

Class III.

Section 1. Frantz Dahl, Copenhagen, Denmark
François Geny, Nancy, France
Hans Kelsen, Cologne, Germany
Julius Makarewicz, Lwów, Poland
Rudolf Stammler, Berlin, Germany
Giorgio Del Vecchio, Rome, Italy
Section 3. Werner Sombart, Berlin, Germany
Section 4. Gösta A. Bagge, Stockholm, Sweden
Sir Josiah Stamp, London, England

Class IV.

Section 1. Georg Elias Müller, Göttingen, Germany
Section 2. Rafael Altamira y Crevea, Madrid, Spain
Friedrich Meinecke, Berlin, Germany

The following communication was presented:

Mr. H. Helm Clayton: "Recent Studies of Solar and Weather Relationships."

Two papers were read by title: "A Suggestion concerning Plato's Atlantis," by W. A. Heidel; "The Significance of Cells as Revealed by their Polyhedral Shapes, with Special Reference to Precartilage, and a Surmise concerning Nerve Cells and Neuroglia," by Frederic T. Lewis.

The meeting was dissolved at 10.20 P. M.

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THE RT. HON. SIR CLIFFORD ALLBUTT (1836–1925).

Foreign Honorary Member in Class II, Section 4. 1922.

Sir Clifford Allbutt was born at Dewsbury in Yorkshire and died at Cambridge, England, February 22, 1925, in the eighty-ninth year of his life.

For years he was the acknowledged leader in British medicine and remained active in medical work up to the time of his sudden death after a few minutes of acute dyspnea. Allbutt, more than any other English speaking physician of his time, was an exponent of a clear, forceful and delightfully scholarly use of the English language. All that he wrote might be read with profit and pleasure both as a literary production and as a scientific presentation.

He entered Gonville and Caius College in 1857 and graduated in the first class of the Natural Science Tripos. His medical education was continued at St. George's Hospital in London and while there he graduated in medicine at Cambridge. Leeds became his residence and there he rapidly became a busy general practitioner, later with a huge consulting practice, while busied with the duties of physician to the Leeds General Infirmary, Leeds Hospital for Women and Children, and Leeds Fever Hospital. In 1868, while still at Leeds, he had an active part in the perfecting of the clinical thermometer as we know it today and was largely instrumental in its becoming generally used in medicine.

A little later in a somewhat similar way he busied himself with perfecting the ophthalmoscope and in impressing upon the medical profession its great value as an instrument for clinical use. In 1871 he published the first book on the use of the ophthalmoscope, in which he stressed particularly its value in the study of diseases of the nervous system and of the kidneys.

He became Commissioner in Lunacy and for three years admirably filled that difficult office just prior to going to Cambridge. He was well over fifty years of age when he went to Cambridge as Regius Professor of Physic, but in a surprisingly short time he took a front place in the academic life of that university. Then, about this time he began the editorship of a System of Medicine, which remains as one of the best of this type of medical work. He was particularly interested in diseases of the circulation and his two-volume work, "Diseases of the Arteries Including Angina Pectoris," remains today a mine of information on this subject. His views on high blood pressure were of great importance in developing our knowledge of cardiovascular disease in its forms which today are so prevalent and important as the cause of death in full vigor of middle age of so many important men from all branches of modern society.

In his long life of such great activity Sir Clifford Allbutt held many of the important offices in the various medical and scientific organizations of Great Britain, received numerous honorary degrees and was foreign member or corresponding member of many organizations.

He left an important impress on British medicine, which is reflected in American medicine. From May 10, 1922, until the time of his death he was a Foreign Honorary Member of the American Academy of Arts and Sciences.

Henry A. Christian.

NATHANIEL ALLISON (1876–1932).

Fellow in Class II, Section 4, 1925.

Nathaniel Allison, M.D., born in Webster, Missouri, was the grandson of Dr. Nathaniel Allison, pioneer physician of Audrain County, Missouri. He was the son of James Allison, successful merchant, and of Addie (Shultz) Allison. His progenitors were English, emigrating to America in colonial days and fighting in the War of Independence. His father, early in Dr. Allison's life, moved to Saint Louis and the son was educated at Smith Academy. When he was seventeen he received an appointment to West Point, but largely owing to admiration for his grandfather, he resigned from the military academy at the end of a year and determined to study medicine. Further study at the Penn Charter School in Philadelphia enabled him to enter Harvard University in 1894 and he was graduated from the medical school in 1901. The specialty of orthopaedic surgery attracted him and after post graduate study at the Boston Children's Hospital he returned to Saint Louis and began practice in his chosen field.

Almost at once his ability for research became apparent and remained of almost chief concern to him throughout his medical career. In 1909 he married Marion Aldrich of Chicago.

The respect for his ability steadily increased until before the great war he had become outstanding in his specialty and chief of the Department of Orthopaedic Surgery of Washington University and of its allied hospital clinics. In 1915, he served with the American Ambulance in Paris and in 1917 was commissioned as captain in the United States Army Medical Corps, sailing for France in May with the Washington University unit for service in a base hospital in Rouen. He was a member of an Army Board appointed to standardize splints and dressings, and after the publication of this manual was chiefly responsible in arranging for the manufacture and delivery of these necessities to our combat troops and to all overseas United States Army hospitals. This meritorious service was splendidly executed and he rose to the rank of Colonel and Assistant Director of the A. E. F. Section of Orthopaedic Surgery. On his return to America in 1919 he was assigned to the Walter Reed Army Hospital at Washington as assistant director of the surgical service. He was honorably discharged from the army and was sent in September 1919 to Rome as representative of the United States Medical Department at the Inter-allied Congress of Surgery. The Distinguished Service Medal was awarded him by the President of the United States.

Civil practice was resumed in Saint Louis and to his old teaching and hospital responsibilities were added those of the deanship of the Washington University Medical School. In 1923 he was offered the post of chief of the Orthopaedic Department of the Massachusetts General Hospital and a place on the faculty of the Harvard Medical School. These positions he accepted and he was made a full Professor of Orthopaedic Surgery in 1924. Dr. Allison was a vital part of the medical life of Boston for six years. In 1929 he was invited to occupy the chair of Professor of Surgery in charge of the Division of Orthopaedic Surgery at the Medical School of Chicago University. He served in this important capacity until 1932, when relapsing illness made it necessary for him to lay down his duties.

Dr. Allison's bibliography encompassed nearly every domain of orthopaedic surgery. He never lost his acute interest in research and exhibited an unique faculty for applying the results of research to the relief of physical handicap and disease.

The list of medical and civic societies to which he belonged is a long one and included the American Academy of Arts and Sciences, to which he was elected on May 13, 1925, The American College of Surgeons, The New England Surgical Society, The American Orthopaedic Association (President 1922), The International Society of Orthopaedic Surgery, The American Medical Association, The Southern Surgical Association, The Association of Military Surgeons, The Massachusetts Medical Society, The Massachusetts Association for Occupational Therapy (President 1925–1929), The St. Louis Surgical Society, The Boston Surgical Society, Honorary Membership in the British Orthopaedic Association, Corresponding Membership in the Société des Chirurgiens de Paris, and many other medical and urban clubs in St. Louis, Washington, and Boston.

It has been said of him that "wherever this genial but self-contained and unhurried physician practised he became a vital element in the professional life of the community and the trusted adviser of important medical institutions. He was independent in thought and action and by the exercise of his own wisdom and effort usually attained the end sought. He broadened the scope of orthopaedic surgery by research and increased its prestige by the respect in which he was held by medical and surgical colleagues; a respect engendered by his knowledge of men and of medicine."

ROBERT B. OSGOOD.

SOLON IRVING BAILEY (1854–1931).

Fellow in Class I, Section 1, 1892.

Born December 29, 1854, at Lisbon, N. H., son of Israel C. and Jane Sutherland Bailey; died at Norwell, Mass., June 5, 1931.

In 1859 the Bailey family moved from Lisbon to Concord, N. H., where Solon attended high school. He later entered Tilton Academy, graduating in 1877.

Degrees: A.B., 1881, A.M., 1884, Boston University; A.M., 1888, Harvard University.

From 1881 to 1884 he was headmaster of Tilton Academy. In 1883 he married Ruth E. Poulter of Concord, N. H. She and their son, Irving W. Bailey, Professor of Plant Anatomy at Harvard, survive him.

In 1887, he came to Harvard, becoming volunteer assistant at the Observatory.

University appointments:

Assistant in Astronomy	1891–1892
Assistant Professor	1893–1898
Associate Professor	1898–1912
Phillips Professor	1912–1925
Acting Director	1919–1921
Phillips Professor Emeritus	1925–1931

In 1888 he was sent to investigate the mountains on the west coast of South America as a possible site for an observatory. On his recommendation Arequipa, Peru (altitude 8,000 feet), was chosen as a site for a permanent station of the Harvard Observatory to be constructed from the Boyden Fund. In 1893 he established a meteorological station on the summit of El Misti at an elevation of 19,000 feet.

Professor W. H. Pickering took charge of the station at Arequipa from 1891 to 1893 while Professor Bailey was absent. Thereafter the station remained virtually in Professor Bailey's care until 1924. He alternated his time between Cambridge and Peru, his last visit to the latter being in 1922.

"His name should always be honorably associated with the great development of astronomy which began during the last decade of the nineteenth century, for his pioneer labors in the photography of the southern stars helped to open up the whole sky to modern astronomers, and his early investigations of variable stars in globular clusters provided an important foundation stone for the structure of the new cosmogony." And in meteorology his study of Peruvian meteorology, based on ten years of records from El Misti and other stations he established at various altitudes down to sea level, is unique.

In 1908 his investigations of the meteorological conditions in South Africa were the basis upon which many observatories selected sites for South African stations.

Memberships: Fellow Am. Acad. of Arts and Sciences; A. A. A. S.;

Washington Acad. Sciences; mem. Nat. Acad. Sciences; Am. Astron. Soc.; Geog. Soc. of Lima; Int. Astron. Union; Astronomische Gesellschaft; Asso. Royal Astron. Soc.

He received an honorary Sc.D. and the title of Professor of Astronomy in the University of San Augustin, Peru, in 1923.

"Those who have known him as the courteous gentleman, the genial host, the wise counsellor, and wholly delightful comrade will sorely miss the inspiration of his great soul."—Professor E. S. King.

L. B. ANDREWS.

SIMEON EBEN BALDWIN (1840–1927).

Fellow in Class III, Section 1, 1912.

Simeon E. Baldwin was born in the year 1840, graduated at Yale in 1861, and studied law at the Yale and the Harvard Law schools. He was a member of the faculty of the Yale Law School from 1869 to shortly before his death. He was a justice of the Supreme Court of Errors of Connecticut in 1893 and chief justice in 1907, until in 1910 he was forced to retire because of the compulsory retirement law of Connecticut. He was one of the founders of the American Bar Association and at one time its president. He received the degree of Doctor of Laws from Harvard University in 1891.

After his retirement from the Supreme Court he was elected governor and served with success for the usual number of terms. He died in 1927 at the age of 87 years, being vigorous to the end.

Judge Baldwin was a man of strong character and firmly-held opinions; an excellent lawyer, a good administrator and a man of very high character and interesting personality. He was a very vigorous man and delighted in walking and climbing. His vigor colored his opinions as judge and his teaching as a professor of law and his action as governor. He could be wrong at times, and he was just as persistent in error as he was in the truth, but being a sound and learned lawyer he was seldom in error. He was greatly loved by his colleagues and his students and there was universal dissatisfaction at the time of his forced retirement from the bench. Judge Baldwin was elected a non-resident member of the Academy in May 1912.

J. H. BEALE.

LOUIS AGRICOLA BAUER (1865–1932).

Fellow in Class I, Section 2, 1912.

He was born in Cincinnati, January 26, 1865. He received the degree of C.E. from the University of Cincinnati in 1888 and the Ph.D. from Berlin in 1895. He was a teacher for some years, first at the University of Chicago, later at the University of Cincinnati, but from 1899 on, most of his work was done in the U. S. Coast and Geodetic Survey or in the service of the Carnegie Institution at Washington. His special field was terrestrial magnetism. His work in this field was conspicuous and gained for him wide recognition both at home and abroad.

EDWIN H. HALL.

MELVILLE MADISON BIGELOW (1846–1921).

Fellow in Class III, Section 1, 1911.

Melville Madison Bigelow was born in Michigan, August 2, 1846. He received his degree of A.B. in 1866, LL.B. in 1868 and A.M. in 1871 at the University of Michigan. Coming to Boston directly after leaving his university, he presented, in 1872, a scheme for the organization of the law school in the Boston University, largely on the current model of the Michigan Law School, and was made professor there and so continued for 50 years. During that period he was dean for 9 or 10 years and upon leaving the deanship he became a graduate instructor. He died in the year 1921.

Professor Bigelow was a lawyer, a teacher, an historian and a poet. For the degree of Ph.D. in Harvard he prepared, and in 1872 published, his Placita Anglo Normanica, and a few years later his History of English Procedure. He was a valued member of the Massachusetts Historical Society and was always interested in the historical aspects of law. He was also an author of professional treatises on the law, and in that way, between 1872 and 1898, he published treatises on Estoppel, Torts, Bills and Notes, Fraud, and Wills, which were kept alive by frequent editions. In his later years he turned to legal philosophy, publishing books on the Legal History of Government, and was one of the authors of Centralization and the Law.

Professor Bigelow was essentially a scholar. He taught all his life a group of students not especially interested in the scholarly aspects of the law, and succeeded in winning their respect, their admiration and their affection. For many years before his death he was invested by his students with a halo of legal sanctitude, and his position in his law school was a happy as well as an honorable one. In the profession at large his place was very high and he was honored for his scholarship in his own country and abroad. He was elected a Fellow of the Academy in October 1911.

J. H. Beale.

ROLAND WILLIAM BOYDEN (1863-1931).

Associate, 1928; Fellow in Class III, Section 4, 1931.

Roland W. Boyden was born in Beverly, October 18, 1863. He received his bachelor's degree in arts from Harvard College in 1885 and his bachelor's degree in law from the Harvard Law School in 1888. He was made doctor of laws by Harvard in 1924.

From 1894 to his death he was a member of the firm of which the latest title was Ropes, Gray, Boyden & Perkins. In 1917 to 1919 he was counsel for the United States Commission, and for several years later was the unofficial observer for the United States with the Reparations Committee, and took an important part in the deliberations of that body. In 1930 he became a member of the Permanent Court of Arbitration at the Hague, and was a member of the Mixed Claims Commission between Germany and America. From 1924 to 1930 he was an Overseer at Harvard College. In granting his honorary degree, President Lowell characterized him as "an American whose patriotism is not bounded by the shores of his native land, who strove to bring order into the financial chaos of a weary world." He died suddenly, October 25, 1931.

Throughout his professional life Boyden was characterized by Yankee shrewdness and good business sense. He was tolerant of others' opinions, but clear and firm in his own. His advice was sought and followed by all his associates, and he became the outstanding man in Massachusetts in all legal relations with Europe.

He received numerous foreign decorations. Just before his death he had been selected as president of the Reparations Arbitral Commission to settle disputes under the Dawes and Young Plan.

Boyden was full of the simplicity and sincerity of his race. Always interested in athletics, he would take the time from a puzzling case to talk baseball or to gossip about old times. It was well said of him that "he liked good wine, good food and human companionship, and he brought to his amusements a zest and energy as great as that he showed in his work." The characteristics of friendliness and simplicity proved very attractive in his public work, and all his associates in that work will greatly miss his cordiality and sympathetic dealing with public problems.

J. H. BEALE.

HENRY ANDREWS BUMSTEAD (1870–1920).

Fellow in Class I, Section 2, 1914.

My personal acquaintance with Henry A. Bumstead dates from a meeting of the British Association in Winnipeg in the summer of 1909. He had studied in Cambridge, England, where his engaging personality, keen intelligence, and unusual *savoir faire* had made him a place in the hearts and homes of English scientists which has been held by few Americans. I was then almost unknown both to him and to them, but I soon learned that if Bumstead was in any gathering I should at once feel at home.

I was walking with him one day through one of the busy streets of Winnipeg when he asked if I would not step into a shop with him while he bought a little memento for Mrs. Bumstead, a "bad habit" which he said he had formed on trips away from home.

I mention these two trivial incidents because they reveal the soul and heart of the man; and what, after all, is either science or art in comparison?

When in 1917 the important and difficult post of scientific attaché in London was created, Bumstead was the only man considered, for no scientist in this country had his tact, his judgment, his knowledge of England, and his ability to assist in bringing about what was then, and what is now, the most important need of the modern world, namely, the coöperation and mutual understanding of the two great branches of the Anglo-Saxon race.

Bumstead's success in London was extraordinary. The British liked and trusted him. Admiral Sims and our own War Department placed large responsibilities upon him, and his office became the center of a very active and very important service. Young American officers who went abroad on scientific missions found him the center of their contacts and the prime source of their usefulness. They all became his devoted admirers. Not one or two but a dozen or more of both British and American officers who came to Washington during the war told me that they owed their success primarily to Bumstead, and counted it the most valuable part of their experience that they had had an opportunity to become acquainted with him. One of these officers described him as the most influential American in England.

As Chairman of the National Research Council, as member of the National Research Fellowship Board, and as participant in other important groups with which he was associated at the time of his death, Bumstead showed the same broad outlook, the same big human interest, the same tact, the same sane intelligence and sound judgment which had characterized his work in England.

He spent practically the whole of the holiday week, 1920, at my home in Chicago in attendance upon the meetings of the Physical Society and of various committees of which he was a member. He was apparently in the best of health and spirits. Indeed, he spent Friday morning, December 31, going over with me the research work of the Ryerson Laboratory, and as we chatted together before he left about future plans he remarked that since his last operation some four years ago he had been feeling in excellent condition. He left me at about 11:30, intending to take the afternoon train for Washington. The next morning Dr. Vernon Kellogg, who occupied the berth opposite him, attempted to waken him and found that he had gone.

His passing has left a big gap in the ranks of American physicists. He was born on March 12, 1870, in Pekin, Illinois, and educated in the public schools of Decatur, from which he went first to Johns Hopkins, where he took his Bachelor of Arts degree in 1891, and then to Yale, which conferred upon him the degree of Doctor of Philosophy in 1897. From 1891 to 1900 he was connected with the Sheffield Scientific School of Yale, first as assistant and later as instructor, and from 1900 to 1906 he was assistant professor at Yale. In 1906 he became a full professor and director of the Sloane Physical Laboratory, which position he held till the time of his death. He was a very influential member of the Yale faculty, a member of the National Academy of Sciences, a member of the American Physical Society, of which he was President in 1918, a member of the Connecticut Academy of Sciences, the American Philosophical Society, and a Fellow of the American Academy of Arts and Sciences. He had a brilliant analytical mind, profound scholarship, exceptional critical capacity, excellent judgment, an extraordinarily winsome personality, the finest culture, and a great heart. His personal scientific contributions were important, though they had been much interfered with by his none too rugged health. His effect upon American physics, however, was not limited to his own scientific papers, but he exerted a powerful influence upon his pupils and upon his fellow physicists.

It is not merely American science, however, which can ill afford to lose him twenty years before his time. American life in all its aspects is sadly in need of men of Bumstead's type. The cause of sanity, of culture, of Anglo-Saxon solidarity, of scholarship, of science, of world civilization, all suffer irreparably through his death.

R. A. MILLIKAN.

FLORIAN CAJORI[1] (1859–1930).

Fellow in Class I, Section 1, 1921.

Florian Cajori, professor emeritus of mathematics at the University of California, died at his home in Berkeley, 14 August 1930, after a brief attack of pneumonia, in the seventy-second year of his age. For forty years his name had been a familiar one throughout the world, recalling a genial personality and a prolific writer on various topics connected with the history of mathematics and physics.

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[1] This sketch is an abridgment of one by the writer which appeared in *Isis* in April 1932, v. 17, p. 386–407 + portrait plate; a complete bibliography of 286 titles is given on pages 389–407. Other published sources of information concerning Professor Cajori are: *American Men of Science*, fourth ed., New York, 1927. *Who's Who in America*, v. 16 (1930–1931). L. G. Simons, *Amer. Math. Mo.*, v. 37, Nov. 1930, p. 460–462. D. E. Smith: (a) *Science*, n. s., v. 72, 19 Sept. 1930, p. 287–288; (b) Amer. Math. So., *Bull.*, v. 36, Nov. 1930, p. 777–780, portrait; (c) *Archeion, Archivio di Storia della Scienza*, v. 12, 1930, p. 369–371.

He was born at St. Aignan near Thusis, Switzerland, 28 February 1859. His mother was Catherine (Camenisch), and his father was Georg Cajöri (the family name was spelled thus, or as Cajoeri), an engineer and contractor, well known in his part of Switzerland for important bridges and highways that he built.

Professor Cajori received his early education at Zillis, and in the canton school at Chur. At the age of 16 years he came to America, an elder brother having preceded him, and made his way at once to Whitewater, Wisconsin, where he continued his education at the normal school during the years 1876–78. After some teaching in a country school he entered the University of Wisconsin where he received the degrees of Bachelor of Science in 1883, and of Master of Science in 1886. From January 1884 to June 1885 he was a graduate student of mathematics at The Johns Hopkins University, and received his first academic appointment in the following autumn. This was at Tulane University, New Orleans, where he was an assistant professor of mathematics 1885–87, and professor of applied mathematics 1887–88. Since, on account of his health, a change of climate seemed very desirable, he spent the next year in collecting material for a forthcoming work and accepted a position at Colorado College, Colorado Springs, where he was first a professor of physics 1889–98, then professor of mathematics 1898–1918, and dean of the department of engineering 1903–18. During these twenty-nine years not only did he always carry a heavy teaching load, but also served in various administrative capacities for which he was ideally qualified. But in spite of all such demands upon his time, even though far removed from adequate library facilities, he achieved a remarkable quantity of publication. Throughout life he possessed to an extraordinary degree the ability to utilize every spare minute, taking up intensively at a moment's notice the consideration of any particular question. He was greatly interested in college athletics and followed enthusiastically both the regular and practice games, especially in football. He always walked a great deal in the mountains near

Colorado Springs, and later, to a more limited extent, in the Berkeley Hills.

Some of Cajori's principal publications of this period were *The Teaching and History of Mathematics in the United States* (Bureau of Education, Washington, D. C., 1890, 400 p.), involving a great amount of careful research; *History of Mathematics* (1894, 422 p.; second ed. revised and enlarged, 1919, 524 p.); *A History of Elementary Mathematics with Hints on Methods of Teaching* (1896, 312 p.; revised and enlarged ed., 1917), translated into Russian (1910) and Japanese (1928); *A History of Physics in its Elementary Branches including the Evolution of Physical Laboratories* (1899, 330 p.; revised and enlarged ed., 1929, 437 p.; two Italian editions, 1909 and 1929); *An Introduction to the Modern Theory of Equations* (1904, 249 p.); and in 1908 the excellent 160-page section on "Arithmetik. Algebra Zahlentheorie" in volume 4 of Cantor's monumental work *Vorlesungen über Geschichte der Mathematik.*

With such a record to his credit the University of California but extended her own prestige by appointing Cajori, 1 July 1918, professor of the history of mathematics, one of the few chairs of the kind in the world. Freed at last from all administrative duties, and enthusiastic over the opportunity presented for devoting all of his energies to the problems of teaching, research, and publication in the field which he loved, his productivity increased by leaps and bounds; 159 notes, papers, memoirs, and books are assignable to the twelve years in California. Since he became professor emeritus 1 July 1929 he was then freed from teaching obligations, and was thus enabled to proceed with even greater singleness of purpose to draw upon the accumulated stores of his erudition. But an illness resulting in a serious operation occurred in February 1930, and greatly sapped his vitality.

During this California period Cajori's most notable publications were in connection with mathematical notations, the history of various matters connected with the calculus, fluxions, and partial differential equation, and work of Newton. His monumental work on *Mathematical Notations* (2 v., 1928–29, 852 p.), original in conception, must be of enduring value. Three somewhat extensive and interesting papers may be mentioned along with this, namely: "The history of notations of the calculus" (1923), "Leibniz, the master builder of mathematical notations" (1925), and "A list of Oughtred's symbols with historical notes" (1920).

Another important book was his *History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse* (1919, 307 p.) published shortly after going to California. With this may be coupled "Grafting the theory of limits on the calculus of Leibniz" (1923), "Newton's fluxions" (1928), and "The early history of partial differential equations and of partial differentiation and integration" (1928) of which a Spanish translation (1929) was also published.

As to further work of Newton Cajori's discussion of the question of the "twenty years' delay in announcing the law of gravitation" (1928), read at the Newton bicentenary celebration, was complete and satisfying. For many years Cajori had talked of bringing out a new edition of Newton's *Principia* with notes. The older translations of Motte and Thorp were difficult of access, and left much to be desired. It was therefore with great satisfaction that his colleagues learned that the complete manuscript, ready for the printer, was found among his papers. It is hoped that this may soon be published by the University of California.

Professor Cajori did comparatively little work with manuscript source material; practically all of his discussions were with reference to printed documents. But in this respect he performed great services. Hundreds of points, and scores of connected accounts of topics, will hereafter be noted and have their proper settings, because of his presentations of the facts.

Supplementary to our earlier list of extra-academic honors it may be noted that Professor Cajori was vice-president of the American Association for the Advancement of Science and chairman of Section L (Historical and Philological Sciences) in 1923; vice-president of the History of Science Society in 1924–25; and vice-president of the Comité International des Sciences in 1929–30.

He was a man, exceeding modest in bearing, lofty in his ideals as to personal conduct and as to the search for scientific truths, especially endowed with good judgement, blessed with the most gentle kindliness of spirit and with intense interest in others and their problems, and a man who could be the finest of friends. The memory of the beauty of his personality lingers as a precious, fragrant, and inspiring possession of hosts of his students and friends, left behind for a space.

RAYMOND CLARE ARCHIBALD.

JAMES COOLIDGE CARTER (1827–1905).

Fellow in Class III, Section 1, 1896.

James Coolidge Carter was born at Lancaster, Massachusetts, October 14, 1827. He was graduated at Harvard College in 1851 and the Harvard Law School in 1853. He went to New York where his ability won quick recognition. He was associated with Charles O'Connor in the prosecution of the Tweed ring in 1875, was a member of the commission to devise a form of municipal administration for the cities in New York in 1888, and a member of the Constitutional Convention in 1893. He was one of the counsel for the United States in the Fur Seal arbitration, and argued many questions of general and constitutional importance. Perhaps his most important argument was that in favor of the constitutionality of the income tax law. He was throughout his life a foe of codification. At the time of his death he had been appointed to give a course of lectures on the subject in the Harvard Law School, and his notes for these lectures were published after his death in the book entitled "Law: its Origin, Growth and Function." He died on February 14th, 1905.

He cannot be better characterized than in the dedication of Professor J. B. Thayer's book on Evidence, "who throughout a professional career which has placed him high among the leaders of the Bar of the United States has never been wanting to the larger interests of his profession and his country, when wisdom, eloquence and generous courage might serve them." He was the first President of the Harvard Law School Association, founded in 1886, and made a masterly address at the first meeting of the association. On the same occasion Mr. Joseph Choate advised the students to come to New York, for, he said, "Mr. Carter will soon retire and leave room for a thousand." Mr. Carter's retirement, however, was delayed for 20 years. Mr. Carter was elected a non-resident member of the Academy in 1896.

J. H. BEALE.

JOHN JOSEPH CARTY (1861–1932).

Fellow in Class I, Section 4, 1915.

General John J. Carty, pioneer in the development of the telephone, died in the Johns Hopkins Hospital, Baltimore, Maryland, on December 27, 1932. He was born on April 14, 1861, in Cambridge, Massachusetts. At the completion of his preparation for college, he was obliged because of a temporary impairment of his eyesight, to discontinue further formal study.

The possibilities of the telephone which Alexander Graham Bell had but recently invented so strongly appealed to his imagination that he sought an opportunity to participate in the development of this new marvel. In 1879 he entered the service of the Bell Telephone Company in Boston and, with the exception of a period of service in the United States Army during the World War, was active in its councils until his retirement on June 30, 1930.

It is difficult, if not impossible; now to visualize the conditions under which the telephone pioneers operated. Man had yet to acquire much of the scientific knowledge he possesses today; there were no schools of electrical engineering to provide trained personnel; primary batteries and magneto generators were the only sources of electrical energy; and adequate theories of the transmission of such energy over wires had not been formulated.

The story of General Carty's life is the story of the technical development of the telephone, not only in the United States but throughout the whole world. His own contributions were notable. He designed and installed the first multiple switchboard, which featured common battery signaling, and later showed how to operate two or more telephone transmitters from a single central battery. These features, foundations of the present common battery system essential to every large telephone switchboard, made practical the commercial development of the telephone in metropolitan areas.

From the beginning of the telephone business, induction between closely adjacent circuits had to be prevented, so that speech transmitted in one could not be overheard in another. Closely related was the task of minimizing electrical disturbances from other sources. When but little was known on this subject, he carried out an important scientific investigation which made it possible to correctly handle wire transpositions and laid the foundation for holding inductive disturbances within limits that permitted the full development of the industry.

In early telephone systems the instruments, including signaling devices, were placed in series in the line. This arrangement with several instruments on the same line impaired telephone transmission seriously and interfered with satisfactory signaling. General Carty concluded from a study of this question that instruments should and could be placed in parallel if correctly designed with high impedance signaling apparatus. This arrangement removed a serious limitation to the development of the business, and made the party line and the rural line a possibility.

After his early experience in the design, construction, and operation of telephone systems in and around Boston and later with the Western Electric Company in New York City, he became in 1889 Chief Engineer of the New York Telephone Company. In that position he was responsible for the company's technical problems in the city which is telephonically the largest in the world. He built up an engineering staff and was the first among the operating executives to recruit technical personnel from the graduates of scientific and engineering schools.

In 1907, he became Chief Engineer of the American Telephone and Telegraph Company. His first task in this position was to reorganize the technical forces of the Bell System. He consolidated the laboratories that were located in Boston, New York and Chicago into what later became the Bell Telephone Laboratories in New York.

To talk across the continent from the Atlantic to the Pacific was the boyhood dream of General Carty and his early associates, and he now had the opportunity of leading a distinguished group of scientists and engineers in an attack upon this problem. It was solved on January 25, 1915, when the New York-San Francisco line was opened to public service. Through the application of the methods employed, telephone service was later extended to every State of the Union, and to Canada, Cuba and Mexico as well.

General Carty inspired and directed the work which resulted on October 21, 1915 in transmitting the first articulate words across the

Atlantic Ocean by radio telephony from the United States Naval Station at Arlington, Virginia to the Eiffel Tower in Paris. This work continued, until today the barrier of the Atlantic Ocean has been overcome and commercial telephone service between the old world and the new is a daily fact.

At the outbreak of the World War, he was ordered to active duty as a Major in the Signal Corps of the United States Army. In 1918, he was promoted to Colonel and ordered to France, where he served on the staff of the Chief Signal Officer, and later was promoted to the rank of Brigadier General in the United States Army Reserve Corps. For his services as an officer in the United States Army, he received the Cross of the Legion of Honor from France and the Distinguished Service Medal from the United States.

Upon his return to the United States in 1919, he became a Vice President of the American Telephone and Telegraph Company and, with the incorporation of Bell Telephone Laboratories in 1925, he became Chairman of its Board of Directors.

General Carty was a member of the National Academy of Sciences and an honorary member of the Franklin Institute, which Society bestowed upon him the Franklin and Edward Longstreth Medals. The American Institute of Electrical Engineers, of which he was President during 1915–1916, awarded him the Edison Medal in 1917. The United Engineering Society awarded him in 1927 the John Fritz Gold Medal "for pioneer achievement in telephone engineering and in the development of scientific research in the telephone art." For his services in connection with Japanese electrical communications, he received from that government the decorations of the Order of the Rising Sun and the Order of the Sacred Treasure.

Stevens Institute of Technology and New York University granted to General Carty the degree of Doctor of Engineering. From Chicago, Bowdoin, Tufts, Yale and Princeton, he received the degree of Doctor of Science, and from McGill and the University of Pennsylvania the degree of Doctor of Laws.

In August, 1930, the American Telephone and Telegraph Company established, under the custody of the National Academy of Sciences, the John J. Carty Medal for outstanding achievement in science in honor of General Carty's noteworthy contributions to the advancement of fundamental and applied science and in appreciation of his great service in the development of electrical communication. Shortly before General Carty's death, the National Academy of Sciences selected General Carty as the first recipient of this Medal. The presentation was made posthumously on April 25, 1933 in Washington at the annual meeting of the Academy.

FRANK B. JEWETT.

EDWARD CHANNING (1856–1931).

Fellow in Class III, Section 3, 1911.

Edward Channing was born in Dorchester, Massachusetts, June 15, 1856, the fifth child of the poet Ellery and his wife Ellen (Fuller) Channing. He was brought up by his grandfather, Dr. Walter Channing, in Boston and Brookline, entered Harvard College in 1874, and took his first degree in 1878. While in college, he became interested in history by Henry Adams, and decided to make that subject his life work. In 1880 he took his Ph.D., writing his dissertation on the Louisiana Purchase. After a year of travel in Europe, Dr. Channing returned to Cambridge and for two years supported himself by writing articles and book reviews. In the fall of 1883 he was appointed instructor in History, and from that year to his death he was a member of the Harvard Faculty, being promoted in the usual course to Assistant Professor and Professor. At first Dr. Channing taught all branches of medieval and modern history, as well as government and international law. After becoming full professor in 1897, he was able to concentrate on his favorite branch of American History. During all this period, he was actively engaged in writing, his prize essay on *Town and County Government in the English Colonies* (1884) having given him scholarly repute and created a demand for his contributions. As early as his junior year in college, Channing determined to write a history of the United States from the beginning to our own day; but it was not until 1897, when he became a full professor, that he had sufficient relief from teaching and from writing of a pot-boiling nature, to give him time actively to prosecute this great design.

The first volume of his *History of the United States* came out in

1905; the sixth and last, covering the Civil War, in 1925. A seventh on the period from 1865 to 1900 was partly completed at the time of his death on January 7, 1931. Although his *History of the United States* will probably remain his chief monument for posterity, Professor Channing performed a no less useful purpose in his teaching. He was a successful teacher both of undergraduates and post-graduate students; hundreds of the latter were trained in his famous seminar, History 23, which moved forward with his *History of the United States* from the Northmen to McKinley.

S. E. MORISON.

JOSEPH HODGES CHOATE (1832–1917).

Fellow in Class III, Section 1, 1900.

Joseph Hodges Choate was born in Salem, January 24, 1832. He graduated from Harvard College in 1852 and was bachelor of laws of the Harvard Law School in 1854. He received the degree of doctor of laws from Amherst, Harvard, Cambridge (England), Edinburgh, Yale, St. Andrews, Glasgow, Williams, Pennsylvania, Union, Magill, Toronto, and of D.C.L. from Oxford. He was the president of the New York Constitutional Convention in 1894, was ambassador to Great Britain in 1899–1905. He had been president of the Harvard Chapter of Phi Beta Kappa, president of the Harvard Alumni Association, of the Law School Association and of the American Bar Association.

These records of achievement are records of his less characteristic activities. He was over and above all a great advocate. For the last half of his life he was distinctly the head of the American Bar so far as the actual trial of cases was concerned. His kindliness and humor, his great intelligence and his knowledge of men together with the professional sense which he derived from a race of lawyers made him eminent as a jury lawyer; not that his arguments on points of law were undistinguished, for he argued and won some of the most important cases of his time before the Supreme Court of the United States. He was elected a non-resident Fellow in January 1900.

J. H. BEALE.

FRANK SHIPLEY COLLINS (1848–1920).[1]

Fellow in Class II, Section 2, 1901.

Frank Shipley Collins was born in Boston, Mass., February 6, 1848. He was the son of Joshua Cobb Collins and Elizabeth Ann (Carter) Collins. Both the Cobb and the Collins families were old colonial families of the Cape Cod Peninsula, while the Carters were of similar stock of Charlestown, Mass. The mother of Frank Shipley Collins and her two sisters were teachers, teaching English, Greek, Latin, French, mathematics, and botany in the Charlestown Female Seminary. Frank, being a delicate child, was first taught at home, and having lost his father very early in life and the rest of the family having removed to Malden, he attended first a very exclusive private school there and later the public schools, from which he was graduated with honors in all his subjects. He completed his formal public education on graduating from the high school of Malden in his sixteenth year (1863). The principal influences in his training were his home studies and these were continued for two years after he finished school, while suffering from severe attacks of asthma. During these years he was able, however, also to study harmony and the pianoforte, and under a talented teacher, a man of ability and charming personality, one who had traveled much, he was introduced to attendance on public recitals, concerts, opera, etc., of the highest type. This interest continued throughout his life and he was a constant attendant on the performances of the Boston Symphony Orchestra, of chamber music, and of virtuosi. It was the ardent desire of his mother and his aunts that he enter Harvard University, but his grandfather Carter, whose means at that time were amply sufficient, but who was a man of severe, practical ideas, decided upon a business career. Finally, after various disappointing experiences, the family means having in the meantime greatly diminished, qualified as an expert bookkeeper, he entered the employ of the Malden Rubber Shoe Company, as ticket clerk. Here his attention was directed to the problems presented by the system of laying out the separate pieces of boots and shoes to be made up into the finished product. He was eminently successful in administering and improving these fundamental processes and continued with this same firm (under its various reorganizations), as manager, for over thirty years, until his retirement. Shortly after that, however, he was recalled by the exigencies of the World War to become efficiency expert at various plants of the company, and his death, after a brief illness, at New Haven, Connecticut, on May 25, 1920, found him still occupied along these lines.

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[1] Compare biography by W. A. Setchell, Amer. Jour. Bot., 12: 54–62, Jan. 1925, with portrait and bibliography.

In 1875, Collins married Anna Lendrum Holmes of Little Falls, New York, a lady of congenial tastes, especially in art and music, who, with the two sons born of the marriage, survived him. Through his wife he became acquainted with Mrs. Maria H. Bray, at Magnolia, Mass., who, because of the cards of named and mounted "Sea Mosses" she sold at her "Ptilota Café," was the direct instrument of attracting his attention to the marine algae of New England. These soon became his principal life interest and raised him to high rank as an authority on the subject.

Collins evidently absorbed the elements of botany from one of his maternal aunts, Mrs. Simon G. Shipley, and this was increased by his association with the members of the Middlesex Scientific Field Club and the Middlesex Institute (incorporated, 1881), especially with L. L. Dame, Walter Deane, C. E. and Edwin Faxon, G. E. Davenport, Clara E. Cummings, and W. G. Farlow. There resulted the "Flora of Middlesex County, Massachusetts," by Dame and Collins, published by the Middlesex Institute, in 1888.

Collins was a business man and during office hours was fully occupied. His unoccupied (leisure?) hours and his short vacation periods were devoted to study, whether of music, art, or science. Nevertheless, his systematic application and tireless energy led him far in several fields and the tangible results are many. His list of published works and articles amounts to close upon 100 titles, the great majority being concerned more or less directly with the algae, and particularly with the marine algae. Beginning in 1880, his notes and monographs followed one another in constant and logical succession. W. G. Farlow, in the first impression of his "Marine Algae of New England (1881)," mentions Collins as one of those to whom he was especially obligated. The publication of Farlow's manual was a great incentive to a serious-minded and progressive amateur like Collins. The New England coast became his especial hunting ground and in his later years, when it was hoped, both by Farlow and others, that he would complete a new edition, or rather substitute for it, he had more than doubled the original content of its known species. Unfortunately, the amplified account of the marine algae of New England never came to definite form.

While Collins was preeminently a regional phycologist, yet his interests and experience went far beyond this, extending particularly throughout the Atlantic coasts of North America and even to the Pacific coasts. In company with Rev. A. B. Hervey, he explored and made known the marine flora of the Bermuda Islands (1917); from extensive collections of various collectors he did the same for the island of Jamaica (1901); for Vancouver Island (1913); as well as the blue-green algae of the Bahama Islands (1920). In a monographic fashion, Collins elaborated reports on the green algae; first on the marine Cladophoras of New England (1901); second, on the Ulvaceae (or Sea-Lettuces) of North America (1903); and finally on the green algae of North America as a whole, both marine and fresh water (1909, and supplements 1912 and 1918).

As a collector and assembler of algal material, Collins had few equals. His herbarium was rich in specimens of his own and of others' collecting and received from other and particularly from authoritative workers. After his death it was purchased by Dr. N. L. Britton and presented to the New York Botanical Garden. One of the important works carried through by Collins was the issue of some 51 fascicles (of 35 to 50 numbers each) of the Phycotheca Boreali-Americana (1895–1919), containing 2300 numbers of smaller and 125 numbers of larger mounted specimens of North American algae. While this was done in collaboration with Isaac Holden and W. A. Setchell, the burden of correspondence, assembling, printing, and issue, involving at least 200,000 specimens, fell upon Collins and was so admirably carried through that this collection is basic to all future study of North American Algae.

Collins was in active consultation, personal or by correspondence, with all of the active phycologists of his time, and was especially in sympathetic relations with W. G. Farlow. He was a member of the Middlesex Institute, of the Boston Society of Natural History, of the Massachusetts Horticultural Society, of the New England Botanical Society (president for three years), corresponding member of the Torrey Botanical Club, and long a member of this Academy. He received the degree of M. A. (*honoris causa*) from Tufts College in 1910, and was appointed an Associate in the University Museum of Harvard University. His memory will be perpetuated by the members of the genus *Collinsiella* (Setchell and Gardner) and particularly by his most interesting namesake, *Phaeosaccion Collinsii* Farlow.

Frank Shipley Collins was a slender man, of medium height, early gray, but with youthful appearance, roundish face, with broad mustache; he was the more resembling (although not closely) the Mark Twain than the James Russell Lowell type. He was strictly the amateur, in that his vocation was far from phycology, but in his accomplishment he carried through the critical and thorough methods of the most accomplished professional. He was kindly towards the errors and shortcomings of others, generous in sympathy and substantial aid to his fellow workers, a friend of enduring loyalty, and ardent in his pursuit of knowledge. He represents a type of scholar of whom the community which possessed him as a member may well be proud.

WILLIAM ALBERT SETCHELL.

ARTHUR MESSINGER COMEY[1] (1861–1933).

Fellow in Class I, Section 3, 1891.

Arthur M. Comey was distinguished as a teacher, author, and director of a large and important industrial research laboratory. His personality and interests outside the field of chemistry attracted to him many friends. His retirement from active work made it possible for him to enjoy for over ten years the pursuit of the things that were always dear to him. He played the piano and the flute, was a skillful amateur photographer and gardener, a great walker, and an experienced woodsman.

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[1] An appreciation of Dr. Comey and his work written by Charles L. Reese was published in "Industrial and Engineering Chemistry," *20*, 438 (1928).

Doctor Comey was born in Boston, November 10, 1861, and came from old New England stock. He was graduated from Harvard in 1882 and went to Germany to continue the study of chemistry. He worked with Victor Meyer at Zurich and received his doctorate from Heidelberg in 1885. He was immediately appointed to an instructorship at Harvard where he organized and served as director of the summer school of chemistry. In 1889 he became professor of chemistry at Tufts College and resigned four years later to take up literary and consulting work. In connection with the latter he showed the folly of the attempt to extract gold from sea water. He spent a summer at Marblehead on this work and succeeded in securing a bead of gold large enough to be seen under a microscope.

The first edition of Dr. Comey's Dictionary of Chemical Solubilities was issued in 1896, and a second edition in 1921. This book continues to be a standard work of reference.

From 1906 to 1919 Dr. Comey was director of the explosive laboratory of the DuPont Company. It is probable that his most important invention was an improved process for the manufacture of dynamite based on the nitration of a mixture of sugar and glycerine. The method brought about much economy in the manufacture of explosives and yielded a product safer to handle. Dr. Comey retired in 1921. He spent his winters in Cambridge and his summers at his beautiful home on Lake Chocorua, New Hampshire. He died on April 6, 1933.

Dr. Comey took an active part in the affairs of the American Chemical Society. During the World War he was chairman of the Sub-Committee on Explosives of the National Research Council.

JAMES F. NORRIS.

LESLIE COLBY CORNISH (1854–1925).

Fellow in Class III, Section 1, 1923.

The descendant of Maine ancestry, Leslie Colby Cornish was born in Winslow in that state on October 8, 1854, and died on June 24, 1925. He attended a district school in his native town, then the Waterville Classical Institute, now the Coburn Classical Institute, and graduated at Colby University, now Colby College, in 1875 before reaching his majority. After teaching for two years and studying for a time in an office, he spent one year at the Harvard Law School and was admitted to the bar in 1880. His residence for the rest of his life was in Augusta. There he practiced law with steadily increasing distinction until he went on the bench. He was married in 1883 to Fannie Woodman Holmes, whom he survived but a comparatively short time. He was a member of the House of Representatives of his state for a single term and served in the common council and board of aldermen of his adopted city. He was a member of the state board of bar examiners and also trustee and president of the Augusta Savings Bank. He was a trustee of Colby College from 1888 and president of the board from 1904 until his death. He was a director of the American Unitarian Association and a president of the Maine Unitarian Association. In 1907, he was appointed a justice of the Supreme Judicial Court of his state and became its twelfth Chief Justice in 1917, a trust which ill health compelled him to resign in March, 1925. He was deservedly held in high esteem as a prompt and efficient magistrate in presiding over jury trials and in promoting just results. He was on the bench more than sixteen years and wrote three hundred and forty-four opinions expressive of the judgment of the court, found in volumes 103 to 124 of the Maine Reports. His style was simple and lucid, a garment fitting for clear thinking and ample learning. He made substantial contributions to the growth of jurisprudence. He was highly regarded by the bench and bar of his state. He possessed the characteristics which should accompany high judicial station, intellectual and moral power, industry, energy, learning, evenness of temper, the spirit of justice. He was recognized widely outside his native state as an able and just judge. In one of the recurring periods of complaint over the law's delay, he wrote a convincing and sparkling article on the "Law's Dispatch" as illustrated by the practice of Maine. He was the recipient of the honorary degree of LL.D. from Colby and Bowdoin, and the University of Maine. He was elected a fellow of this Academy on May 9, 1923. Endowed with a melodious voice and a ready wit, he was a public speaker of unusual grace and force. Somewhat above the average height, he was erect of figure and dignified in bearing, an impressive personage on any occasion and in any company.

Culture and refinement, strength and good will were expressed in his appearance. Attributes of heart and mind endeared him deeply to many friends. He was rich in the human qualities and in the intellectual achievements which mark a man for large usefulness and wide influence in the promotion of the public welfare.

ARTHUR P. RUGG.

CHARLES AMBROSE DECOURCY (1857–1924).

Fellow in Class III, Section 1, 1924.

Charles Ambrose DeCourcy was born in Lawrence, Massachusetts, on September 23, 1857, and died in New London, New Hampshire, on August 22, 1924. He was the son of parents who emigrated to this country from Kinsale in southern Ireland. He attended the public schools of his native city and was graduated at Georgetown University in 1878 and at the law school of Boston University in 1880. He practiced his profession in Lawrence. He was assistant district attorney for the Eastern District comprising Essex County for six years beginning in 1884, and was city solicitor of Lawrence in 1892. By ability, industry and energy he became a leader of the bar. He was married in 1886 to Elizabeth M. Roberts, who with two sons, Harold and John, survived him. In 1902, at the age of forty-four, he was appointed a judge of the Superior Court by Governor Crane. He achieved a high degree of success as a trial judge. Although experienced in every branch of the law, he was deeply interested in the development and extension of the probation system in the administration of criminal law, and regarded it as susceptible of becoming a highly effective method in solving the problem of crime. By his own constructive efforts and wise counsel he made signal contribution to this aspect of the public welfare. In 1911, with general professional and public approbation, he was appointed by Governor Foss a justice of the Supreme Judicial Court. His service on this court covered a period of about thirteen years. He wrote seven hundred and eighteen opinions expressive of the judgment of the court, to be found in thirty-nine volumes of the reports of its decisions from 210 Mass. to 249 Mass., both inclusive. His opinions possess a distinctive excellence. They are brief but adequate, characterized by clarity of thought, thorough investigation and mature consideration of underlying principles. The lucidity of his style was marked by rare literary finish. He performed his judicial work with ardent enthusiasm and all his might in the consuming aspiration that he be worthy of his great trust. His tastes were scholarly. He had a keen aptitude for public speaking. His melodious voice was carefully cultivated to this end. He was elected a Fellow of the Academy on May 14, 1924. The honorary degree of LL.D. was conferred upon him by Georgetown University in 1904. He was a vice president of the American Institute of Criminal Law and Criminology, a chairman of the committee on criminal law of the American Prison Association, and president of the Massachusetts State Conference on Charities. He was one of the Board of Regents of Georgetown University and a visitor in the Department of Philosophy at Harvard University. He had rare personal charm. In manner he was animated, winsome, attractive. By nature and by practice he was religious, and he was strictly observant of the rites of his church. He was delightful in conversation and companionship, with sparkling and kindly humor. Lover of the classics, widely read in poetry, literature and history, he was on terms of friendship with large numbers of people of quite varied accomplishments. His life enriched the civic life and the judicial annals of the Commonwealth.

ARTHUR P. RUGG.

ALBERT VENN DICEY (1835–1922).

Foreign Honorary Member in Class III, Section 1, 1901.

Albert Venn Dicey was born February 4, 1835. He was educated at Oxford where he graduated first class in 1858. Entering the Bar he got immediate employment as a junior and soon became junior counsel to the Inland Revenue. In 1882 he became Vinerian Professor of the Common Law at Oxford and remained such until his resignation in 1907. He died April 7, 1922, at the age of 87 years. He was one of the founders of the English Law Quarterly Review and a frequent writer in its columns. In the year 1897 he delivered a series of lectures before the Harvard Law School which were published as "Law and Public Opinion in the 19th Century."

Professor Dicey was the author of a number of important and well-written law books of which the principal was his Conflict of Laws, the first edition of which was issued in 1896. He was a sound lawyer, an accomplished advocate and a clear writer. Few men in England have surpassed him as a legal author. He was elected a Foreign Honorary Member in March 1901.

J. H. Beale.

CHARLES LEAVITT EDGAR (1860–1932).

Fellow in Class I, Section 4, 1924.

Charles L. Edgar was born December 23, 1860, at Griggstown, N. J., the son of Thomas and Annie Veghte Edgar, of Scottish and Dutch colonial ancestry.

His early education was in the district schools of Middlesex County, N. J. His summer vacations from 1875 to 1880 were spent on his uncle's farm near Menlo Park, N. J., where the great American inventor, Thomas A. Edison, had his laboratory, and where the incandescent electric lamp was being successfully developed. As a justice of the peace, his uncle paid frequent professional visits to Menlo Park in connection with Mr. Edison's patent applications. Young Edgar came into contact there with Mr. Edison, which changed the course and direction of his life. The inventor's enthusiasm, energy, and kindly zeal inflamed Edgar's aspirations, as indeed they did those of many other young men of that period.

After a short preliminary course at the Rutgers Preparatory School, Edgar entered Rutgers College, N. J., in 1878, and graduated four years later with the degree of B.A. He then continued with a short post-graduate course in applied electricity, which was, at that time, a rapidly advancing subject.

Early in January 1883, Mr. Edgar left Rutgers, armed with a letter of recommendation from the Dean, and went in search of employment, to call at Menlo Park on Mr. Edison, whom he had not seen for four years. Mr. Edison was well pleased with the interview, and gave him a letter to the manager of the Edison Electric Light Co., in New York City. This was shortly after the start of the first Edison incandescent lighting station at Pearl St., in downtown New York City.

Mr. Edgar entered the service of the Edison Companies forthwith, and took an active share in starting and operating Edison lighting and power systems in a number of towns and cities. In September 1887, he was sent to Boston as manager of the Edison Electric Illuminating Co.'s station in that city. He remained in Boston, devoted to the service of that company, until his death in 1932. In 1889, he was promoted to be general manager of the Boston Company; in 1891, he was elected a director; in 1892, he became vice-president; and in 1900, Mr. Edgar became president of the company, holding that position until the last.

When he was manager of the station in 1887, the number of its customers, using electric light service in Boston, was 260, in a territory not exceeding 30 hectares (0.11 sq. mile). In 1932, the number of customers had risen to more than 400,000 over an area of more than 1680 sq. km. (650 sq. miles).

Edgar's outstanding characteristics were unflagging industry, calm judgment, executive ability, and unfailing courtesy. He won the affection of all his subordinates and the esteem of all those with whom he came in contact by his honesty and fair dealing. He was a great admirer of Mr. Edison, and a close student of the business of electric light and power distribution, based on scientific principles. The new lighting stations, that he successively built, were all models at their respective dates of installation. The high standard of electric lighting service in Boston is largely the result of his work.

In 1927, Mr. Edgar received the honorary degrees of Sc.D. from his alma mater, Rutgers College, and of LL.D. from Tufts College. He was elected to membership in the American Academy of Arts and Sciences in 1924. Dr. Edgar was a director or trustee in many New England organizations, and was the first chairman of the Boston Section of the American Institute of Electrical Engineers.

He died April 14, 1932, six months after the death of his beloved leader, Thomas A. Edison. The friendship between these two remained to the end a source of mutual comfort.

Dr. Edgar is survived by his widow, Mrs. Annette Duclos Edgar, and by his son, Leavitt L. Edgar.

A. E. KENNELLY.

BENJAMIN KENDALL EMERSON (1843–1932).

Fellow in Class II, Section 1, 1895.

Born in Nashua, N. H. December 20, 1843, died at Amherst, Massachusetts, April 7, 1932, nearly ninety years old.

One of the most distinguished geologists of the United States,—the main facts of his life (society memberships and offices held) are given in *Who's Who,* Volume XI, 1920–21, and in the memorial by F. B. Loomis (Bulletin Geological Society of America, Volume 44, p. 317). He was reported to have been killed in a railroad accident in 1893 and the *Springfield Republican* published his obituary and a tribute of which he remarked, "It will be a job to live up to it." But the collision left him more robust. In the Amherst Graduates' Quarterly for August 1932, p. 259–265, Horatio Smith describes *Professor Emerson, the Man.* In the class notice in the same number, p. 282, is an account of his honors and achievements.

For over 50 years he taught geology at Amherst College, and Loomis lists 17 distinguished geologists who were his pupils. It was not therefore surprising that the Geological Society of America of which he had been President and many times officer and toastmaster presented him with a loving cup on his retirement in 1924–5. His writings are mainly given in the memorial by Loomis. But his Göttingen Ph.D. thesis is omitted:—"Die Liasmulde von Markoldendorf bei Einbeck."

He was recognized to be the best of his time in all around acquaintance with the geology of Massachusetts and his U. S. G. S. Bulletin 597 has not been replaced.

Though he used the microscope Professor Emerson was a keen field observer. He first called my attention to the puckered appearance around allanite which characterizes the radio-active minerals.

Characteristic in more than one way is the story that he hired a horse for a geological trip, and leaving the horse hitched while he followed the geology across field and dale, finally walked home, forgetting the horse which the livery man had to retrieve.

My most vivid recollection of the first dinner of the Geological Society that I attended was of him as toastmaster. His wit lasted to the end. No one who heard it will forget the shout at the dinner at which he was given the loving cup. John M. Clarke had been saying how much he appreciated Emerson's course and had never cut. Then Kemp (for whom he prepared the Academy memorial) presented his compliments but owned that he could not say as much. At which Emerson broke in with that deep voice "Why not? Clarke said it."

There are many such stories of which Smith preserves a few. Some of his absent-mindedness may have been deliberate for he was not above "spoofing." See the stories on p. 262. One I like, not mentioned, is the dialogue with the student conditioned in mineralogy set down before a tray.

"It seems to me, Professor, that we would get along faster if you'd lecture."

"Young man, I believe your major subject is philosophy?"

Yes, he had done well in philosophy. He liked it better than science.

"Did you take the course in Psychology?"

Yes, he had a good mark in Psychology.

"Did the professor of Psychology tell you about the theory of grooves in the brain?"

Yes, he remembered well that interesting theory of the association of ideas and the repetition of experiences producing grooves in the brain.

"Yes? Well, did the Professor tell you that the deepest groove in a sophomore's brain is from one ear to the other?"

Like all geologists he loved to travel and was a circumnavigator. Wherever he went he was a welcome and delightful guest. His grandchildren had a little chant:

"When I was in Russia
With my little hammer
And dined with the Czar. . ."

In print his jokes might sound rasping. But in life it was clear that they were fun.

Professor Emerson was of an old and distinguished New England family with many collateral relatives known in educational and diplomatic circles, and the sturdiness of the stock showed in that when he died he had eight surviving children and twelve grandchildren, having lost none by death. It was well said that he was "characterized by vigor and serenity," one may add by keenness of insight and wit in expression, good nature and optimism in spirit. No wonder he was an inspiring teacher.

ALFRED C. LANE.

WILLIAM WALLACE FENN (1862–1932).

Fellow in Class IV, Section 1, 1909.

William Wallace Fenn, born in Boston, February 12, 1862, died in Cambridge, March 6, 1932, was the son of William Wallace and Hannah Morrill (Osgood) Fenn. His father died when he was seven weeks old, but his mother with rare courage managed to keep her home in Boston and give her only child the educational advantages of the city. He attended the public schools, prepared for college at the Boston Latin School, leading his class, graduated from Harvard in 1884, *summa cum laude*, and from the Harvard Divinity School in 1887. He accepted a call to Unity Church, Pittsfield, Massachusetts, where he met and married Faith Huntington Fisher, in 1891. From 1891 to 1901 he was minister of the First Unitarian Society of Chicago, Illinois; and in 1901 he was appointed Bussey Professor of Theology in Harvard Divinity School, a position which he held until his death.

Upon graduation from the Divinity School he received the M.A. degree in recognition of his high standing; and in 1908 he received the S.T.D. degree from Harvard. He was a member of the Phi Beta Kappa Society. He served two terms as College Preacher, 1896–1898 and 1902–1905. From 1906 to 1922 he was Dean of the Harvard Divinity School Faculty. For almost thirty-two years he held his professorship, taking but one half of one sabbatical and rarely missing a lecture. His last illness was mercifully brief; four days after the lecture that was destined to be his last he was gone. He became a Fellow of the American Academy of Arts and Sciences in 1909.

A lively human interest coupled with unusual intellectual power made him an illuminating teacher. His school-boy passion for the classics developed into a special interest in New Testament Greek, and at one time it was his dream to teach the New Testament in some divinity school. Throughout his own divinity school career he had assisted Professor Joseph Henry Thayer in the preparation of the "Greek-English Lexicon of the New Testament,"—an exceedingly profitable experience. With his appointment to the Chair of Theology came the necessity of transferring his interests to a new field. No one realized better than himself that Dean Everett's shoes would be pretty big ones for him to fill. Modest and conscientious to the last degree, he threw himself with the utmost assiduity into his new work, and before many months won the place he will ever hold in the hearts of his colleagues and pupils,—one of the most thorough scholars and one of the most quickening teachers the Harvard Divinity School has ever boasted.

He found an especially congenial field in the New England Theology, and became the foremost authority in that subject. A thorough New Englander himself, he insinuated his sympathy into the intricacies of the New England Theology in a way to make that whole tortuous fabric of thought live again. Probably no man of his generation knew as much as he about the church polity, the customs, and the literature of New England Puritanism.

He steadfastly refused to write and publish a book. Many sermons and addresses were issued in pamphlet form or as articles in various periodicals. Could these be collected, they would make several imposing volumes, but even these lapses into print he "viewed with alarm." When asked why he was so determined not to publish, his favorite retort was "I have not yet stopped thinking." And as the desolating implications of that reply sank into the hearer's mind, he would throw back his head and indulge in a hearty laugh. No man ever treated books with truer discrimination. There was no trace of false reverence for printer's ink, as such. A worthy book received from him instant and grateful appreciation. But in too many instances, the appearance of a book seemed to him to synchronize with a stoppage in its author's thought processes, and he had no intention of adding to that number. Yet to hundreds who came under his influence, what a treasure a book from his pen would be!

He was very reticent, very reserved, and exceedingly sensitive. His friendship was not easily won; once it was won, it could never be lost. Young men he loved, and they always had quick access to his heart. With all his power of intellect, he was as free from intellectual snobbery as one could be. Among his most intimate friends were those who could never hope to give him intellectual companionship. And he revealed the breadth and balance of his judgment when he remarked once of a favorite of his, "No, he will never set the world on fire; but, thank the Lord, he will never do a mean thing."

CHARLES E. PARK.

FREDERICK PERRY FISH (1855–1930).

Fellow in Class III, Section 1, 1913.

Born in Taunton, Massachusetts, January 13, 1855; died in Brookline, Massachusetts, November 6, 1930. His early education was obtained in the public schools of Taunton; he entered Harvard College from the Taunton High School, received his bachelor's degree with the Class of 1875, then entered the Harvard Law School, leaving that school in 1876 to enter into active practice in the offices of Col. Thomas L. Livermore and former U. S. Senator for New Hampshire, Bainbridge Wadleigh, in Boston.

The law firms with which Mr. Fish was connected were, in succession, Wadleigh and Fish; Wadleigh, Fish and Wellman; Livermore and Fish; Livermore, Fish and Richardson; Fish, Richardson and Storrow; Fish, Richardson, Storrow and Herrick; Fish, Richardson and Herrick; Fish, Richardson, Herrick and Neave; and Fish, Richardson and Neave.

Mr. Fish's practice during substantially the whole of his professional life, was mainly in patent cases. This period was one in which the industries of the United States were rapidly expanding under the stimulus of invention and discovery in every department. In nearly all the patent cases of primary importance during his forty years of active practice Mr. Fish appeared as counsel. These cases comprised patents relating to the telephone, air-brakes, the steam turbine, automobiles, shoe machinery, radio-apparatus; Mr. Fish's industry, intelligence, and versatility mastered the technique of them all; he was recognized by the Federal Courts in all parts of this country as a master-advocate and became the acknowledged leader of the American Bar in patent causes.

In the mastery and solution of problems of business, Mr. Fish also manifested abilities of the highest order, and, as general counsel of the Thomson-Houston Electric Company, and, later, of the General Electric Company, he performed an influential and directive part in the development and organization of the electric lighting industry in the eighties.

In July, 1901, Mr. Fish was made president of the American Telephone and Telegraph Company, and held that position for six years, during which time he infused into the Company's personnel his own enthusiasm and his conviction that the full success of the telephone industry depended on national service by a national organization. Under his leadership, the Telephone Company practically completed and unified the vast network of telephone lines which now reaches into every part of the country.

The larger diversity of interests afforded by patent law practice drew Mr. Fish back to his law firm, which he rejoined in 1907. His mind and energy were such as not to be satisfied with a single task, however great it might be. This reason, which led to his resignation of the presidency of the American Telephone and Telegraph Company, also determined his refusal of the presidency of the Massachusetts Institute of Technology.

Mr. Fish served as Vice-President of the Bar Association of the City of Boston from 1909 to 1920 and as President of the Massachusetts State Bar Association for the year 1919–1920.

Mr. Fish possessed a tremendous vitality, an active and creative imagination, and an extraordinarily tenacious memory. He never remitted his work of self-education. His knowledge of the arts, sciences, history and literature was as nearly complete as possible for a man of his age and generation.

He was actively interested also, in the cause of education for others. For years he served as a member of the Corporation of the Massachusetts Institute of Technology, of the Governing Board of Radcliffe College, of the Board of Overseers of Harvard College. He was a trustee of the American School of Classical Studies in Athens, and Chairman of the Massachusetts State Board of Education. In all these positions he gave unstintingly of his labor and counsel.

In 1916 he founded the National Industrial Conference Board, which has comprised the leading industrial figures of this country.

In his seventy-fifth year, Harvard College conferred on Mr. Fish the honorary LL.D. President Lowell said of him, on that occasion:

"Leading patent lawyer of the country, who, amid the work of an exacting profession, has wrung time for public service in directing education."

Frederick P. Fish was a broad scholar, a great barrister, a public spirited citizen, a generous-minded and warm-hearted man.

Odin Roberts.

JOHN WILKES HAMMOND (1837–1922).

Fellow in Class III, Section 1, 1914.

John Wilkes Hammond was made a Fellow of the American Academy of Arts and Sciences on May 13, 1914, in Class III, Section 1.

His Fellowship was well earned. His career fully sustained the wisdom of his election. His ancestors came early to Plymouth Colony, thence to Massachusetts Bay. William Hammond, the father of the first Hammond to settle here, married a sister of Admiral Sir W. Penn, an aunt of the William Penn who founded Pennsylvania. An ancestor on his mother's side, Constance Southworth, was born in Leyden, Holland, in 1615. For this we have the authority of Attorney General Benton and a committee of the Bar in 1923. Ancestors on both sides had been persons held in high respect in their communities. Born in Mattapoisett on December 16, 1837, his father's death left him, when five years old, to the care of his mother, the daughter of the village physician. Educated by life in the country village, by its public schools, and by what he called "the one educational institution of the town—Harlow & LaBaron's store," he entered Tufts College, and took its degree of Bachelor of Arts in 1861, graduating at the head of his class. Then, after teaching school at Stoughton and Tisbury, he enlisted and served as private in the third regiment of Massachusetts Volunteers in the Civil War, until mustered out in June of 1863. For a time he taught in high schools at Wakefield and Melrose. Then, in 1865, he entered the Harvard Law School where he took his degree of LL.B. in 1866.

Admitted to the bar in 1866, after taking his law degree and after having had practical training since 1864 in the law office of Sweetser & Gardner, he opened an office in East Cambridge. Later he moved the office to Cambridgeport and to Boston. Business came to him. In 1873 he was made City Solicitor of Cambridge, an office which he held, and adorned, until his appointment to the bench of the Superior Court in March, 1886. For a short time in 1879—to fill a vacant term—he was, by appointment, district attorney of the northern district. He served Cambridge in its common council, as its representative in The Great and General Court in 1872 and 1873, and upon its school committee. He served Tufts College as one of its Board of Trustees. From March, 1886, until Governor Wolcott, on September 7, 1898, appointed him an Associate Justice of the Supreme Judicial Court, he rendered exceptional service to the Commonwealth as a Judge of the Superior Court. Lawyers loved to try before him. Jurors remembered their terms of service with joy and pleasant recollections of witty sayings; dignified handling of trial problems; shrewd suggestions in puzzling situations; wise, sensible, helpful charges that yet never took from them their sense of personal responsibility for the proper verdict. Parties left the court room satisfied that they had had an impartial trial guided by sound rulings of law from a judge wise, not only from the books, but from sound common sense and broad human understanding.

His service on the Supreme Judicial Court added to his reputation. His opinions are clear, accurate and expressed in English written as only few can write. Several of them are likely to stand as landmarks of the law to which they relate. His colleagues found him as helpful as he was delightful in the work of the Court over the consultation table. His work, like that of most judges, is of greatest value in the parts least open to the eyes of the people. But the people generally did trust, honor and believe in him. Tufts, in 1891, and Harvard, in 1911, conferred on him their degrees of LL.D. He was no ascetic. He loved society. He played chess. He played golf. He greatly loved to watch a base ball game, and he attended league games whenever he could. He was loved by wife, children, and many friends. He resigned from the court in December, 1914, to live in enjoyable retirement till his death, March 26, 1922. As was said of him by

Chief Justice Rugg: "His career has added substance and lustre to the judicial annals of Massachusetts." The Academy may be proud of its Fellow.

WILLIAM C. WAIT.

EDWARD WASHBURN HOPKINS (1857–1932).

Fellow in Class IV, Section 3, 1915.

Edward Washburn Hopkins was born in Northampton, Massachusetts, on September 8, 1857, and died in Madison, Connecticut, on July 16, 1932. He was for half a century recognized as one of the leading Indological scholars of the world.

He graduated from Columbia University in 1878, and immediately went abroad, to Germany, where he spent three years studying Sanskrit, Iranian, and comparative philology at Berlin and Leipzig. He took the doctorate at Leipzig, under Ernst Windisch, in 1881, with a dissertation on the relations of the four castes in Manu. Returning to this country he was appointed Tutor in Latin at Columbia, which position he held from 1881–5, giving instruction also in Sanskrit and Avistan. From 1885–95 he was Professor of Greek, Sanskrit, and Comparative Philology in Bryn Mawr College. In 1895 he was called to Yale University as Salisbury Professor of Sanskrit and Comparative Philology, in succession to William Dwight Whitney. This position he held until 1926, when he retired and was made Professor Emeritus.

He spent one year (1896–7) in study and travel in India. In 1911 he visited the International Congress of Orientalists at Athens, and was given an honorary degree by the University of Athens. He also received honorary degrees from Columbia and Yale.

His scholarly activities covered a very wide field, but were especially noteworthy in sociological and religious subjects. Besides his doctoral thesis, mention may be made of his important monograph on the 'Ruling Caste' in ancient India; his books on *The Religions of India* (1895), *The Great Epic of India* (1901), *India Old and New* (1901), *Epic Mythology* (1915), *The History of Religions* (1918), *Origin and Evolution of Religion* (1923), and *Legends of India* (verse translations; 1928). His articles in scholarly journals and in encyclopedias (notably Hastings' *Encyclopedia of Religion and Ethics*) are far too numerous to permit mention even of illustrative examples; yet not a few of them contain enough matter to justify a book. His scholarship was thorough and painstaking, always based on sound and far-reaching acquaintance with original sources. In his two abovementioned books on religions in general, he went far beyond the field of Indology and showed a mastery of the whole vast subject of the history of religion.

He was elected a Fellow of the Academy on May 12, 1915.

FRANKLIN EDGERTON.

HECTOR JAMES HUGHES (1871–1930).

Fellow in Class I, Section 4, 1915.

Hector James Hughes was born at Centralia, Pa., on October 23, 1871, the son of James H. and Mary (Miller) Hughes. He studied at the public schools in Williamsport, Pa., and at Harvard College, where he graduated A.B. in 1894, specializing in languages, history, and economics.

After graduation, he entered civil engineering work until 1897, when he returned to Harvard and pursued professional engineering studies in the Lawrence Scientific School, from which he received an S.B. degree in Civil Engineering, June 1899. He then entered the engineering office, in Chicago, of the Chicago, Burlington and Quincy Railroad.

He returned to the Harvard Engineering School in 1902 as instructor, and remained there as a teacher of engineering until his death on March 1, 1930. He was appointed successively assistant professor, professor of Civil Engineering, Director of the Harvard Engineering Camp at Squam Lake, N. H., and Dean of the Harvard Engineering School (in 1920).

In all his work, Dean Hughes showed marked executive ability and conscientious care. In the Summer Engineering Camp, he exacted strict discipline and required hard work from the students without loss of their respect and esteem. In the Engineering School, he insisted on the maintenance of high standards in work and scholarship.

Dean Hughes married Elinor Lambert of Cambridge, Mass., who, with their two daughters, Katherine Porter and Elinor Lambert, survive him.

A more detailed Minute of his professional work, publications, and accomplishments was published in the Gazette (June 14, 1930, page 220) of Harvard University, where he was held in high esteem and affection by his colleagues.

Dean Hughes was elected a Fellow of the American Academy in 1915.

ARTHUR E. KENNELLY.

WILLIAM DEWITT HYDE (1858–1917).

Fellow in Class III, Section 1, 1915.

William DeWitt Hyde was born on September 23, 1858 and died in 1917. He was elected a member of this Academy in 1915.

After preparing for college at Phillips Exeter Academy, he entered Harvard College in the autumn of 1875 and took his A.B. degree in 1879. Immediately on graduation he entered the Union Theological Seminary in New York, and after a year at that institution spent additional years in the Andover Theological Seminary. In 1883 he became minister of a Congregational church in New Jersey, where he performed his parish duties with energy and devotion, and with extraordinary success. Very shortly, in 1885 his career took an entirely new turn. He became president of Bowdoin College and also professor of Philosophy in the institution. It is no secret that the appointment was made largely on the recommendation of his friend Professor George Herbert Palmer. Hyde had been in close contact with Palmer during and after his college course, and a friendship arose, based upon a degree of mutual respect and regard not often seen between men of different generations. Hyde remained President of Bowdoin College until his death on June 29, 1917.

His leadership of Bowdoin College was beyond praise. Though a young man, at the outset a stranger to the college and to the alumni, and without experience as a teacher, he became at once the very heart of the institution. By nature and temperament he was a teacher and a spiritual leader. He had a remarkable gift of language, spoken and written, a vigorous personality, and all the qualities of an inspiring teacher. He had executive ability as well, and both in dealing with faculty and students was prompt, just, sound in judgement, unhesitating in decision. From the start his views on the fundamental questions of academic policy were settled. He believed that the college of moderate size, distinct from a university, had a place of the first importance in American life and culture. Bowdoin College was to be maintained and developed as an institution which should do its full duty in accord with that tradition. As part of this settled policy he aimed first of all to secure men of the right kind as students and as teachers. In his selection of teachers and professors weight was laid not only on intellectual and scholarly competence, but upon soundness of character, and effectiveness of personality and of teaching gift. His judgment in the selection of teachers was proved by the fact that those whom he caught young, so to speak, and brought to Bowdoin, were frequently asked to accept positions elsewhere at larger institutions, with higher salaries and wider opportunities. He was entirely content that this should happen, believing it to be better to have a first-rate man for a few years than one of less quality for life. To the writer of the present memoir he once remarked that if he succeeded in appointing a first-rate young professor once a year, he believed he was achieving his chief task as president.

President Hyde had an extraordinary gift of statement, oral and written. He was a wide reader, and a lover of all good literature in verse and in imaginative writing as well as in philosophy and history. His preaching and his public addresses were distinguished by fluency, ease, and roundness of statement. The same facility appeared in his books. Of these there was a constant succession. Among them may be mentioned: Practical Ethics, 1892; From Epicurus to Christ, 1905; Practical Idealism, 1897; The Quest of the Best, 1913. They made no show of being products of research or the results of profound scholarship. While the writings of a man imbued with scholarship, they were addressed to the general public, and always were homiletic in character.

Hyde took an active part in social movements, and might have been enlisted in political life if he could have been brought to consent to such a change; but he was rightly convinced he could be most useful as a teacher, educator, and preacher. A convinced believer in the Christian faith as it was held by his ancestors, he was none the less catholic as regards the beliefs of others, and strongly favored union among the denominations.

Strong, earnest, direct, equipped with the full culture of his profession, eager to help in every good work, conservative by nature and training yet tolerant and forward looking, he won universal admiration as a noble representative of the best of the New England which even in his own day was becoming old.

F. W. Taussig.

CHARLES WILLISON JOHNSON (1863–1932).

Fellow in Class II, Section 3, 1917.

Mr. Johnson, naturalist of the old school, came to Boston in 1903 to take charge of the collections of the Boston Society of Natural History. His earlier activities, with biographical data and portrait, have been published elsewhere, from several sources.[1] They all record singleness of purpose, friendliness, joy in scientific work, and a contented life such as is rarely found. The following note, read at a memorial meeting, describes him as Curator of Insects and Molluscs:—The door of Mr. Johnson's room was always open, as he sat close by the window with a tray of insects at hand. There might be a fire, a traffic jam or riot in the thoroughfare outside. He saw it not, but deftly transferred, examined and recorded his insects, never breaking a leg or antenna in the tangled mass. Whenever any one—school-boy, expert, old friend or stranger—crossed the threshold, his work was laid aside. His time and wealth of information were at every one's disposal. Often I have visited there, and never without learning much and developing a taste for more, as with the utmost simplicity Mr. Johnson drew upon his boundless lore of insects and of molluscs.

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[1] Brooks, W. S., Bull. Bost. Soc. Nat. Hist., Oct. 1932, No. 65, p. 3–5.
Brues, C. T., Entomological News, 1933, vol. 44, p. 113–116.
Gray, A. F., The Nautilus, 1933, vol. 46, p. 129–134.
Melander, A. L., (Entomological publications of C. W. Johnson, 180 titles) Psyche, 1932, vol. 39, p. 87–99.
Rehn, J. A. G., Science, 1932, vol. 76, p. 226–227.

Of course the collections in both branches, of which he had charge, were in perfect order. He knew most of the thousands of specimens by name, and produced immediately what was relevant to the matter in hand. Suppose you came with some beetle which had attracted your attention, common, it turns out, as English sparrows. "That," he would say, "is an interesting specimen, though not rare." Then he might mention that in your locality back in '69 some one had found a beetle never collected since. The Society's specimen of that other insect was a poor one—perhaps you might find a better—and with anecdotes of catches and collectors, invariably kindly told and entertaining,—with delightful stories of amazing habits and fascinating peculiarities of sundry insects, all suggested by your specimen,—with questions of nomenclature raised it seems by every myriad midge, to be disposed of incidentally,—the visitor left, to collect more wisely, and to plan another such pleasurable encounter. Should the specimen be a rare one, his beaming smile was glorious. "By Jove, that little fly is new to me," I heard him say but once; and the next day he could report that some one had taken three specimens in Michigan, with no other record to date. His detailed knowledge of the groups in which he was specially interested was impressive.

For him there was no idle time. When absent from the Museum, he was surely off collecting, and in his patient work he took unending satisfaction. It was Edward Everett Hale, speaking at a meeting of the Boston Society of Natural History, who said,—"I like these meetings. I sometimes think that people who care nothing about plants and insects will have nothing to talk about when they go to heaven. For the good Lord is interested in these things, or He wouldn't have made so many of them." So Mr. Johnson must have thought. In a sentence he tells the secret of his fine work as curator,—"I dearly love to assist and encourage others in the study of nature." He did exactly that, with a modesty and simplicity concealing the deep, wide knowledge that made his help effective.

FREDERIC T. LEWIS.

WILLIAM WILLIAMS KEEN (1837–1932).

Fellow in Class II, Section 4, 1901.

William Williams Keen was born in Philadelphia, January 19, 1837, the son of William Williams Keen and Susannah Budd. He was graduated from Brown University in 1859 and returned for a postgraduate year in chemistry, physics and English literature.

Upon matriculating at Jefferson Medical College in 1861, he was fortunate in coming under the influence of S. Weir Mitchell. After receiving his degree in 1862 he was appointed Acting Assistant Surgeon in the United States Army and was assigned to work under Mitchell. Between 1862 and 1864 their studies on injuries to peripheral nerves were made and he soon became known through his publications in association with Mitchell and Morehouse. The next two years were spent in study abroad.

Upon his return in 1866 he was elected to the staff of St. Mary's Hospital in Philadelphia where he served until 1889 and at the same time was a chief in the Medical Clinic at the Jefferson Medical College. He also took over the private Philadelphia School of Anatomy where he lectured on anatomy and operative surgery until 1875. He stopped his work in the Medical Clinic at Jefferson in 1869 but continued his lectures on pathology until 1875. His lectures at the Philadelphia School of Anatomy brought him considerable renown as a teacher and he was appointed Professor of Artistic Anatomy at the Pennsylvania Academy of Fine Arts in 1876, a position which he held until 1890.

When the International Medical Congress was called in Philadelphia in 1876 he was one of the secretaries. Lister was made the chairman of the section on surgery. In his address he outlined his antiseptic system. Doctor Keen later wrote:—"As a result of this debate I was completely convinced of the value of Lister's methods. From the first day of October, 1876, I never swerved in my allegiance to antiseptic surgery and its later development, aseptic surgery." He was the first surgeon in Philadelphia to introduce the newer surgery.

The following year, Doctor Keen became a Consulting Surgeon and Lecturer on Clinical Anatomy at the Woman's Medical College and was appointed Professor of Surgery at this institution in 1884.

He resigned the position upon his election to the chair of Surgery at Jefferson in 1889. He held the chair until his retirement in 1907.

Although his interest in neurology and the surgery of nerves started from earlier researches with Weir Mitchell, his attention was focused on surgery of the brain by a series of circumstances which he always regarded as the "hand of Providence." He was given the task of revising Gray's Anatomy in 1886. Since the previous edition a number of new facts on cerebral localization and the pathways of impulses in the central nervous system had been discovered. Doctor Keen found it necessary to make a thorough study of these recent developments in order to revise the text book. His edition was printed in 1887. Shortly after publication a patient presented himself with symptoms which led Doctor Keen through his recent knowledge of cortical localization to diagnose a brain tumor. Removal of this tumor, the largest successfully removed up to that time, added to his reputation and he became known as America's first and most prominent brain surgeon.

His professional interest was not limited to neurosurgery, but embraced the entire field of medicine. His original contributions, numbering more than 250, spread over 62 years. Although he never performed animal experiments he was eager to further the development of experimental research and strenuously opposed all efforts of the anti-vivisectionists. Many of his writings in later years were directed against them and he willingly became the target of their invective.

His lectures and clinical instruction were given with painstaking thoroughness from hand-written notes. He never read his lectures and was most meticulous in his use of English. Towards the close of his life he frequently spoke over the radio on questions of public health. On one occasion his voice was recognized by a physician in Saskatchewan who had been his pupil 40 years before.

Thoroughness pervaded his whole being, combined with wide and eager interest in his surroundings. He was an omnivorous reader and took careful notes. His chief relaxation came from this source and from personal contacts. Up to the last year of his life he was actively engaged in correspondence with hosts of friends throughout the world on scientific, political and religious matters.

As an operator he was bold but very painstaking and careful. The welfare of his patient was always his first concern. In the words of Lord Moynihan: "His work upon all branches of surgery—and he practiced all with immense success—was perhaps the most considerable and the most inspiring of that of any surgeon in the States."

In addition to his original contributions in the field of surgery he was the author of a number of books both of medical and of general interest. The following are some of his better known works: Early History of Practical Anatomy, 1870; Surgical Complications and Sequelae of Typhoid Fever, 1898; History of the First Baptist Church, Philadelphia, 1898; Animal Experimentation and Medical Progress, 1914; The Surgical Operations on President Cleveland in 1893, 1917; Medical Research and Human Welfare, 1917; Everlasting Life, 1924.

He also edited in 1870 Practical Anatomy—Manual of Dissections; 1879, American Health Primers; 1887, Gray's Anatomy; 1892, An American Text Book of Surgery; and in 1905 to 1921, Keen's Surgery.

Honors were showered upon him: he was President of the American Philosophical Society, the American Surgical Association, the American Medical Association, the College of Physicians of Philadelphia, the Congress of American Physicians and Surgeons, and the Fifth International Surgical Congress.

He was an honorary member of most of the European medical and surgical societies as well as an Honorary Fellow of the Royal College of Surgeons of England, Edinburgh and Ireland. He received honorary degrees from many universities both at home and abroad and was decorated by France and Belgium. Many medals were bestowed upon him, including the Henry Jacob Bigelow Gold Medal of the Boston Surgical Society for contributions to the advancement of surgery. Doctor Keen was also very proud of his Loyal Legion Medal for service as a Lieutenant during the Civil War and his Victory Medal for service as a Major in the Medical Reserve Corps of the United States Army during the World War.

His interest in Brown University never flagged. He was a Trustee from 1873 until 1895 and then a Fellow until his death. Deeply religious, he took an active part in the Baptist Church as a Trustee of Crozer Theological Seminary for 65 years and as Deacon and Trustee of the First Baptist Church of Philadelphia.

The last months of his life were spent as an invalid but he still maintained his active interest in his surroundings and in the developments of science and medicine. Always cheerful, he greeted friends with his customary smile and ready wit. At the time of his death, June 7th, 1932, he was in his 96th year. In his will, besides personal bequests, he endowed a Fellowship at Brown University for research in one of the fields of pure science.

NORMAN E. FREEMAN.

OLIVER DIMON KELLOGG (1878–1932).

Fellow in Class I, Section 1, 1921.

Oliver Dimon Kellogg was born on the tenth day of July, 1878, at Linnwood, Pennsylvania. His father was the Rev. Day Otis Kellogg and his mother, Sarah Cornelia Hall. The family was far from affluent so that Kellogg learned stern lessons in economy. In college he was largely on his own resources and had to meet his expenses by his own efforts. This training gave him an appreciation of the difficulties of the poor student which was invaluable to him in later life when he was father confessor to many a young man of high ambitions but slender purse.

Kellogg's first venture away from home was at the preparatory school in Lawrenceville, N. J. Later he entered Princeton and began specializing in mathematics. He was especially happy in his association with Edgar Odell Lovett, now President of the Rice Institute, and the much-beloved Henry Burchard Fine. He graduated in 1899, a member of Phi Beta Kappa, and received the degree of A.M. a year later. There followed three years in Germany, divided between Göttingen and Berlin. He took his doctorate in the former place in January 1903, writing his thesis under David Hilbert on the application of linear integral equations to Potential Theory. As the twig was bent, so grew the branch. His interest in Potential Theory lasted throughout life.

He returned to Princeton for two years in the autumn after his German promotion, but there was no permanent vacancy in his own university, so he transferred to the University of Missouri where he occupied the positions of assistant professor and professor until 1918.

During these busy years when he obtained valuable experience in problems of university administration and helping undergraduates with their personal difficulties, he married Edith Taylor on August 26, 1911, and she, with her daughter, Margaret Philbrick, survives him. In 1918 he departed on leave of absence for New London, Connecticut, to study problems of submarine detection with a group of physicists who were occupied with this pressing question. This work was done under the pressure of great haste, but Kellogg's directness, promptness, and accuracy were of fundamental advantage to all the scientists employed. His leave of absence in 1918 was extended another year to enable him to come to Harvard as lecturer. It was never renewed, for his connection with the University of Missouri terminated at the end of that year and he made his final move to Harvard. He began as an associate professor and was appointed to a full professorship in 1927.

Kellogg's doctor's thesis at Göttingen dealt with an application of linear integral equations to Potential Theory. The subject of Potential Theory, which is of fundamental importance both in mathematics and physics, was to occupy much of his future attention. If a region of space be permeated by a system of gravitational or similar forces, the mathematical handling of the situation is simplified by the fact that there exists a function depending upon position alone from which it is possible to deduce the distribution of forces. Such a function is called a potential function. To Kellogg are due a number of results of importance in this domain, in particular some concerning the dependence of the potential function upon the boundary values; it is significant of the originality of his ideas that an illuminating example constructed by him was the direct occasion of a fundamental theoretic advance. His book, "The Foundations of Potential Theory," published in Berlin in 1929, is regarded as the principal text in the field. At the time of his death he was eagerly making preparations for an advanced companion volume dealing with the same topic.

His mathematical interests were, however, by no means limited to Potential Theory. Not only did he have an unusual grasp of many branches of mathematical physics, but he wrote original papers in such varied fields as linear differential equations, series of orthogonal functions, function-space, the theory of the gyroscope, and the theory of numbers. His work was characterized by persistence and ingenuity, and he seldom abandoned a problem which had once occupied his attention, until he had definitely contributed to its solution.

The outstanding traits in Kellogg's character were sympathy and sensitiveness. He was equally at home in dealing with a charming but inconsequential freshman who had yet to learn that life was a serious matter, with the overworked and over-worried graduate student who had more financial anxieties than professional prospects, or with the distinguished colleague in Europe or America who was puzzled by a question of priority or a matter of professional etiquette. His sympathy was poured out impartially on old and young. Old and young loved and trusted him in return. He died of heart failure Friday, August 26, 1932, while climbing alone on the hills of Maine. He died as he lived, looking forward and climbing upward.

JULIAN L. COOLIDGE.

EDWARD SKINNER KING (1861–1931).

Fellow in Class I, Section 1, 1915.

Born at Liverpool, N. Y., May 31, 1861, son of Nathaniel and Cornelia C. King; died at Cambridge, Mass., September 10, 1931.

Degrees: A.B., 1887, A.M., 1890, Sc.D., (Hon.), 1927, Hamilton College.

As an undergraduate Professor King concentrated in mathematics, studying under Professor C. H. F. Peters, to whom is to be attributed his entrance into astronomy. Professor Peters sent him in 1887 to Professor Pickering who, in 1889, placed him in charge of the Harvard Station near the summit of Mt. Wilson. In 1890 he returned to Cambridge, and on July 23 of that year married Kate Irene Colson of Batchellerville, N. Y. Mrs. King and two children, Harold Skinner and Margaret Wight, now survive. The third, Everett Tryon, died in 1918.

From 1893 until his death he superintended the photographic work of the Harvard Observatory.

His major scientific work was concerned with standard photographic photometry of stars and planets and with tests of photographic plates.

University appointments:
1887–1913 Observer
1913–1926 Assistant Professor of Astronomy
1926–1931 Phillips Professor

Tenure of this latter office ended on September 1, 1931, with his retirement.

Memberships: Fellow Am. Acad. Arts and Sciences; A.A.A.S.; Am. Astron. Soc.; Int. Astron. Union; Société Astronomique de France; hon. mem. Nantucket Maria Mitchell Assn.; Phi Beta Kappa.

A fine tribute to Professor King is contained in a letter to him from Professor S. I. Bailey sent shortly before the latter's death. "To have done work which is widely recognized, to have gained the sincere esteem of many and the real love of even a few, surely these are sufficient reasons to look on life as well worth the living."

L. B. ANDREWS.

ROBERT WILLIAMSON LOVETT (1859–1924).

Fellow in Class II, Section 4, 1922.

Dr. Robert Williamson Lovett was elected a Fellow of the American Academy of Arts and Sciences on May 10th, 1922. At this time he occupied the John Ball and Buckminster Brown chair of Orthopaedic Surgery at the Harvard Medical School. He was also in charge of the Orthopaedic Department of the Boston Children's Hospital, surgeon-in-chief of the New England Peabody Home, consulting surgeon to the New York State Board of Health, chairman of the Harvard Infantile Paralysis Commission, corresponding member of the British, French and Italian Orthopaedic Associations, member of the Société Internationale de Chirurgie, Fellow of the American College of Surgeons, member of the American Orthopaedic Association (Pres. 1897), American Medical Association, Massachusetts Medical Society, Boston Surgical Society (Pres. 1920–'22), and of many social organizations.

Dr. Lovett was born in Beverly, Massachusetts, on November 18th, 1859, the son of John Dyson and Mary (Williamson) Lovett. He was graduated from Harvard College in 1881, from the Harvard Medical School in 1885, and received his surgical training at the

Boston City Hospital, becoming later a member of its surgical staff. He practised general surgery in Boston until 1899, when he began to limit his private work to orthopaedic surgery and his hospital work to the Boston Children's Hospital. During the World War, Major Lovett was placed in charge of the training of surgeons to fit them to care for bone and joint casualties. Under his direction, army teaching centers were organized throughout the country and an important national service was performed.

His contributions to medical literature were many. Books on Lateral Curvature of the Spine, Hip Disease, Infantile Paralysis, text-books on Orthopaedic Surgery in collaboration at first with Dr. Edward Hickling Bradford and subsequently with Sir Robert Jones, Bart., of England, ran through many editions.

Robert Lovett may be said to have been an aristocrat of medicine, keeping his own counsel, maintaining and demanding high standards of research and clinical practice; not above compromise but at the last ditch, fighting if necessary with adroitness and courage. Friends held him fast. He possessed an unusual charm of companionship in diversion. The personal tribute of Sir Robert Jones in the British Medical Journal well epitomizes his character: "He represented the highest ideals of our art and brought to bear upon it the rich power of a cultured mind."

The fragile health of his later years broke under the strain of public and private work. He was taken ill on a voyage to England in the summer of 1924, and died in Liverpool at the house of Sir Robert Jones on July 6th, 1924.

Robert B. Osgood.

NATHAN MATTHEWS (1854–1927).

Fellow in Class III, Section 1, 1914.

Nathan Matthews was born March 28, 1854, and died in 1927. He took his bachelor's degree at Harvard in 1875, his degree as bachelor of law in 1880 and he was made doctor of laws in 1909. All his life he was a lawyer in busy practice, but he could always spare the time for public service. He became Mayor of Boston in 1891 and served as such for four terms until the end of the year 1895.

His mayorality was a very notable one and is still regarded as a model for city administration. He served later on several commissions for the government of the city, but particularly as chairman of a Commission for the Proposal of a New Charter for the City of Boston. The commission studied the situation with great fullness and recommended the charter under which the city of Boston is governed today. Nathan Matthews was a man of true scholarly nature and applied to current problems of practice and of politics the attitude which another might have applied to the investigation of abstract truth. He studied with the greatest care every problem brought before him, whether as counsel or as mayor, and proved in his own life the power of scholarship to solve concrete problems. He was elected a Fellow in May 1914.

J. H. Beale.

CLIFFORD HERSCHEL MOORE (1866–1931).

Fellow in Class III, Section 2, 1910.

Clifford Herschel Moore was born on March 11, 1866, in Sudbury, Massachusetts. Of old New England stock, he was the seventh of his line in as many generations to be born on the same farm. His grandfather was for forty consecutive years moderator of the Town Meeting. He himself was proud of the New England tradition which he well exemplified even when his influence extended far beyond New England.

After preparation at the Framingham High School, Moore entered Harvard College, and graduated in 1889, *summa cum laude*. Though his chief field of scholarly interest, then as always, was in the classics, he read widely in other subjects, developed his interests in music and literature, and made many friends. Three years of experience in teaching at the Belmont School, in California, and two years at Phillips Academy, Andover, contributed not a little to his later success as a college teacher. In spite of the brevity of his stay at Andover he left such a mark that eight years later he was made a Trustee of the Academy, and held his trusteeship for the rest of his life.

From 1894 to 1898 Moore served at the University of Chicago as Instructor and as Assistant Professor of Latin, except for the year

1896–97, which he spent at the University of Münich, winning the degree of Ph.D. *summa cum laude*. In 1898 he returned to Harvard as Assistant Professor of Latin; in 1905 he became Professor, and in 1925 was named Pope Professor of Latin. Among the temporary appointments that he held were those of Annual Professor in the American School of Classical Studies in Rome (1905–06), Harvard exchange professor at certain western colleges (1913), lecturer of the Lowell Institute (1914), and Ingersoll Lecturer on the Immortality of Man, at Harvard (1918). He received from Colorado College the honorary degree of Litt.D. (1914). Among learned societies of which he was a member mention may be made of the New England Classical Association, of which he was president in 1910–11, the American Philological Association, of which he was president in 1919–20, the American Philosophical Society, the American Historical Association, and the Dante Society. He was elected a Fellow of the American Academy of Arts and Sciences on March 9, 1910 (Class III, Section 2), and enjoyed attending its meetings whenever he was able.

Although Moore was a capable scholar in many fields of classical scholarship, and gave instruction and published works in the fields of literature, history, grammar, epigraphy, and antiquities, his chief interest was in Greek and Roman religion and philosophy and their influence on Christian thought. Probably his most widely read works are his edition of the *Odes* of Horace and his Lowell Institute lectures on "The Religious Thought of the Greeks." His last published work, which appeared within a few months of his death, was on "Theories Regarding the Immortality of the Soul." As a teacher he was incisive and stimulating; he was impatient of slipshod work, and knew how to call forth his students' best efforts.

Moore's executive ability, early recognized, caused him to devote an increasing proportion of his time to the administrative duties that were laid upon him. For some years he was Chairman of his Department; the critical year 1918 found him discharging the duties of acting Dean of the Graduate School of Arts and Sciences, Chairman of the Committee on Instruction, and Director, under the Federal Government, of the S. A. T. C. first of Harvard and then of all the New England colleges. In 1925 he became Dean of the Faculty of

Arts and Sciences, and was thus responsible for the budget and for many other tasks which had hitherto been borne by the President. In most of the educational policies that matured during the last twenty-five years of his life he had a large share,—entrance requirements, the modification of the elective system, the institution of the general examination and the tutorial system, the increase both of tuition fees and of salaries, the organization of the Houses. More and more his energy and tact, his judgment and practical idealism, his friendliness and courtesy in dealing with people, won respect and affection and achieved abundant results. Though he knew his own mind, he could work with others, and could sink personal preferences in a common undertaking. He knew how to lead discussion, or, on occasion, how to cut it short. It can not be said that he suffered fools gladly; nevertheless, he gave of his time and strength generously to many who had little claim on him. In him the man was never lost in the official; and younger men, whom he went out of his way to encourage, were put at their ease when in company with him. If he invited their opinions, it was because he genuinely valued them.

Moore was married in 1890 to Miss Lorena Leadbetter, of Charlestown, Massachusetts. Their home on Brattle Street, Cambridge, gave hospitality to young and old, and to many outside the immediate academic circle. Here were to be found a gracious welcome, broad human sympathies, and much good talk. Moore's fondness for the out-of-doors found expression both in his summer vacations in the Adirondacks and in winter outings. He was an accomplished figure-skater. The illnesses of his later years were borne with a cheerful fortitude.

On August 31, 1931, Moore died in Cambridge. In him there was something of the Greek, something of the Roman, and much that was drawn from his native New England.

WILLIAM C. GREENE.

GEORGE FOOT MOORE (1851–1931).

Fellow in Class III, Section 2, 1906.

George Foot Moore, Fellow of the Academy, Class III, Section 2 from 1906 onwards and President from May 1921 till May 1924, was born at West Chester, Pennsylvania, on October 15, 1851. He received his education at West Chester Academy, at Yale and at Union Theological Seminary. After some years of teaching and ministerial work he was in 1883 called to the chair of Hebrew at Andover Theological Seminary, where he taught till 1901, being President from 1899 to 1901. In 1902 he passed to Harvard as Professor of the History of Religion, a position which he occupied till his retirement in 1928. In 1909–10 he was Harvard Exchange Professor at the University of Berlin. He was President of the Massachusetts Historical Society, Vice-President of the Colonial Society of Massachusetts, and Doctor honoris causa of Harvard, Yale, Göttingen, and several Jewish institutions. He died in Cambridge on May 16, 1931.

He started life as a scholar with an admirable linguistic foundation and grew up in a time of very rapid advance in the study of various fields of the history of religious thought and practice. His marvellous powers of work and his rare zest for knowledge enabled him to keep abreast of these advances in a manner which was unique and can hardly be seen again. This width of range went hand in hand with intensive work of a very high order. In all that he did, in his writing, in his teaching, in his remaking of the Harvard Theological Review, there was an unhurried thoroughness. He was by nature a severe critic, to none so severe as to himself, impatient of all humbug and all half knowledge, but always ready to extend the greatest generosity to any honest student, reserved but with a remarkable gift of friendship and a delicate humor.

His editions of Judges (1895, 1898, 1900) and his articles in the Encyclopaedia Biblica established his reputation as a Semitist of eminent distinction. His History of Religions (1913/9) is the only notable survey of the whole field made by one man, and in its brevity represents the final crystallization of very long thought and study; it is one of the few general treatises of which specialists praise the sections referring to their own topics. A shorter book, The Birth and Growth of Religion (1923) remains the best introduction to the subject as a whole. These works would in themselves have justified his great reputation, but his last book, Judaism (1927/30), is in a higher class. It belongs to that rarest type of work in which something of the past comes to life in its own flesh. The reader knows intuitively that it is not another book about Judaism, but something which a Rabbi of the second century of our era might have recognized as a worthy Summa Theologica of his own belief. He had given to the dead his blood, and they had drunk, and could speak.

Arthur Darby Nock.

THOMAS BURR OSBORNE (1859–1929).

Fellow in Class I, Section 3, 1914.

Thomas Burr Osborne was born in New Haven, August 5, 1859, his early ancestors on both sides having been of good English stock who came to New England in the early colonial years. His paternal ancestor, Richard Osborne, settled at Hingham, Massachusetts in 1635, removing to New Haven four years later, while his mother, Frances Louisa Blake, was a descendant of William Blake who settled at Dorchester, Massachusetts in 1630.

From his mother's side Osborne undoubtedly derived many of the traits of character that helped in the development of his life-work. His great-great uncle, Eli Whitney, was the inventor of the cotton gin, while his grandfather, Eli Whitney Blake, was the inventor of the stone crusher. Further, through his maternal grandmother, he was sixth in descent from the Reverend James Pierpont who graduated from Harvard in 1681 and who shortly thereafter became the pastor of the first church at New Haven. On the side of his father, Arthur Dimon Osborne, there was a long line of Yale graduates beginning with his great-great-great grandfather, Ebenezer Dimon of the Class of 1728, down to the subject of this sketch, who gained his Bachelor's degree in 1881.

During the following five years, Osborne's time was spent mainly in the study of chemistry at Yale, doing some teaching and some research, but in 1886 he joined the scientific staff of the Connecticut Agricultural Experiment Station and was soon embarked on his life work, the investigation of the fundamental chemistry of proteins, especially the proteins of vegetable origin. For twelve years Osborne studied in great detail the proteins of thirty or more grains and seeds, beginning with the oat kernel, inventing new methods of procedure in an effort to obtain the proteins in the purest possible forms. At that date it was the generally accepted belief that there were only a few chief types of vegetable proteins, and that chemical composition was the sole criterion of purity and individuality. Further, physiologists held the view that the different vegetable albuminoids were practically alike in their nutritive or biological value.

As Osborne's work proceeded it became clear that the number of different proteins in nature was not so limited as had been believed, for seed after seed yielded apparently pure substances that plainly were not identical, such, for example, as edestin from hemp seed, wheat, rye, etc., avenalin from the oat kernel, conglutin from lupine seed, excelsin from Brazil nut, amandin from the almond and corylin from the walnut, six distinct proteins which previously had been looked on as identical and grouped together under the name of vitellin. Some of these globulins Osborne obtained in well defined crystalline forms, thus emphasizing that they were without doubt definite chemical individuals.

Next followed an investigation, extending through several years, of the partition of nitrogen in these vegetable proteins, using the Hausmann method with some modifications. By this method Osborne aad his associates hydrolyzed some thirty odd individual proteins, representing various types from different sources, comparing the yield of the various *a*-amino acids, basic amino acids, etc., by which striking differences in the chemical make up of the vegetable proteins were revealed. As a simple illustration, Osborne found that leucosin, a protein of the wheat kernel, yielded 6.73 per cent of glutamic acid, while gliadin of wheat yielded 37.33 per cent of this acid. Again, leucosin gave on hydrolysis 2.75 per cent of lysine, while gliadin yielded none of this base. This demonstration that many of the vegetable proteins differ widely in their amino acid composition introduced a new factor in the identification of individuality, but even more important was the suggestion, which Osborne quickly recognized, that the character and amounts of the various substances represented by the basic and non-basic nitrogen of the protein molecule may have an important bearing on the physiological and nutritive value of the individual proteins.

This led to Osborne's work on nutrition and growth, in which he was associated with Lafayette B. Mendel; work which continued from 1909 to 1928 and which was concerned primarily with the biological value of individual proteins, as revealed by feeding small animals, under carefully defined conditions, with pure proteins of known constitution. By such methods of study it was soon found that with certain proteins growth could not be maintained, while with others of different chemical constitution growth was vigorous. Thus in time Osborne was able to differentiate between adequate and inadequate proteins, lack of growth being due to the absence of certain amino acids in the protein fed. Thus, for example, edestin and wheat glutenin were found to be adequate proteins, animals growing well on such a diet, while on zein from maize there was failure of growth, due to the fact that this protein is lacking in the amino acids lysine and tryptophane. In this manner there was opened up a new chapter of the greatest physiological importance bearing on the synthetical production of protein by the animal body from the crystalline fragments which result from the breaking down of food proteins in the processes of digestion, etc. Again, in their work on nutrition Osborne and Mendel brought to light much that contributed to our knowledge of those factors of growth which are now associated with the vitamins, as well as the effects of the individual inorganic constituents of the diet.

While the problem of protein differentiation was helped greatly by use of the methods of amino acid analysis, Osborne in collaboration with H. Gideon Wells, by several years of close study of the anaphylaxis reaction of vegetable proteins, was able to make differentiation between proteins that had hitherto shown no chemical differences. Osborne believed that the specificity of the anaphylaxis reaction depended on the chemical structure of the protein molecule, "but structures sufficiently alike to give rise to this reaction are only observed in proteins derived from plants that are botanically closely related."

Osborne's work on the vegetable proteins, extending from 1889 until his retirement from active service in 1928, constitutes a remarkable example of persistent effort in the scientific study of a single group of substances and their derivatives; work that required great faith, painstaking accuracy, clear thinking and great originality. The results of his investigations have had a profound influence upon many problems of biochemistry. Dr. Osborne died at New Haven on January 29, 1929. An adequate biographical memoir with a complete bibliography may be found in Volume XIV of the National Academy Biographical Memoirs, by Hubert Bradford Vickery; also in Bulletin 312, Connecticut Agricultural Experiment Station, February 1930, under the title "Thomas B. Osborne, A Memorial."

Russell H. Chittenden.

WILHELM OSTWALD (1853–1932).

Foreign Honorary Member in Class I, Section 3, 1905.

Professor Ostwald was a great teacher and a great organizer of scientific knowledge. He was the great leader in the development of physical chemistry. Many in the older generation of chemists in America today were his pupils.

He was born in Riga in 1853; he entered the University of Dorpat in 1872; his first scientific paper was published in 1875 and his doctor's thesis in 1878. In 1881 Ostwald was called to the Polytechnic at Riga as Professor of Chemistry, and here he began the preparation of his Lehrbuch der Allgemeinen Chemie. In 1887 he took the chair of Physical Chemistry at Leipsic and it was in the same year, just before he went to Leipsic that he began the publication of the Zeitschrift für Physikalische Chemie. Ostwald's important work was done at Leipsic between 1887 and 1906. His laboratory grew in fame and became so crowded that it was necessary to provide larger quarters. The Physical Chemistry Institute was built for him and dedicated in 1897. Besides directing this Institute he continued to edit the Zeitschrift für Physikalische Chemie. Ostwald's books, which are almost too many to enumerate, were nearly all of great importance and were powerful factors in spreading the doctrines of physical chemistry not only in Germany, but through translations into other countries.

In 1900 Ostwald invented, with the aid of his assistant Brauer, the process that bears his name, for the catalytic oxidation of ammonia by air in the production of nitric acid. Although this has since developed into a commercial process, Ostwald had no thought of immediate commercial profit; his motive was to ensure the military security of his country.

By 1906 Ostwald had begun to feel the strain of his many activities and he resigned his professorship. But this did not mean for him a rest. It was simply a transfer of his activities to other fields. He continued his interest in philosophy, which had been active since he had started in 1901 a new journal, the "Annalen der Naturphilosophie." A few of the books which he published from his home in Gross Bothen in Saxony give an idea of the activities of his later years. In 1909 a book entitled "Grosse Männer" in which he attempts to classify men of genius; in 1912 a book called "Der energetische Imperativ," and another entitled "Monumentales und dekoratives Pastell"; in 1923 "Farbkunde." He devoted much of his time in the later years to painting and to a study of the theory of color. In 1909 Ostwald received the Nobel prize for Chemistry.

Ostwald died April 4, 1932.

Ostwald's great service to science was made during his years at Leipsic, during which time he organized and consolidated that part of the field of chemistry which was opened by the brilliant theoretical work of Van't Hoff, Arrhenius, and Willard Gibbs. With his students he widened the experimental exploration of this field; with the Zeitschrift he recorded and in a way led the world's progress, and in his lectures and his books he organized this field into a well rounded branch of science.

One of the most interesting and appreciative accounts of Ostwald's life is by Wilder D. Bancroft in the Journal of Chemical Education, September and October 1933, pages 539 to 542 and 609 to 613.

ARTHUR A. BLANCHARD.

WILLIAM BARCLAY PARSONS (1859–1932).

Fellow in Class I, Section 4, 1914.

William Barclay Parsons was the son of William Barclay and Eliza Glass (Livingston) Parsons. He was educated in part in England and in part while travelling on the continent. He entered Columbia College in 1875, graduating in arts in 1879. Later he graduated from the School of Mines, as Civil Engineer, in 1882.

Beginning practice with the Erie Railroad, he reached the rank of Division Engineer, and wrote two books,—"Turn-outs" and "Tracks."

From 1885 to 1891 he was in private practice with his brother, H. de Berkeley Parsons, in New York.

After serving on two commissions which planned subways for Manhattan, he became, during the Spanish American War, Chief of Engineers of the New York National Guard, with the rank of Brigadier-General.

In 1898–9, he planned nearly one thousand miles of railroad in China. When this was built, he became its Chief Engineer. This work is described in his book, "An American Engineer in China."

Returning to New York in 1899, he began work on New York's first subways, his outstanding work, and continued as Chief Engineer of the Rapid Transit Commission until 1904.

This year be became a member of the Isthmian Canal Commission, a majority of which, including the foreign members and General Parsons, reported in favor of a sea-level canal.

Resuming practice in New York, in 1905, under the name of William B. Parsons, Consulting Engineers, he was active in public works of many kinds, the firm continuing under various names until his death.

In 1904, he served as a member of the Advisory Board of Engineers for the Royal Commission on London Traffic. His paper before the Institution of Civil Engineers of Great Britain, in 1908, describing the New York subways, brought him the Telford Gold Medal.

General Parsons' list of notable engineering accomplishments is long, and includes such water works as those for Vicksburg and Natchez, Mississippi; such hydro-electric works as those at Spier Falls on the Hudson River, at McCall Ferry (Pennsylvania) and at Colliers (New York) on the Susquehanna River, at Ephratah (New York), at Altmar (New York) for the Niagara, Lockport and Ontario Power Company, at Diamond Creek (Arizona) on the Colorado River, various Hudson River projects, and the general Niagara development; such transit problems as those which confronted San Francisco, Detroit, Cambridge, Baltimore, Chicago, Philadelphia, Toronto, and other cities; the Havana Docks; and the Cape Cod Canal. His paper describing the latter was awarded the Norman Medal by the American Society of Civil Engineers in 1918.

He was the author of the article on "Rapid Transit" in the Eleventh Edition of the Encyclopædia Britannica.

In 1915, General Parsons was appointed Chairman of the Joint Commission of Engineering Societies to urge the organization of an engineering reserve in the Army. He achieved success when the Officers Reserve Corps was created. Later he was commissioned a major and helped to organize the regiment known during the World War as the 11th Engineers.

Later on, as a member of the Military Railways Commission, he included the first military units sent overseas. He performed signal service, returning to the 11th Engineers, then in England, as its Lieutenant-Colonel, and later fighting in France, notably near Cambrai and Peronne, and later in the Meuse, St. Mihiel and Argonne offensives. He ended the war as its commanding officer when the regiment disbanded in May, 1919.

Following the war, he was transferred to the Engineers Reserve Corps, and in 1922 was appointed Deputy Chief Engineer with the rank of Brigadier-General. For his services as Chief of the Military Railways Commission, and for his valor, General Parsons received the Victory Medal with five clasps, the Distinguished Service Medal and a citation for conspicuous service from the United States, the Distinguished Service Order from Great Britain, the Order of the Crown from Belgium, and the Legion of Honor from France.

Since the war, he has written "American Engineers in France," "Robert Fulton and the Submarine," and at the time of his death had almost completed his last work, "Engineers and Engineering of the Renaissance."

He was a member of the Institution of Civil Engineers of Great Britain, honorary member of the Institution of Architects, and a member of the Société des Ingenieurs Civils de France, as well as being a past director and honorary member of the American Society of Civil Engineers, and a member of many scientific bodies.

As Trustee of the Carnegie Institution, he took a special interest in the explorations made in Yukatan. As a Vestryman and Warden of Trinity Church, New York City, as Trustee of Columbia University, and as a member of various clubs and societies, his interests were large. He died in the Presbyterian Hospital, the last rivet in whose structure he drove on May 24, 1926. He was Chairman of the Joint Administrative Board of the Medical Center which built this hospital.

The degree of Doctor of Science was conferred upon him by Princeton University in 1920 and by Trinity College in 1921. He was made a Doctor of Laws by St. Johns College (Annapolis, Md.) in 1909, and a Doctor of Engineering by Stevens Institute in 1920.

General Parsons was finely trained, practical, broad-minded, and cosmopolitan in view. He had the imagination to conceive, the ability to express, and the enthusiasm and courage to carry projects to completion, as well as that fine mental balance which made him so useful in "controlling the forces of nature to the use of man." Gifted and honored, he always was modest and seeking to serve others.

General Parsons is survived by his wife, Anna DeWitt Reed of New York City, also by his son, Dr. William Barclay Parsons of New York City, and his daughter, Mrs. Rudolph Weld of Boston.

R. S. Weston.

LOUIS VALENTINE PIRSSON (1860–1919).

Fellow in Class II, Section 1, 1917.

Louis Valentine Pirsson, professor of Physical Geology in the Sheffield Scientific School of Yale University and member of this Academy since 1917, died on December 8, 1919. He was born on November 3, 1860, in Fordham, New York. His early education was received under the tutelage of Rev. William J. Blain, of Amsterdam, New York, in whose family he lived for a number of years and until he was sixteen. He then attended school at Amenia Academy in Dutchess County, New York, subsequently moved to New Marlboro, Massachusetts. His interests even as a youth were strongly toward the natural sciences. It was therefore quite natural that he should enter the Sheffield Scientific School in 1879 for the continuation of his education. He graduated from this school with honors in the class of 1882. While an undergraduate his major field of study was chemistry and in the six years following his graduation he continued his chemical studies, supporting himself by tutoring and by acting as laboratory assistant for Professors Allen, Mixter, and later H. L. Wells.

During the year 1888–89 he served as professor of chemistry at the Brooklyn Polytechnic Institute. In the summer of 1889 he secured through the influence of Professor Brush of Yale a position as field assistant to Dr. Arnold Hague on a survey party of the United States Geological Survey, and the season was spent in making a survey of the Yellowstone National Park. The experience crystallized his interest in geology and he determined to follow this field of work. In order to better prepare himself for this career, he returned to New Haven and studied under Professor S. L. Penfield and others. A second summer was spent in geological work in Montana under the direction of J. P. Iddings and W. H. Weed. After some further work in New Haven, he spent the next year and a half abroad, partly in travel and partly in study at Heidelberg under H. Rosenbusch and in Paris under Professors Fouqué, Millard and Lacroix. It was during this period that he was thrown into contact with a number of men who later, with himself, became leaders in petrographic work.

In the fall of 1892 he accepted a position to teach mineralogy and lithology in the Sheffield Scientific School under Professor Penfield. The following year he extended his teaching activities to include instruction in general geology. At this time he considered a proposal to accept a position left open by the death of Professor George H. Williams of Johns Hopkins University, but finally decided to remain in New Haven, and was appointed an assistant professor of inorganic geology in the Scientific School. In 1897 he was promoted to the grade of full professor with membership in the Governing Board of the Sheffield Scientific School, a position which he held until his death.

In the teaching of the course in general geology he was during the earlier years associated with Professor Beecher and after the latter's death with Professor Schuchert. His chief interest, however, lay in advanced instruction and research in the general field of petrology, and a number of geologists received their training in petrology under him during the next two decades.

One important outcome of his teaching work in introductory geology was a textbook. This was first planned in cooperation with Professor Beecher and after his death was carried on with Professor Schuchert. This text was issued in two parts; the first dealing with inorganic geology was written by Pirsson, the second, covering historical geology by Schuchert. This excellent text has been widely used for many years in American colleges. Shortly after the publication of this text Professor Pirsson issued another book entitled "Rocks and Rock Minerals." While this book dealt primarily with rocks from the megascopic point of view, it contained an excellent general treatment of the principles of geology. He had in mind a more advanced text in petrology and had prepared a considerable portion of it, but increasing physical infirmities during the latter years of his life prevented him from completing what would certainly have been a most important treatise.

As an original investigator of igneous rocks, Pirsson occupies a very high position. His most important contributions centered about the smaller mountain complexes, chiefly laccolithic in character, occurring in Montana and lying easterly from the Rocky Mountains proper, the Castle, Little Belt, Highwood and Judith Mountains. The field work in these areas was done in part of W. H. Weed and Pirsson, or by Weed alone. The petrographic studies were entirely by Pirsson. The results of these investigations appeared in several bulletins of the U. S. Geological Survey and in independent papers, chiefly in the *American Journal of Science*. These studies contributed very largely to our knowledge of the laccolithic type of the igneous intrusion, explaining and confirming Gilbert's earlier classic work in the Henry Mountains. Much information was added regarding new rock types, while the characteristics of already known types were extended and light was thrown on the principles of rock classification. The work did much to substantiate and illustrate the relationships of igneous rocks leading to the concept of "petrographic province" which has played so important a role in the theories of the genesis of the igneous rock types. Pirsson later extended his earlier work by studies of igneous rock occurrences in New Hampshire, in which work he was associated with Dr. Henry S. Washington and Professor W. N. Rice. Professor Pirsson had a profound and intimate knowledge of igneous rocks. He was an accurate and painstaking observer, a clear thinker with a highly discriminating judgment. His descriptions of rocks and their occurrences were models of precise and accurate form. His long and thorough training in chemistry and mineralogy and his skill as an analytical chemist were valuable assets in all of these investigations and contributed very largely to the completeness and extraordinarily valuable quality of his petrographic work.

He was always alert to the more philosophic aspects of his science and contributed a number of papers in this field. With other petrographers he realized very keenly the difficulties and confusions in petrographic classification, and in 1898 he joined forces with J. P. Iddings, H. S. Washington and Whitman Cross in developing the so called quantitative classification of igneous rocks. In common with his colleagues he devoted an enormous amount of thought and labor to this classifications, and there is little doubt but that he exercised a great if not dominating influence in determining the final form which this classification assumed. The basis of this classification was essentially a chemical-mineralogical one, and this fact made Pirsson an invaluable member of the group in working out the principles which were to be used in the classification. Whatever its shortcomings as a classification of rocks may be, it undoubtedly served the important purpose of directing the attention of geologists in general to a more critical evaluation of the quantitative chemical and mineralogical characteristics of rocks.

Recognition of the importance of his contributions to petrography brought him membership in many learned societies, among them being the National Academy of Sciences, the Geological Society of Sweden, the American Academy of Arts and Sciences, and the Geological Society of America, of which he was vice president in 1915. A full bibliography of his scientific papers is given in the *Transactions* of the National Academy of Sciences in a memoir prepared by J. P. Iddings.

Professor Pirsson in 1898 was appointed secretary of the Governing Board of the Sheffield Scientific School and from that time on he devoted an increasingly large amount of time to the committee and administrative work for the school. He had a very deep interest in all educational matters and this, combined with his naturally sound judgment, rendered him a valuable man in school administration.

In 1902 he married Miss Eliza Trumbull Brush, daughter of Professor George J. Brush.

Professor Pirsson was a man of the highest ideals and character, and a man of broad cultivation, widely read and interested in many things. His students found in him a sympathetic and competent teacher; his associates, a delightful companion in a social way and an inspiring collaborator in scientific work. A very full and admirable account of his life and work has been written by his intimate friend and eminent geologist, Dr. Whitman Cross, of the United States Geological Survey, for the *American Journal of Science*, where it may be found in the issue of September 1920.

Charles H. Warren.

IRA REMSEN[1] (1846–1927).

Fellow in Class I, Section 3, 1875.

Ira Remsen was for many years the outstanding figure in American chemistry. He was the first to make possible the acquisition of a thorough and broad training in chemistry in this country. The laboratory which he established soon after his return from study in Europe attracted at once many ambitious young men to pursue graduate study and research, and its success led to the development of similar facilities in other institutions.

Ira Remsen was born in New York City, February 10, 1846 and died in Carmel, California in 1927. Before the completion of the required work at the Free Academy, now the College of the City of New York, he began the study of medicine and was granted the degree of M.D. at the age of twenty-one by the College of Physicians and Surgeons. The little he learned of chemistry in his medical course aroused his interest and he forsook medicine and went to Europe to study the more exact science. He began work in Liebig's laboratory in Munich and received there under Volhard his first systematic training in methods of analysis. As Liebig had given up the instruction of students, Remsen went to Göttingen, carried out a research with Fittig in organic chemistry, and was awarded the degree of Ph.D. in 1870. He went with Fittig to Tübingen to serve as his assistant for two years. On his return to America Remsen was appointed professor of chemistry and physics at Williams College. He set up a laboratory for himself, continued his research work, translated Wöhler's "Organic Chemistry" and wrote his "Theoretical Chemistry."

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[1] Biographical sketches of Ira Remsen have been published in Science, *66*, 243 (1927), J. Am. Chem. Soc., *1927*, 3282, J. Am. Chem. Soc. Proceedings, *50*, 67 (1928) and Biographical Memoirs, Nat. Ac. Sci. Vol. XIV. The last two references contain a bibliography of Remsen's scientific papers.

The publications of Remsen led to a call to head a department of chemistry at the recently established Johns Hopkins University. For twenty-five years he held the department in the first place among those devoted to graduate work in chemistry.

Remsen took an active part in the administration affairs of Johns Hopkins University and was called to the Presidency in 1901 on the retirement of President Gilman. He resigned his post in 1913, but his love of chemistry forced him back into the work and he accepted a consulting position in the petroleum industry. His insistence upon the need for research culminated in the establishment of a research laboratory to study the chemical problems of the industry.

Remsen and his students published a large number of papers of importance which can not be reviewed here. The training and inspiration of future chemists held a high place in Remsen's mind. He lived to see his efforts rewarded. Many of the leaders in research in chemistry in America are the products of his care; the ability to express the principles of chemistry acquired by his disciples has made many of them successful teachers.

Remsen extended his influence as a teacher by means of his text-books. His Organic Chemistry was a masterpiece of lucidity and logic. The book was used over the world in translations in several languages. He wrote three text-books on inorganic chemistry which remained for many years the standard texts in chemistry in America. He wrote in all eight text-books and laboratory manuals.

The accomplishments of Remsen brought him many honors. He was a Fellow of the American Academy of Arts and Sciences and honorary member of the Société Chimique de France, the Pharmaceutical Society of Great Britain, and the American Chemical Society, and honorary fellow of the Chemical Society (London). He served as President of the American Chemical Society (1902), the Society of Chemical Industry (1910), the National Academy of Sciences (1907–13) and the American Association for the Advancement of Science (1903). He was the medalist of the Society of Chemical Industry in

1904, and received the Willard Gibbs medal in 1914 and the Priestley medal in 1923. He was awarded many honorary degrees: LL.D., Columbia University, 1893; Princeton University, 1896; Yale University, 1901; University of Toronto, 1902; Harvard University, 1909; Pennsylvania College, 1910; University of Pittsburgh, 1915; D.C.L., University of the South, 1907.

Ira Remsen's ashes rest in a beautiful new laboratory dedicated to his memory and named Remsen Hall. The men who were reared as chemists by his painstaking hands have established a fund in his honor to be used in furthering research in chemistry at the university.

JAMES F. NORRIS.

WILLIAM NORTH RICE (1845–1928).

Fellow in Class II, Section 1, 1917.

A biography of Dr. Rice by L. G. Westgate will be found in the Bulletin of the Geological Society of America, Volume 40, pages 50–55, and in Volume 35, page 117, an account of the testimonial presented to him by his geological colleagues. The Wesleyan University Bulletin of January, 1929, Volume 23, Number 1, was entirely given up to appreciations of him and includes a reprint of the obituary notice in the American Journal of Science, January, 1929.

He was the first Doctor of Philosophy in Geology at Yale and was said to be the favorite pupil of James D. Dana, whose Geology he edited after his death. From 1903 to 1916 he was Superintendent of the Connecticut Geological and Natural History Survey. He was connected with Wesleyan from 1867 to 1928, being three times acting president. Other data will be found in Who's Who.

His scientific work was largely connected with Connecticut. He also wrote upon Bermuda and upon certain broader aspects of geology, especially with regard to the use of some of Dana's terms and the classification of mountains. He was prominent in education and religion as well as in geology and was a favorite after dinner speaker of the Geological Society, combining broad philosophical views with genial humor. Few that heard it will forget when he was rather teasingly called, about the time of the Scope trial, to speak at the Ann

Arbor dinner of the Geological Society as a clergyman, his response that Bryan was not much more wrong on evolution than on economics.

At the annual meetings of State Geologists his sound judgment and open mind, combined with sane conservatism in the use of scientific terms, was notable. It was noteworthy that a professor other than one at Yale should be superintendent of the Connecticut Survey. This showed not only the broad mindedness of his Yale colleagues but his scientific leadership and high standing.

His eminently synthetic mind is shown in those of his works which combine the religious and geological point of view: "Science and Religion," "The Return to Faith," "The Poet of Science." It was due perhaps as much to him as to any other man that the great religious denomination to which he belonged has not assumed an attitude hostile to evolution. Talking to me he one time humorously alluded to himself as the "Lord High Inquisitor," referring to the fact that he was Chairman of the Examining Committee of the New York Conference of the Methodist Episcopal Church, and told me that he knew of but one in that Conference who did not believe in evolution.

Up to nearly the very end he was an active and valued participant in the Intercollegiate Geological Excursions of New England.

ALFRED C. LANE.

JAMES HARDY ROPES (1866–1933).

Fellow in Class IV, Section 1, 1909.

Professor James Hardy Ropes, who was elected a Fellow of the Academy on May 12, 1909, died at his home in Cambridge on January 7, 1933, in his sixty seventh year.

He was born in Salem, Massachusetts, September 3, 1866, the son of William Ladd Ropes, Librarian of Andover Theological Seminary, and of Harriet Lawson (Pierson) Ropes. He graduated from Phillips Academy, with first class classical honors in 1885, and from Harvard College, *summa cum laude*, in 1889. After a year spent in field work with the United States Geological Survey, he turned to what was to become his life work as a theologian. Three years in Andover Seminary were followed by a year of graduate work in German universities.

He began his academic life as an instructor in Harvard, 1895, receiving his full professorship in 1903. For seven years from that date he was Bussey Professor of the New Testament, and in 1910 was appointed to the Hollis Professorship of Divinity, the oldest chair in the University.

Professor Ropes was ordained to the ministry of the Congregational Church in 1901, but never served as a parish minister. He was however for many years one of the unofficial ministers of religion to the University community.

Professor Ropes' name had become in these latter years a synonym for New Testament scholarship of the first rank. His published works, consisting of books and frequent articles in learned journals, were crowned by his *Text of Acts* published in 1925. This volume was instantly received and recognized in this country and in Europe as being the definitive statement and treatment of one of the most debated problems in New Testament studies. It stands in the great tradition of Biblical scholarship and is an honor to its author. Harvard University and the Academy share, humbly and gladly, in its distinction.

As a teacher Professor Ropes was a man of single mind and of austere requirements. He refused to equate pure scholarship with pious edification, and by the rigors of life in his class room did much to restore to American theology its lost lustre. Lazy men did not understand him, sentimental men could not follow him, but for those men with whom "truth is carried alive into the heart by passion" he was a trusted guide.

James Hardy Ropes was one of those men who found it hard to defend the solitude necessary to constant and productive scholarship. His friendships were wide, his circle of acquaintances much wider. The roots of his life went down deep into the whole New England community and the branches of his sympathy spread far over its present variety of interests. He was therefore drawn into many activities other than those of the study, and being a man of conscience he never allowed these collateral duties to become matters of hollow form. In such capacity he had served the University as Dean of its Department of University Extension, the corporations of both Radcliffe and Simmons Colleges as trustee. More particularly he had given himself whole-heatedly to his duties as a trustee of Phillips Academy, Andover, and latterly as Chairman of that board he gave freely of time which was always precious and strength which was limited. He was a member of many learned societies and, in particular, of this Academy. His fidelity to its ideals, concerns, and companionships remains to us a gracious memory.

In 1897 Professor Ropes married Alice, the daughter of Edward Jackson Lowell of Boston, and is survived by Mrs. Ropes and by two children, Harriet and Edward Jackson Lowell.

W. L. Sperry.

JOSIAH ROYCE (1855–1916).

Fellow in Class III, Section 1, 1891.

I.

A summary of the events and appointments in the life of Josiah Royce is contained in *Who's Who in America*, 1917. In "Papers in honor of Josiah Royce on his 60th Birthday," published in *The Philosophical Review* for May, 1916 (Vol. XXV, No. 3), a good deal of biographical material is included. His own autobiographical sketch published as the final chapter of *The Hope of the Great Community* (Macmillan, 1916), contains valuable data. George Santayana contributes his view of Royce in *Character and Opinion in the United States* (Scribner's, 1920). The *National Cyclopedia of American Biography* contains an article on Royce (Vol. XI, p. 171), and a volume yet to be published of the *Dictionary of American Biography* will include an article on Royce by Ralph Barton Perry. The best short sketch of him that I know is that by G. H. Palmer in Morison's *Development of Harvard University*, pp. 9–15.

II.

Royce was a diligent writer. Benjamin Rand enumerates twenty-three volumes and ninety-four articles written by him. Of this output the books most apt to be read by students of philosophy in the coming years are:

The Religious Aspect of Philosophy, 1885.
The Conception of God, 1897.
The World and the Individual, 1900–1901.

By this last book his permanent importance as a philosopher will, I think, be determined.

The complete bibliography of his writings above mentioned was published by Dr. Benjamin Rand in *The Philosophical Review* for May, 1916.

III.

Josiah Royce was a shy, awkward, intensely conscientious, deeply loyal and affectionate person. From his sixteenth birthday till his death at sixty-one, he poured all the strength that he had into his work as a teacher and writer. He eked out his meagre salary by lecturing whenever opportunity offered, so as to be able not only to do his best for his wife and his two boys, but to send money to relatives in California.

Almost his only recreation was in listening to music, especially orchestral music, which he enjoyed intensely. Sometimes he spoke of symbolic logic as another recreation.

He was an omnivorous reader not only of philosophy but of general literature, novels, magazines, and newspapers. His friendships were close and enduring. He had an especial kinship and affection for children.

In philosophy he was, I believe, the most original and creative mind that American has produced. Logic and the strict analysis of meanings was the centre of his interest. The applications of this analysis in metaphysics, epistomology, cosmology, religion, and ethics make up the bulk of his writing. He never touched aesthetics, but his interest in psychology and in the history of philosophy was strong and productive.

As a teacher he did most of his work by lecturing, though in seminaries he was expert in leading and conducting discussion and at interpreting the ideas of his pupils. In his earlier writings he attained, I think, considerable literary skill, but this declined as he grew more and more interested in putting his meaning so that it *could not be misunderstood*, and his later works, while more profound and exact, are less readable.

The wide range of his interests is suggested by the fact that he wrote a novel, and a history of California, worked in child study, wrote a book on psychology and a volume of essays on provincialism, race questions, and other American problems. His patriotism was intense and active. In his last years it broke out in a series of fiery essays on the World War and America's duty to take part in it.

Disease cut short his life while still he had abundant powers and plans for future work.

RICHARD C. CABOT.

GEORGE AUGUSTUS SANDERSON (1863–1932).

Fellow in Class III, Section 1, 1930.

George Augustus Sanderson was born at Littleton, Massachusetts, on July 1, 1863, and died on June 11, 1932. His earliest Sanderson ancestor on this continent came to the Massachusetts Bay Colony in 1645 and settled in Watertown. A great grandson of this Puritan immigrant bought in 1750 the farm in Littleton which has ever since been in the possession of the family. On the distaff side, his earliest forebear settled in Ipswich in 1635 and a descendant removed to Littleton early in the eighteenth century. Every successive generation of these families was disciplined by lives of simplicity, industry and frugality. Most of them were tillers of the soil. His father, George W. Sanderson, was a man of great public spirit who was selectman, member of each branch of the Legislature, and in other ways strove to promote the general welfare.

George A. Sanderson received his early education in the public schools of his native town and in the Lawrence Academy at Groton. He graduated from Yale in 1885 and two years later from Boston University Law School. He was married in 1893 to Annie S. Bennett of Ayer. They had five children—two daughters, one of whom died in childhood, and three sons. He was survived by his widow, the sons and one daughter.

After being admitted to the bar in 1887, he practiced law in Boston. In 1893 he was appointed assistant district attorney and in 1901 elected district attorney for the northern district, which comprises Middlesex County. He continued to be district attorney until 1907, when he was appointed by Governor Guild a justice of the Superior Court. In October, 1924, he was appointed by Governor Cox a justice of the Supreme Judicial Court and discharged the duties of that position until his death.

He was an admirable prosecuting officer. He brought to its important and difficult functions a deep sense of fairness, a heart full of sympathy, and a keen consciousness of the demands of public safety. He was a close approach to the ideal as a judge of the trial court. With great patience and unruffled courtesy, he endeavored to understand the point of view of every party, witness and counsel, and to see to it that justice should prevail. He was firm in his rulings, prompt in the disposition of all incidental matters, and moved forward without delay and without haste to a final decision.

As a member of the court of last resort he manifested the traits which make for the strength and usefulness of that tribunal and which assure professional esteem and public confidence in its decisions. He possessed unremitting industry, scholarly attainments, broad and exact knowledge of the law, a logical mind, the judicial temperament. His thought was clear and his expression was lucid. He was steadfast in his convictions, but always open to the force of more cogent reasoning. Throughout his life he continued to grow in intellectual and judicial strength and power. He was a member of the Supreme Judicial Court almost eight years. He wrote four hundred ninety-eight opinions expressive of the judgment of the court, to be found in the thirty-one volumes of Massachusetts Reports from 250 to 280, both inclusive. There is a uniformity of fine excellence in these visible contributions to the body of our jurisprudence. He was thorough and comprehensive. No argument escaped his attention. His analysis was searching, his balancing of conflicting contentions accurate, his perspective correct. He abounded in sound common sense. He discerned between the accidental and the essential, between the dominant and the ancillary. His grasp of governing principles was sure. He was a wise and upright judge.

His tall, erect, muscular figure gave him in every aspect the appearance of a distinguished judge. He was deeply attached to his native soil. His recreations kept him in close touch with the country and farm life. He enlarged his acres, and delighted in the planting and nurture of orchards. In the midst of other activities he took intense interest in the affairs of the town in which he lived. He was moderator of the town meeting and member of the school committee. He served, also, as a trustee of Lawrence Academy, a trustee of the public library of Ayer, and a trustee of the North Middlesex Savings Bank. His family life with wife and children and kindred was most exemplary. He was a spiritually-minded man. He realized his ideals more than most men because he brought every motive and act to the test of an enlightened conscience. He was deeply religious, a constant attendant upon public worship, and senior warden of St. Andrews Episcopal Church of Ayer. He was elected a Fellow of this Academy on May 14, 1930, was frequently present at its stated meetings and much interested in its proceedings. It was the intention of the trustees of Boston University to confer upon him the degree of Doctor of Laws in 1932. Impressive reference was made to him at the Commencement although he had passed away a few days before. His life was a reflection of the best in our citizenship, raised by the merit of personal achievement to high judicial station. He left the world better for his having lived.

ARTHUR. P. RUGG.

HENRY NEWTON SHELDON (1842–1926).

Fellow in Class III, Section 1, 1913.

Henry Newton Sheldon was elected a Fellow of the Academy on May 14, 1913, while one of the Justices of the Supreme Judicial Court of Massachusetts.

He was a son of David Newton Sheldon, a Unitarian minister who for ten years was President of Waterville, later Colby, College. He was an older brother of Edward S. Sheldon, Professor of Romance Philology at Harvard. He was born at Waterville, Maine, June 28, 1842. After a year at Bowdoin College he entered the sophomore class at Harvard in 1861 and graduated at Harvard at the head of the class with the bachelor's degree in arts in 1863. His studious habits and brilliant scholarship earned him the title "The Sluggard," by which his classmates always remembered him. Among those classmates were John Fiske, Francis Amory, Charles Pickering Bowditch, Frederic T. Greenhalge, John T. Hassam, George S. Morison, George B. Shattuck, Clement L. Smith, John Collins Warren, men difficult to surpass even in their college years. On leaving college he taught for a time, but the Civil War claimed him. On his birthday anniversary in 1864 he was commissioned Second Lieutenant in the Fifty-fifth Massachusetts regiment. In December of that year he was made first lieutenant. He remained with his regiment till mustered out in September, 1865 at the close of the war. His soldiers, colored men, revered him. One can feel them gathering renewed courage after a shattering repulse at Cold Harbor from looking at him as he sat on the breast-works amid the missiles of death calmly smoking his pipe while they rallied and again took up the fight. Small in body, he was great in soul.

He had studied law while teaching and in the army. He was admitted to the bar March 12, 1866. He entered the office of Judge Josiah G. Abbott. He was married to Clara P. Morse in 1867. Their son Wilmon Henry Sheldon is now Professor of Philosophy at Yale. He associated himself with General Wilmon W. Blackmar in the law firm of Blackmar & Sheldon at Boston. Both, in 1865, had been among the founders of the Loyal Legion, formed upon the death of President Lincoln and dedicated to the ideals for which that great man had lived and died.

He was more successful in rendering notable service to clients than in gathering clients about him. Nevertheless his office did a good business. The ease and rapidity with which he dealt with his clients' affairs, left him time to write for the American Law Register, to prepare his work on Subrogation published in 1882, still a standard on the subject, and to edit an edition of Bateman, on "Auctions." Perhaps his modesty would have prevented a general recognition of his great learning and ability as a lawyer, had not his classmate, Greenhalge, been elected Governor of Massachusetts. Called on to nominate a Judge for the Superior Court, Governor Greenhalge presented the name of Henry N. Sheldon and urged the friends of "The Sluggard" to get together in his behalf. The nomination was confirmed; and, in 1894, Judge Sheldon began his judicial career. His merits were speedily recognized. Rapid, clear, solid, his mind dominated his court room; and parties, lawyers, juries, were satisfied that here was a judge of ideal excellence. In 1897 Governor Wolcott appointed him chairman of a commission of three to investigate and report upon a plan for the simplification of criminal pleadings and to prepare a schedule of forms. This work, to which he contributed largely, brought about a vast improvement in criminal pleading which has placed the Commonwealth as the leader in one of the reforms most needed by the law. In 1905 he was appointed to the Supreme Judicial Court. In 1908 Harvard conferred upon him its degree of Doctor of Laws. His work in writing of opinions, in consultation, and in such *nisi prius* matters as came to the judges of that court, all added to the reputation for learning and ability earned in his earlier positions.

His memory was remarkable. His reading in all branches of knowledge was extensive. His powers as a mathematician were extraordinary. A matter once passed through his mind was there to stay; and to be recalled and utilized at will on a moment's notice. But there was nothing of the pedant about him. The record of a base ball player was as readily placed at the service of a friend as the words of a Latin or Greek classic or of a great poem in English, French or German, or the volume and page of a law case.

He loved the national game and knew its players and their records. He was a skilful player of chess. The writer recalls the wonder—and humiliation—he felt as he moved chessmen about and saw Judge Sheldon play with him and them.

He retired from the court in 1914; but continued to render service to the law. In 1915 he was President of the Massachusetts Bar Association. He took a leading part in the work of the grievance committee of the Bar Association of the City of Boston, aiding in clearing up a blot on our records of the Bar. In 1919 he served as chairman of a judicature commission which made valuable suggestions for the improvement of our judicial system.

He died January 14, 1926, a scholar, a soldier, a judge, and a man.

WILLIAM C. WAIT.

ALEXANDER SMITH (1865–1922).

Fellow in Class I, Section 3, 1914.

Alexander Smith was born in Edinburgh in 1865. He was the son of Alexander Smith and Isabella Carter Smith. His grandfather of the same name was a sculptor and his father, after studying modeling in clay, became a musician. Many members of our Academy will remember that the father of Theodore William Richards was a distinguished artist. There is certainly a close relation between those qualities of initiative which characterize a great artist and similar qualities which contribute to scientific work of originality and importance.

Alexander Smith graduated from the University of Edinburgh with the degree of B.Sc. in chemistry in 1886. His studies had been devoted to astronomy more than to chemistry. He found, however, that there would be almost no chance for him to secure a position which would give him an independent career in that science and he went to the laboratory of Adolph v. Baeyer in Munich to study organic chemistry. At Edinburgh and Munich he secured thorough training in inorganic and analytical chemistry and after completing an experimental study of the 1, 3-diketones under the direction of Ludwig Claisen he received the degree of Ph.D. in 1889. He returned to Edinburgh for a year as an assistant in qualitative analysis. He also gave a course of lectures on organic syntheses.

During the summer of 1890 Dr. Smith came to America and was proposed as a candidate for the chair of chemistry at Wabash College, Crawfordsville, Indiana. Wabash College was founded under Presbyterian influences and Dr. Smith, with his thorough training, Scotch descent and attractive personality, proved very acceptable to the Board of Trustees and was appointed. He spent four very successful years at Crawfordsville, paying very careful attention to his work as a lecturer and teacher. At the same time he continued research work in organic chemistry and published four papers of his own and two with students upon work in that field begun at Crawfordsville.

In 1894 he was called to the University of Chicago as assistant professor of chemistry in charge of the instruction in elementary inorganic chemistry. He now turned his attention to a very careful study of the methods of instruction for beginners in chemistry and to researches in inorganic and physical chemistry. His previous thorough training in mathematics, physics and astronomy gave him an admirable training for work in these fields.

In 1900 he published "A laboratory outline of general chemistry," which soon passed through four editions and was translated into

German, Russian and Italian. In 1902 he and Edwin H. Hall of Harvard published "The teaching of chemistry and physics in the secondary school." This is still one of the best books we have on the subject. In 1906 he published his "Introduction to general inorganic chemistry." In this the modern theories of solutions and of equilibria were adequately presented for the first time in an introductory English text-book. This book was phenomenally successful and has passed through many editions. Since Dr. Smith's death the book has been revised by James Kendall. It has been translated into German, Russian, Italian and Portugese. Several other books which he wrote still have good sales.

His first important research in physical and inorganic chemistry was an exhaustive and classical study of the forms of sulfur. By the application of Raoult's law he was able to explain the relation between the melting points and composition of mixtures of insoluble amorphous sulfur and soluble sulfur, and by a study of the catalytic effects of various substances he cleared up many inconsistencies in the reports of previous observers.

With A. W. C. Menzies he devised new and accurate methods for the determination of vapor pressures, which were especially useful at high temperatures. By means of these methods it was demonstrated that mercurous chloride dissociates completely to mercury and mercuric chloride when it is changed to the vapor phase.

In recognition of these two researches, Professor Smith was awarded the Keith prize and medal of the Royal Society of Edinburgh in 1912.

In 1911 he was called to the chair of chemistry and head of the department of chemistry at Columbia University, succeeding Professor C. F. Chandler, who had retired the previous year. In 1919 he retired because of failing health. He died in Edinburgh, September 8th, 1922.

At Columbia University he and his collaborators devised methods for determining the pressure of the *saturated* vapor of ammonium chloride and demonstrated that the dissociation to ammonia and hydrochloric acid does not exceed 67 percent at 280° to 330°.

Professor Smith published 10 papers giving an account of researches in organic chemistry, 26 in inorganic and physical chemistry, 4 in astronomy and 21 on miscellaneous topics, chiefly on the teaching of chemistry. A detailed list of his papers and books will be found in the biographical memoir published by the National Academy of Sciences in 1927.

He was President of the American Chemical Society in 1911, was elected a member of the Royal Society of Edinburgh in 1891, an honorary foreign member of the Sociedad Española de Fisica y Quimica of Madrid in 1911, a member of the American Academy of Arts and Sciences in 1914 and of the National Academy of Sciences in 1915. In 1919 he was given the degree of LL.D. by the University of Edinburgh.

In 1905 Professor Smith married Sara Potter Bowles, of Memphis, Tennessee. There are two children, Isabella Carter Smith and William Bowles Smith.

The following characterization is quoted from the biographical memoir published by the National Academy of Sciences:

"To those who knew him best Professor Smith was a valued and loyal friend, a kind and considerate husband and father, with the highest ideals in his personal life and with a broad human interest in art and literature and in many fields of science other than chemistry. His knowledge was broad and profound, and his keen wit, quickness of repartee, and the epigrammatic quality of his remarks made conversation with him stimulating in a high degree. He has left a deep impression on many students who worked with him. The science of chemistry in America and in the world has been enriched by his labors as a teacher and as an investigator."

William Albert Noyes.

WILLIAM HOWARD TAFT (1857–1930).

Fellow in Class III, Section 1, 1914.

William Howard Taft, twenty-seventh President of the United States, was born in Cincinnati, Ohio, September 15, 1857. He went from Woodward High School, Cincinnati, to Yale University from which he graduated, the second in his class, in 1878. Returning to Cincinnati, he studied law in the Cincinnati Law School, receiving the degree of LL.B. in 1880. Immediately admitted to the bar, for a year he was law reporter for two Cincinnati newspapers and for two years, from 1881 to 1883, was assistant prosecutor of Hamilton County, Ohio. He then engaged in the private practice of the law until 1887, when he was appointed by Governor Foraker to fill a vacancy upon the bench of the Superior Court of Cincinnati. Upon the expiration of the term for which he had been appointed, Mr. Taft was elected as his own successor, the only office which he ever held by election until he became President of the United States in 1909.

In 1890, President Harrison appointed Mr. Taft, Solicitor General of the United States. His success in this office was so marked that upon the passage of the Federal Court of Appeals Act in 1892, Mr. Taft was made a Circuit Judge for the Sixth Circuit, comprising Ohio, Michigan, Kentucky and Tennessee. For nearly the whole time before he went to the Philippines, Mr. Taft was presiding judge of the Circuit Court of Appeals, and frequently sat in the Federal District Court in the trial of jury cases.

In 1896, Mr. Taft was one of a group of judges and lawyers, who, dissatisfied with the legal education given in the Cincinnati Law School, established the Law Department of the University of Cincinnati. The new school, of which Mr. Taft was made the dean, adopted the Langdell case method, one of the earliest expressions of the advancing esteem for Dean Langdell's ideas which has now become country wide. Within the year the success of the new school was so great that a consolidation was effected. Mr. Taft continued as dean and professor of the law of Property until he resigned his judgeship to go to the Philippines. His interest, notwithstanding the pressure of judicial duties, never flagged and he seldom missed his class appointments.

In 1900, Mr. Taft was appointed as the head of the Second Philippine Commission, charged with the duty of establishing civil government in the Islands. The succeeding year he became the first Governor General and remained in that position until 1904, when he became Secretary of War in President Theodore Roosevelt's cabinet.

In 1908, Mr. Taft was the candidate of the Republican Party for the Presidency of the United States and was elected. In 1912, he was again a candidate but met defeat. Upon the expiration of his term, Mr. Taft became Kent Professor of Constitutional Law in Yale University and in the same year was elected President of the American

Bar Association. His work as teacher, lecturer and advocate of causes, such as the League to Enforce Peace, occupied him until in 1921 he was appointed Chief Justice of the United States. He resigned, owing to failing health, on February 3rd, 1930 and his death followed quickly on March 8th.

Mr. Taft was elected a Fellow of the American Academy of Arts and Sciences on January 14, 1914, and continued as such until his death.

A bald recital such as this is not an adequate tribute to Mr. Taft's great public services, but space will not permit the consideration of his work as judge, administrator and statesman, nor does it allow anything more than a mention of his warm and attractive personality.

In addition to many addresses and lectures printed in scattered places, Mr. Taft was the author of the following books:

Presidential Addresses and State Papers, 1910.
Popular Government, 1913.
The Anti-Trust Act and the Supreme Court, 1914.
The United States and Peace, 1914.
Ethics in Service, 1915.
Our Chief Magistrate and His Powers, 1916.

His opinions as a member of the Circuit Court of Appeals for the Sixth Circuit are to be found in volumes 49–100 of the Federal Reporter and in volumes 1–40 of the United States Circuit Court of Appeals Reports. Mr. Taft's opinions as Chief Justice of the United States are contained in volumes 257–280 of the United States Reports.

ELDON R. JAMES.

ROLAND THAXTER[1] (1858–1932).

Fellow in Class II, Secuion 2, 1892.

Roland Thaxter was born in Newton, Mass., on Aug. 28, 1858, and died in Cambridge, Mass., April 22, 1932. His father, Levi Lincoln Thaxter (A.B., Harv. 1843, LL.B. 1845), aside from his professional training in the law, was of literary tastes and an authority on the life and works of Browning. His mother, Celia (Laighton) Thaxter, was well known for her poems of nature and her deep religious sympathies. Thaxter attended various schools in Boston and Cambridge, entered Harvard in 1878 and received the degree of A.B. *magna cum laude* in 1882. After nearly a year of illness, but during which he published two of his short papers on insects, he completed (1883–1884) a year at the Harvard Medical School. Receiving a two-year Harris Fellowship, however, he transferred to the Graduate School of Arts and Sciences, concentrating on research work in cryptogamic botany under William Gilson Farlow. From 1886–1888 he was Assistant in Cryptogamic Botany at Harvard and from 1888 to 1891 Mycologist to the Connecticut Agricultural College at New Haven, whence he was recalled to Harvard as Assistant Professor of Cryptogamic Botany. He was advanced to Professor in 1901 but retired to the position of Professor Emeritus and Honorary Curator of the Farlow Herbarium in 1919, in order to devote his entire time and energy to his life work. Thaxter was married on June 8, 1887, to Mabel Gray Freeman of Springfield, Mass., who, with three of their four children, Katherine, Elizabeth, and Edmund Lincoln (A.B., Harv. 1922), survives him. The Thaxter home was at Kittery Point, Maine, a locality rendered classic by him because of the numerous types of fungi he described from it, but his Cambridge residence was long fixed at 7 Scott Street.

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[1] Compare the biography by W. H. Weston, Jr. *in* Mycologia, **25:** 69–89, pl. 16, 17, Ma.-Ap. 1933.

Thaxter, inheriting literary, artistic and nature-loving tendencies, early developed along biological lines, his major attraction at first being entomology. Between 1877 and 1884 he published 5 short papers on insects. His attention then turned towards fungi and his cultures of the rusts on alternating hosts of *Cupressoideae* and *Pomoideae*, commonly known as "Cedar-apples" (*Gymnosporangia* and *Roesteliae*), undertaken as a problem under Farlow, occupied some of his attention during 1886 and 1888 and even extended to 1890.

In 1888 Thaxter began what we may suppose he expected to complete in a comparatively few years, but which became his life work, with the publication of "The Entomophthoreae of the United States," presented as his doctor's thesis. With the separates, Thaxter enclosed a printed slip announcing his desire "of extending the present paper so as to include all the Entomogenous fungi of North America" and his wish "to procure by exchange or otherwise as large an amount of material for comparison as possible" and "information concerning fungus epidemics among noxious insects." In this paper Thaxter extended the species of Entomophthores from the four previously known to 27, the great majority of which were new. The Entomophthoreae were further results of his labors during 1886 and 1887, and the culturing, preparation of slides and execution of the remarkable series of drawings, were all of the high grade of care and skill that characterized his later work.

The second group of entomogenous fungi was the *Laboulbenia* group, which in his first publication (in the Proceedings of this Academy) he evidently considered to be "a small group of fungi," consisting of 15 described species distributed through five genera, only one species of which was known from North America. The number of new species and genera, based upon discovery of an increasingly varied morphology, segregation of sex, and behavior on the part of these amazing ectoparasites of beetles, flies, etc., led to the publication by this Academy of some 20 papers in the Proceedings and five portions of the projected monograph in the Memoirs. All of his *Laboulbenia* papers are thus seriately to be found in the Academy's publications. He was working on the final portion of the monograph (Part VI) at the time of his death.

Thaxter brought the "small" *Laboulbenia*-group to recognition as an order, segregated into at least three distinct families and containing somewhere about 80 genera and, in turn, represented by about 1500 well marked species. The gigantic task of collecting or assembling from the larger and smaller entomological collections of the world, this immense number of species, of many individuals each, of mounting these minute plants in fixed positions on the microscope slides, of outlining them with the camera lucida, of filling in the details of the structures, of preparing the final plates (each drawn complete in itself) could not have been accomplished in so exemplary a fashion except by skill, ingenuity, keen insight and untold application. The interpretation appears most masterly and there is provided knowledge of a large group of entomogenous Ascomycetes in such detail and closely approximate completeness as to furnish a new link in the discussion of the phylesis and relations of ascomycetous fungi. The published drawings alone, restricted to 1, 2, or very few for each species, number approximately 3367 and yield to scrutiny of other mycologists a wealth of detail of variance as a basis for study and generalization. It is largely on account of the contributions toward a knowledge of Laboulbeniales that Thaxter was, early in his work, awarded the "Prix Desmazieres" by the French Academy.

The third great group in the trilogy of entomogenous fungi, that of *Cordyceps* and its conidial stage of *Isaria*, never arrived at publication. Thaxter published (1914) a single short paper on the ascosporic condition of *Aschersonia*. His collections and cultures were, however, numerous and to others interested in the study of these plants, he communicated notes and criticisms of great service and value.

While Thaxter's life work was largely on the Laboulbeniales, his interests were extended to all Cryptogamic groups. On algae, he published a single paper (*Compsopogon*, 1900); on lichens, of which he had large knowledge, also a single paper (*Myxotheca*, 1927); on mosses, which he also knew well, nothing. Primarily he was a mycologist, collecting, culturing, and following through life histories of innumerable forms, only a few of whose results came to publication. He discovered, illuminated and championed the remarkable and puzzling group of the Myxobacteriaceae (1892, 1893, 1897, 1904); in Zygomycetes he published three papers (1895, 1897, 1914) on new or peculiar forms, and a revision of the peculiar Endogonacae (1922); in Oomycetes, on aquatic fungi, five papers (1894, 1895, 1896) and a new American *Phytophthora* (1889) on lima beans; on Hyphomycetes three papers (1891, 1903); on fleshy Ascomycetes two papers (1905, 1922); and one on a peculiar phalloid genus (*Phallogaster*, 1893). As mycologist to the Experiment Station of Connecticut he published (1889, 1890) on "injurious fungi," "fungi and fungicides," on "potato scab," and other more economic aspects of mycology.

As a teacher, both of more elementary cryptogamic botany and of graduate and research grades, he was meticulous as to method and exposition. Himself a master of delicate and skilful technique, he was appalled by clumsy methods and slovenly attack. With research students of ability he was most appreciative and helpful, but sternly exacting. Nearly one thousand students of various grades felt his contact.

His travels were ever for botanizing or for inspection of extensive collections of insects or fungi, particularly the former in search of

Laboulbeniales. The larger collections of Europe and America were visited. From Newfoundland to Florida, he carried on his work and into the West Indies. Southern South America saw him in 1905–1906, active in cryptogamic pursuits, collecting algae, lichens, and mosses, but most particularly fungi. His eye was keen and his attention minutely concentrated on forms to be expected or more or less vaguely indicated by others.

Honors came early and often to Roland Thaxter. Of the more usual American professional societies of which he was a member, he was chosen as President by his colleagues of the Botanical Society of America, the New England Botanical Club, and the American Mycological Society. He was early chosen a Fellow of this Academy, was a member of the American Philosophical Society and of the National Academy of Sciences. His European memberships were numerous, including the Linnean Society of London, of Lyons, the British Mycological Society, the Botanical Society of Edinburgh, the Royal Botanical Society of Belgium, the Deutsche Botanische Gesellschaft, the Russian Mycological Society, the Danish Royal Academy, the Swedish Royal Academy, and the Institut de France (Academy of Sciences).

William Albert Setchell.

WILLIAM SYDNEY THAYER (1864–1932).

Fellow in Class II, Section 4, 1921.

Born in Milton, Massachusetts, June 23, 1864, son of James B. and Sophia Ripley Thayer, William Sydney Thayer soon moved to Cambridge, where he was educated in a Harvard atmosphere, his father being a professor in the Harvard Law School, graduating in arts with membership in Phi Beta Kappa in 1884 and receiving his M.D. in 1889. Throughout life Dr. Thayer evidenced a scholarly attitude of mind. Familiar with the classics and facile in acquiring the use of modern languages, such as German, French, Italian and Russian, in his medical career all that he said or did showed the influence of the arts on his science and withal in a man by nature courteous and courtly, obviously reared in a family of culture and refinement.

After receiving his M.D. from Harvard Dr. Thayer served as house pupil in Medicine at the Massachusetts General Hospital and shortly after this migrated to Baltimore to become intimately associated with Sir William, then Doctor, Osler. For nearly eight years he resided in the Johns Hopkins Hospital, for seven being resident physician, at that time an unheard of experience in American Medicine, for only at the Johns Hopkins Hospital had been established then a system of a senior group on the resident staff, permitting to any one that thorough clinical training possible to the man living his life within the hospital walls, witness of the innumerable phenomena of disease, that occur at any part of the twenty-four hours in the many beds within the hospital. It might be added that, since then, it has remained rare for anyone to spend so long a time a member of the resident staff of a hospital.

With this foundation is it any wonder that Dr. Thayer became the clinician par excellence, a teacher of clinical medicine to generations of students as associate professor, professor of clinical medicine, professor of medicine and professor of medicine emeritus in the Johns Hopkins Medical School and physician and physician-in-chief to the Johns Hopkins Hospital and a widely sought consultant in internal medicine?

Dr. Thayer was an active contributor to medical literature with 179 titles in his collected reprints covering a wide range of medical subjects. His earlier work were studies of the blood, notably in leukemia, typhoid and malaria. Later his attention was turned to cardio-vascular disease with important studies published on the circulatory complications of typhoid fever, on the third heart sound, on chorea, arteriosclerosis, heart block, and various forms of endocarditis. Medical history and essays of non-medical interest, charming verse and significant public addresses are included in his bibliographic titles.

Dr. Thayer received many honors at home and abroad. He was president of the Association of American Physicians, of the American Medical Association, and of the Phi Beta Kappa at Harvard; member of all of the important American medical societies; associate member of the Academy of Medicine, Paris; honorary fellow of the Royal College of Physicians of Ireland, Royal Society of Tropical Medicine and Hygiene, London, Royal Society of Medicine, London, Royal Medical Society, Budapest, and New York Academy of Medicine; honorary member of Association of Physicians of Great Britain and Ireland, Therapeutic Society of Moscow, and International Society of Medicine and Hydrology; honorary corresponding member of Sociétié des Hôpitaux, Paris; corresponding member Wiener Gesellschaft für Innere Medizin and Kinderheilkunde, Soc. des Hôpiteaux de Lyon and Soc. Royale des Sc. med. et nat. de Bruxelles. He was a Fellow of the American Academy of Arts and Sciences and American Philosophical Society. He was Commander of the Legion of Honor of France. He received numerous honorary degrees; LL.D. from Washington College, Edinburgh University, McGill University; honorary Dr. from Univ. of Paris; ScD. from University of Chicago. He was a member of the Board of Overseers of Harvard and a trustee of the Carnegie Institution of Washington.

During the war he became Chief Medical Consultant of the United States Expeditionary Force in France with the rank of Brigadier General and received a Distinguished Service Medal in 1919. He served as the head of an American Red Cross Mission to Russia in 1917–18.

In his later years Dr. Thayer came to be very generally known as America's best representative of clinical medicine and as such was called upon to represent America in important foreign medical or academic functions. In such capacity he delivered an address on the contributions of Pasteur to Medicine and Humanity at the Sorbonne in 1923, on Laennec at the Paris celebration of the centenary of Laennec's birth in 1926, on Richard Bright on the occasion of Bright's centenary celebration at Guy's Hospital, London, in 1927, and in 1930 he delivered the Gibson lectures at Edinburgh.

In 1901 Dr. Thayer married Susan Chisolm Read, of Charleston, South Carolina, a lovely woman of great charm, who graced his home and cemented all the stronger the many ties of friendship that Dr. Thayer was ever forming. In her latter years invalided by reason of heart disease she carried on cheerfully and in 1917 faced the supreme test of insisting that Dr. Thayer serve his country by going to Russia on a Red Cross Mission realizing full well that in all probability she could not live until his return but believing that she in this was doing her bit to the cause dear to her heart by sending the one dearest to her to do what he best of all could do. Her anticipation was all too true, for she died in 1917 while Dr. Thayer was in Russia. In 1927 wishing to do honor to Dr. Thayer a group of his freinds and former pupils endowed the "William Sydney Thayer and Susan Read Thayer Lectureship in Clinical Medicine" at the Johns Hopkins, most appropriately linking in this way in perpetuity these two names.

Dr. Thayer was a rare man of unique personality—simple, courtly, courteous, lovable, just, tolerant and of unalterable integrity; to many patients he was both wise physician and helping friend; to physicians and particularly young physicians he was the exemplar of all that is best and at the same time an approachable loved person ready to help with sane advice and with wisdom from his great store of clinical knowledge; to his former pupils he was all this and more, the ever cherished friend.

HENRY A. CHRISTIAN.

FREDERICK JACKSON TURNER (1861–1932).

Fellow in Class IV, Section 2, 1911.

Frederick Jackson Turner was born of New England stock on November 14, 1861, in Portage, Wisconsin. His father, Andrew Jackson Turner, was a prominent newspaper editor and state legislator. Frederick's early life was spent in a community that had but recently been on the frontier, a background that gave realism to his later historical thought. He was educated in the public schools of his state and graduated with honors from the University of Wisconsin in 1884. At Johns Hopkins University he was introduced to the seminary method by Herbert Baxter Adams, and was granted the Ph.D. degree in 1890. For eighteen years (1892–1910) he was Professor of American History at the University of Wisconsin, and from 1910 until his retirement in 1924, Professor of History at Harvard.

As a scholar, Turner's influence upon the writing of American history was profound. In 1893 he read before the American Historical Association an essay on the "Significance of the Frontier in American History." In this short paper, the most influential ever written on American history, Turner stated his thesis: that the continuous movement of civilization into the wilderness, the progressive advance of the frontier, released those forces which have produced the most characteristic features in American history and life. This essay made necessary a reappraisal of the forces of American history, a process of revision which is still going on.

A second seminal idea contributed by Professor Turner was the concept of the geographical section. He conceived of the section as the product of the interaction of man and environment, of population stock and physiographic province. The nation he considered a union of sections. He described American political and constitutional history as the result of the interplay of sectional forces.

As an author Turner's genius was expressed less in extended volumes than in brief suggestive essays, which induced others to continue their researches. In 1920 he gathered together a number of his essays, dealing with the concept of the frontier, into a volume entitled *The Frontier in American History.* His volume on *The Rise of the New West* (1906) is one of the outstanding works in the American Nation Series. After his retirement from active teaching in 1924, he suffered much from ill health; at the time of his death (at San Marino, California, March 14, 1932) his last book, *The United States, 1830–1850, The Nation and its Sections,* was nearly completed.

As a teacher, no less than as a writer, Turner's influence was exceptional. At the University of Wisconsin and at Harvard, he trained successive generations of scholars, who gave proof of his powers as a teacher by their contributions to the field of history. He was particularly effective in the training of graduate students who were capable of seizing his pregnant suggestions and developing them. He imbued them with his own conceptions of work and thought: the multiple hypothesis in history, the tentative nature of historical conclusions, and the imperative need of integrating the historical with the other social sciences. It was one of his distinctive modes of historical presentation to correlate political with geographic, economic, and social forces by specially prepared maps. He enhanced by a singularly gracious personality the intellectual stimulus of his thought.

S. E. Morison.

HENRY PICKERING WALCOTT (1838–1932).

Fellow in Class II, Section 4, 1889.

Henry Pickering Walcott, A.B., LL.D., M.D., was born in Hopkinton, Mass., December 23, 1838, the son of Samuel B. and Martha (Pickering) Walcott. In 1865 he married Charlotte E. Richards of Boston, who died in 1878. A son, Robert Walcott, associate justice of the Third Eastern Middlesex Court of Massachusetts, survives.

Dr. Walcott died at his home in Cambridge on November 11, 1932, aged 93 years. He inherited a sturdy frame and rugged health and his longevity permitted him to serve long and well up to his retirement from active life, which was but a few years before his death. His sterling worth was soon recognized by those with whom he came in contact. He saw with unusual clearness into the core of difficult problems and complex situations. He hated sham and possessed in full measure a rare quality of common sense. This served him and his community to good purpose, for he had many important decisions to make during a trying time of rapid change and progress in the science of sanitation and the art of medicine.

Dr. Walcott dedicated his life to service to his fellow men. His many positions of trust and importance attest the vigor of his industry as well as the variety of his responsibilities. His long career of usefulness was in the varied fields of medicine, education, sanitation, hygiene, economics, horticulture and hospital administration.

Chief among his interests and foremost among his contributions were his constructive leadership as Chairman of the Massachusetts State Board of Health, which under his guidance achieved international recognition; as pilot of the destinies of the Massachusetts General Hospital, which under his presidency became a hospital of hospitals, a teacher of teachers; and his services to Harvard University as Overseer, Fellow and Acting President. Dr. Walcott became a member of the State Board of Health only thirteen years after its formation and served thirty-three years, twenty-nine of them as Chairman. He served through that difficult transitional stage of progress in public health which saw bewildering progress in the sanitary sciences and which called for calm judgment and judicial balance. He gave communities clean water supplies, looked after the proper disposal of sewage and wastes, and promoted studies in these subjects which made the work of the State Board of Health a model and its publications classics which had influence upon health administration far and wide. The value of his work to the Commonwealth is expressed in a sentence taken from a letter written by Calvin Coolidge, then Governor of Massachusetts, to Dr. Walcott on his retirement: "If the public health is good and the sanitary laws wise, to you belongs the credit."

Further recognition of the eminence he reached in the minds and hearts of his colleagues is attested by the fact that he was President of the Massachusetts Medical Society, American Public Health Association, Massachusetts Horticultural Society and the American Academy of Arts and Sciences; President of the Fifteenth International Congress of Hygiene and Demography; Honorary Fellow of the Royal Sanitary Institute of Great Britain; trustee of the Carnegie Institution of Washington. He had been President of the Cambridge Hospital; Health Officer of the Massachusetts Board of Health, Lunacy and Charity, Chairman of the Metropolitan Water and Sewerage Board, a member of the first Metropolitan Drainage Commission and of the commission on the drainage of the Blackstone River, City Physician of Cambridge, and a member of the commission appointed to study the water supply of that city. He was honored with the Boylston medical prize.

Dr. Walcott's patriotism, zeal and spirit may well be inferred by his impatience at the age of twenty-three to serve in the Civil War. To accomplish this he left Harvard for Bowdoin that he might the sooner graduate in medicine. He had many friends because he was a friend. He had a scholarly habit of mind, a keen appreciation for research, an intuitive faculty for finding the best, and keen discrimination. He had an unusual memory, the saving grace of humor and was fond of companionship. Dr. Henry Pickering Walcott has taken an abiding place as a star of first magnitude in the galaxy of Massachusetts' distinguished sons.

M. J. ROSENAU.

HERBERT LANGFORD WARREN (1857–1917).

Fellow in Class III, Section 4, 1905.

Herbert Langford Warren was born in Manchester, England, March 29, 1857, and died at his home in Cambridge, Massachusetts, June 27, 1917.

He was the son of Samuel Mills Warren, of New England Colonial ancestry, and Sarah Anne (Broadfield) Warren of the Broadfields of Bridgenorth, Shropshire, England.

His school days were spent in Manchester except for two years (1869–71) in Germany where he received a thorough grounding in the German language. He studied at Owen's College, Manchester, from 1871 to 1875, and then entered the office of a Manchester architect as draughtsman. He came to this country in 1876.

From 1877 to 1879 he studied architecture at the Massachusetts Institute of Technology, then entered the office of H. H. Richardson in Brookline where he remained until 1884. He also at this time took courses in the fine arts as a special student at Harvard College. The year of 1884 was spent in travel in England, Italy and France, studying, sketching and measuring, particularly the country churches of the region in France which has been devastated by the World War. During the last of his life he felt deeply the destruction of these beautiful buildings and dreamed of sharing in the work of re-building.

On his return to Boston in 1885 he started an independent practice as architect under the firm name of Warren, Smith & Biscoe, later Warren & Smith, with an office also in Troy, New York. Many church edifices, institutional buildings and important residences were designed by his firm, and for nearly forty years he maintained a connection with the practice of his profession,, but during the later years much of his time was given up to what he considered the greater work of administrative officer in Harvard University and teacher there of his art. In 1893–94 he served as Instructor, from 1894 to 1899 as Assistant Professor, from 1899 to 1903 as Professor, and from 1903 to his death as Nelson Robinson, Jr., Professor of Architecture. He also lectured on Architectural History at the Massachusetts Institute of Technology.

Under his leadership instruction in architecture grew in importance and the number of staff and students at Harvard increased largely, until from a small beginning in the Lawrence Scientific School, the School of Architecture and School of Landscape Architecture became established and housed in a building designed to serve their needs. At his death he had served the University nearly twenty-five years. Not so much as a practitioner, therefore, will his memory live, but as scholar and teacher and eminent authority on the historical development of architecture.

His active participation in various professional and quasi professional organizations showed the breadth of his interests and sympathies. He served for three years as Director of the American Institute of Architects, for four years as Secretary of the Boston Society of Architects, was a trustee of the American Academy in Rome, President of the National League of Handicraft Societies for four years, and President of the Boston Society of Arts and Crafts from 1904 until his death.

He was also an interested member of the Archæological Institute of America and a Fellow of the American Academy of Arts and Sciences. In 1902 Harvard University conferred upon him the honorary degree of Master of Arts.

It was my good fortune as a younger man while Warren was at H. H. Richardson's office to be closely associated with him and to know him most intimately, in fact, he had a private room at the end of the office and he took me in to work under him and with him. What greatly impressed me was his tremendous enthusiasm and concentration in the architectural problem on which he was working and his singlemindedness and devotion to beauty and his art.

I always had a feeling that Warren was a Goth at heart, and yet the work at Richardson's office was Romanesque and it was quite interesting to see how by purely intellectual processes he produced such good Romanesque drawings, and he worked on many of Richardson's important buildings. In after years what was closest to his his heart was propounding and elucidating to his pupils the intricacies and beauties of the Gothic Cathedrals, which he did in a most masterly manner.

In our discussions even at that early time he displayed a surprising familiarity with general history and literature besides architectural knowledge, which in after years fitted him so peculiarly well as a scholar and teacher to develop the Architectural School at Harvard and stamp it with the highest standard in the history and principles of architecture and kindred arts.

He was a man of intense feeling, so much so that at times it drew heavily on his vital energy, and with a slight physique it was only his indomitable will and desire to do the highest and best thing possible that carried him successfullly through his many and distinguished accomplishments.

CHARLES A. COOLIDGE.

JOHN WARREN (1874–1928).

Fellow in Class II, Section 4, 1921.

John Warren, the second of that name, was born in Boston in 1874, the elder son of Dr. John Collins Warren. After early preparation at Mr. Noble's School, he graduated from Harvard College with the class of 1896. With his long line of medical progenitors—Dr. John Warren, the first Professor of the Harvard Medical School; the first Dr. John Collins Warren; Dr. Jonathan Mason Warren; the late Dr. John Collins Warren—with such ancestors it was almost inevitable that he, too, should follow the medical profession, and he entered immediately the Harvard Medical School to graduate in 1901. The practice of medicine or surgery, however, was of little interest to him, and he turned to the study and teaching of anatomy.

In the Department of Anatomy of the Harvard Medical School, under Professor Thomas Dwight, a kinsman of the Warrens, John Warren rose from Demonstrator in 1901, Assistant Professor in 1908, to Associate Professor in 1915, when at Dr. Dwight's death he took charge of the anatomical teaching. He showed his deep knowledge of his chosen subject in his clear and forceful lectures and in his skillful dissections.

The pervading idea of his work was always to help the student in mastering the science which he recognized as difficult, but which seemed to him so important in medical and surgical practice. His was always the point of view of a teacher, and this led him to plan and install a teaching anatomical museum, named by him the Dwight

Room, close to the dissecting rooms and immediately available to the students. Most of the specimens displayed are the result of his own skill, and form the basis of the illustrations for an Atlas of Anatomy, unfinished at his death, but published later by his associates.

For original research in anatomy he showed little taste. Two papers on the development of the paraphysis appeared in the American Journal of Anatomy, and show careful and painstaking work. But his first publication, "A Sketch of the Life of John Warren, M.D., the First Professor of Anatomy in the Medical School," printed by the Harvard Medical Alumni Association, showed his interest in historical subjects, and this soon led to studies of the early history of the school, and later to the broader subject of the history of anatomy in general. In his later years he became an enthusiastic collector of old anatomical works. His considerable collection of old books on anatomy, together with the entire medical library of the Warren family, handed down from father to son, he left at his death to the Library of the Harvard Medical School, where it is now impressively displayed as a valuable and interesting memorial to five generations.

John Warren's administrative ability was soon recognized. In 1903 he became the youngest member of the Faculty of the Medical School, and was always called upon to serve on numerous faculty committees. In 1911, President Lowell appointed him University Marshal. During the World War he was called to administrative duties, serving as Adjutant in several of the medical camps in this country, with the rank of Major.

His great height made him a conspicuous figure and induced a certain amount of shyness and reserve; and though he was always ready for a congenial companion, whether a smoking-room acquaintance for an hour or an old friend for a month, he seemed always self-sufficient, to contain within himself the means for his own enjoyment, which he was ready to share or not as occasion offered. With his family and his few personal friends the sweetness of his nature showed to best advantage, but his worth was appreciated by a great number of his students and colleagues, who recognized in him a true and scholarly gentleman and a masterly teacher.

JOHN L. BREMER.

American Academy of Arts and Sciences.

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[1] Deceased.

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OF THE
FELLOWS AND FOREIGN HONORARY MEMBERS.

(Corrected to November 8, 1933.)

FELLOWS.—731.

(Number limited to eight hundred.)

Class I.—*Mathematical and Physical Sciences.*—207.

Section I.—*Mathematics and Astronomy.*—54.

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Jesse Douglas	Cambridge
Philip Fox	Evanston, Ill.
Fabian Franklin	New York, N. Y.
Philip Franklin	Belmont
Edwin Brant Frost	Williams Bay, Wis.
William Caspar Graustein	Cambridge
Einar Hille	New Haven, Conn.
Frank Lauren Hitchcock	Belmont

Edward Vermilye Huntington	Cambridge
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CLASS I., SECTION II.—*Physics*.—57.

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Harvey Nathaniel Davis	Hoboken, N. J.
Clinton Joseph Davisson	New York, N. Y.
Arthur Louis Day	Washington, D. C.
William Johnson Drisko	Winchester
William Duane	Devon, Pa.
Alexander Wilmer Duff	Worcester
Arthur Woolsey Ewell	Worcester
Harry Manley Goodwin	Brookline
George Ellery Hale	Pasadena, Cal.
Edwin Herbert Hall	Cambridge
Arthur Cobb Hardy	Wellesley
George Russell Harrison	Cambridge
Hammond Vinton Hayes	Boston
John Charles Hubbard	Baltimore, Md.
Gordon Ferrie Hull	Hanover, N. H.
Charles Clifford Hutchins	Brunswick, Me.
Frederic Eugene Ives	Philadelphia, Pa.
James Edmund Ives	Washington, D. C.
Edwin Crawford Kemble	Cambridge
Norton Adams Kent	Newton Center
Frank Arthur Laws	Brookline
Henry Lefavour	Boston
Robert Bruce Lindsay	Providence, R. I.
Theodore Lyman	Brookline
Ernest George Merritt	Ithaca, N. Y.

Dayton Clarence Miller	Cleveland, O.
Robert Andrews Millikan	Pasadena, Cal.
Harry Wheeler Morse	Stanford University, Cal.
Edward Leamington Nichols	Ithaca, N. Y.
Charles Ladd Norton	Boston
Otto Oldenberg	Cambridge
George Washington Pierce	Cambridge
Michael Idvorsky Pupin	New York, N. Y.
Frederick Albert Saunders	Cambridge
John Clarke Slater	Cambridge
John Stone Stone	San Diego, Cal.
Maurice deKay Thompson	Brookline
Elihu Thomson	Swampscott
Manuel Sandoval Vallarta	Brookline
David Locke Webster	Palo Alto, Cal.
Edwin Bidwell Wilson	Brookline
Robert Williams Wood	Baltimore, Md.
John Zeleny	New Haven, Conn.

CLASS I., SECTION III.—*Chemistry.*—54.

Roger Adams	Urbana, Ill.
Wilder Dwight Bancroft	Ithaca, N. Y.
Gregory Paul Baxter	Cambridge
James Alexander Beattie	Brookline
Arthur Alphonzo Blanchard	Cambridge
Marston Taylor Bogert	New York, N. Y.
William Crowell Bray	Berkeley, Cal.
Russell Henry Chittenden	New Haven, Conn.
James Bryant Conant	Cambridge
Tenney Lombard Davis	Norwell
Henry Fay	Brookline
Louis Frederick Fieser	Waltham
George Shannon Forbes	Cambridge
Edward Curtis Franklin	Palo Alto, Cal.
Louis John Gillespie	Cambridge
Lawrence Joseph Henderson	Cambridge
Charles Loring Jackson	Boston

Walter Louis Jennings	Worcester
Grinnell Jones	Cambridge
Frederick George Keyes	Cambridge
George Bogdan Kistiakowsky	Cambridge
Elmer Peter Kohler	Cambridge
Charles August Kraus	Providence, R. I.
Arthur Becket Lamb	Cambridge
Irving Langmuir	Schenectady, N. Y.
Gilbert Newton Lewis	Berkeley, Cal.
Warren Kendall Lewis	Newton
Arthur Dehon Little	Brookline
Duncan Arthur MacInnes	New York, N. Y.
Kenneth Lamartine Mark	Boston
Lafayette Benedict Mendel	New Haven, Conn.
Edward Mueller	Cambridge
Samuel Parsons Mulliken	Newburyport
Charles Edward Munroe	Forest Glen, Md.
Lyman Churchill Newell	Brookline
James Flack Norris	Boston
Arthur Amos Noyes	Pasadena, Cal.
William Albert Noyes	Urbana, Ill.
William Albert Noyes, Jr.	Providence, R. I.
Samuel Cate Prescott	Brookline
Robert Hallowell Richards	Jamaica Plain
Martin André Rosanoff	Valencia, Pa.
Allan Winter Rowe	Boston
George Scatchard	Cambridge
Walter Cecil Schumb	East Milton
Miles Standish Sherrill	Winchester
Harry Monmouth Smith	Brookline
Julius Oscar Stieglitz	Chicago, Ill.
Richard Chace Tolman	Pasadena, Cal.
Henry Weeden Underwood, Jr.	Cambridge
William Hultz Walker	Pasadena, Cal.
Willis Rodney Whitney	Schenectady, N. Y.
Robert Seaton Williams	Cambridge
Alpheus Grant Woodman	Watertown

Class I., Section IV.—*Technology and Engineering.*—42.

Comfort Avery Adams	Belmont
Harold Kilbrith Barrows	Winchester
Charles Harold Berry	Belmont
William Hubert Burr	New Canaan, Conn.
Vannevar Bush	Belmont
Harry Ellsworth Clifford	Newton
Theodore Harwood Dillon	Cambridge
Philip Drinker	Brookline
Gano Dunn	New York, N. Y.
William Frederick Durand	Palo Alto, Cal.
Gordon Maskew Fair	Cambridge
Frederic Harold Fay	Boston
Glennon Gilboy	Cambridge
Albert Haertlein	Watertown
John Hays Hammond	New York, N. Y.
George Leonard Hosmer	Woburn
William Hovgaard	Brookline
James Robertson Jack	Watertown
Dugald Caleb Jackson	Cambridge
Frank Baldwin Jewett	New York, N. Y.
Lewis Jerome Johnson	Cambridge
Arthur Edwin Kennelly	Cambridge
Ralph Restieaux Lawrence	Brookline
William Henry Lawrence	Jamaica Plain
Charles Thomas Main	Winchester
Lionel Simeon Marks	Cambridge
Arthur Edwin Norton	Cambridge
Frederick Law Olmsted	Brookline
Charles Francis Park	Taunton
Langdon Pearse	Chicago, Ill.
Harold Pender	Philadelphia, Pa.
Greenleaf Whittier Pickard	Newton Center
George Edmond Russell	Arlington
Albert Sauveur	Cambridge
Henry Lloyd Smyth	Watertown
Charles Milton Spofford	Brookline

John F. Stevens	Baltimore, Md.
Edward Pearson Warner	New York, N. Y.
Arthur Edward Wells	New York, N. Y.
Robert Spurr Weston	Brookline
Frank Edward Winsor	West Newton
Joseph Ruggles Worcester	Waltham

Class II.—*Natural and Physiological Sciences.*—206.

Section I.—*Geology, Mineralogy, and Physics of the Globe.*—51.

Wallace Walter Atwood	Worcester
Norman Levi Bowen	Washington, D. C.
Isaiah Bowman	New York, N. Y.
Charles Franklin Brooks	Milton
Kirk Bryan	Cambridge
Frank Morton Carpenter	Cambridge
Henry Helm Clayton	Canton
Herdman Fitzgerald Cleland	Williamstown
Reginald Aldworth Daly	Cambridge
Edward Salisbury Dana	New Haven, Conn.
William Morris Davis	Palo Alto, Cal.
William Ebenezer Ford	New Haven, Conn.
James Walter Goldthwait	Hanover, N. H.
Louis Caryl Graton	Cambridge
Herbert Ernest Gregory	Honolulu, T. H.
William Jackson Humphreys	Washington, D. C.
Ellsworth Huntington	New Haven, Conn.
Robert Tracy Jackson	Peterborough, N. H.
Thomas Augustus Jaggar	Honolulu, T. H.
Douglas Wilson Johnson	New York, N. Y.
Arthur Keith	Washington, D. C.
Alfred Church Lane	Cambridge
Esper Signius Larsen, Jr.	Cambridge
Andrew Cowper Lawson	Berkeley, Cal.
Charles Kenneth Leith	Madison, Wis.
Waldemar Lindgren	Brookline
Frederic Brewster Loomis	Amherst

Alexander George McAdie	Hampton, Va.
George Francis McEwen	La Jolla, Cal.
Donald Hamilton McLaughlin	Cambridge
Kirtley Fletcher Mather	Cambridge
Edward Bennett Mathews	Baltimore, Md.
John Campbell Merriam	Washington, D. C.
William John Miller	Los Angeles, Cal.
Frederick Kuhne Morris	Cambridge
Charles Palache	Cambridge
Percy Edward Raymond	Lexington
Austin Flint Rogers	Palo Alto, Cal.
Robert Wilcox Sayles	Chestnut Hill
Waldemar Theodore Schaller	Washington, D. C.
Charles Schuchert	New Haven, Conn.
William Berryman Scott	Princeton, N. J.
Hervey Woodburn Shimer	Hingham
Frank Bursley Taylor	Fort Wayne, Ind.
Thomas Wayland Vaughan	La Jolla, Cal.
Charles Hyde Warren	New Haven, Conn.
David White	Washington, D. C.
Herbert Percy Whitlock	New York, N. Y.
Bailey Willis	Palo Alto, Cal.
John Eliot Wolff	Pasadena, Cal.
Frederick Eugene Wright	Washington, D. C.

CLASS II., SECTION II.—*Botany.*—35.

LeRoy Abrams	Stanford University, Cal.
Oakes Ames	North Easton
Joseph Charles Arthur	Lafayette, Ind.
Irving Widmer Bailey	Cambridge
Liberty Hyde Bailey	Ithaca, N. Y.
Edward Wilber Berry	Baltimore, Md.
Nathaniel Lord Britton	New York, N. Y.
Douglas Houghton Campbell	Palo Alto, Cal.
George Perkins Clinton	New Haven, Conn.
Bradley Moore Davis	Ann Arbor, Mich.
Edward Murray East	Jamaica Plain

Rollins Adams Emerson	Ithaca, N. Y.
Alexander William Evans	New Haven, Conn.
Joseph Horace Faull	Cambridge
Merritt Lyndon Fernald	Cambridge
Richard Thornton Fisher	Petersham
Robert Almer Harper	New York, N. Y.
Albert Spear Hitchcock	Washington, D. C.
John George Jack	East Walpole
Willis Linn Jepson	Berkeley, Cal.
Ivan Murray Johnston	Jamaica Plain
Jacob Goodale Lipman	New Brunswick, N. J.
Burton Edward Livingston	Baltimore, Md.
Elmer Drew Merrill	Berkeley, Cal.
Winthrop John Vanleuven Osterhout	New York, N. Y.
George James Peirce	Palo Alto, Cal.
Alfred Rehder	Jamaica Plain
Benjamin Lincoln Robinson	Cambridge
Karl Sax	Jamaica Plain
William Albert Setchell	Berkeley, Cal.
Elvin Charles Stakman	St. Paul, Minn.
William Trelease	Urbana, Ill.
Charles Alfred Weatherby	Cambridge
William Henry Weston, Jr.	Cambridge
Ralph Hartley Wetmore	Belmont

CLASS II., SECTION III.—*Zoology and Physiology.*—66.

Nathan Banks	Cambridge
Thomas Barbour	Boston
Philip Bard	Baltimore, Md.
Francis Gano Benedict	Boston
Henry Bryant Bigelow	Concord
Robert Payne Bigelow	Brookline
William T. Bovie	Chicago, Ill.
Edward Allen Boyden	Minneapolis, Minn.
John Lewis Bremer	Boston
Charles Thomas Brues	Jamaica Plain
John Wymond Miller Bunker	Belmont

Walter Bradford Cannon	Cambridge
Thorne Martin Carpenter	Boston
William Ernest Castle	Belmont
Charles Value Chapin	Providence, R. I.
Lemuel Roscoe Cleveland	Jamaica Plain
Edwin Joseph Cohn	Cambridge
Edwin Grant Conklin	Princeton, N. J.
Manton Copeland	Brunswick, Me.
William John Crozier	Cambridge
Joseph Augustine Cushman	Sharon
Charles Benedict Davenport	Cold Spring Harbor, N. Y.
Hallowell Davis	Belmont
Alden Benjamin Dawson	Cambridge
Samuel Randall Detwiler	New York, N. Y.
Gilman Arthur Drew	Eagle Lake, Fla.
Herbert McLean Evans	Berkeley, Cal.
Henry Clinton Fall	Tyngsboro
Alexander Forbes	Milton
William King Gregory	New York, N. Y.
Joseph Grinnell	Berkeley, Cal.
Samuel Henshaw	Cambridge
Leigh Hoadley	Cambridge
Samuel Jackson Holmes	Berkeley, Cal.
Roy Graham Hoskins	Newton Center
Leland Ossian Howard	Washington, D. C.
Herbert Spencer Jennings	Baltimore, Md.
Charles Atwood Kofoid	Berkeley, Cal.
Frederic Thomas Lewis	Waban
Frank Rattray Lillie	Chicago, Ill.
Ralph Stayner Lillie	Chicago, Ill.
Richard Swann Lull	New Haven, Conn.
Edward Laurens Mark	Cambridge
Ernest Gale Martin	Palo Alto, Cal.
Albert Davis Mead	Providence, R. I.
Axel Leonard Melander	New York, N. Y.
Gerrit Smith Miller	Washington, D. C.
Thomas Hunt Morgan	Pasadena, Cal.

Herbert Vincent Neal	Tufts College
Henry Fairfield Osborn	New York, N. Y.
George Howard Parker	Cambridge
Raymond Pearl	Baltimore, Md.
John Charles Phillips	Wenham
Henry Augustus Pilsbry	Philadelphia, Pa.
Frederick Haven Pratt	Wellesley Hills
Herbert Wilbur Rand	Cambridge
David Rapport	Cambridge
Alfred Clarence Redfield	Milton
William Emerson Ritter	Berkeley, Cal.
Alexander Grant Ruthven	Ann Arbor, Mich.
Percy Goldthwait Stiles	Newtonville
Arthur Wisswald Weysse	Boston
William Morton Wheeler	Boston
Edmund Beecher Wilson	New York, N. Y.
Frederick Adams Woods	Rome, Italy
Jeffries Wyman, Jr.	Chestnut Hill

Class II., Section IV.—*Medicine and Surgery*.—54.

Joseph Charles Aub	Belmont
James Bourne Ayer	Milton
Franklin Greene Balch	Boston
George Blumer	New Haven, Conn.
Charles Macfie Campbell	Cambridge
Alexis Carrel	New York, N. Y.
William Bosworth Castle	Boston
David Cheever	Boston
Henry Asbury Christian	Boston
Stanley Cobb	Milton
Rufus Cole	New York, N. Y.
Harvey Cushing	New Haven, Conn.
Elliott Carr Cutler	Brookline
Eugene Floyd DuBois	New York, N. Y.
Reginald Fitz	Brookline
Simon Flexner	New York, N. Y.
James Lawder Gamble	Brookline

Joseph Lincoln Goodale	Boston
Robert Battey Greenough	Boston
Ross Granville Harrison	New Haven, Conn.
Percy Rogers Howe	Belmont
William Henry Howell	Baltimore, Md.
Edgar Erskine Hume	Washington, D. C.
Reid Hunt	Boston
Elliott Proctor Joslin	Boston
Roger Irving Lee	Brookline
Edwin Allen Locke	Boston
Warfield Theobald Longcope	Baltimore, Md.
Fred Bates Lund	Newton
George Burgess Magrath	Boston
Frank Burr Mallory	Brookline
William James Mayo	Rochester, Minn.
George Richards Minot	Brookline
William Lorenzo Moss	Augusta, Ga.
John Howard Mueller	Boston
Robert Bayley Osgood	Boston
Joseph Hershey Pratt	Boston
Milton Joseph Rosenau	Brookline
Andrew Watson Sellards	Boston
George Cheever Shattuck	Brookline
Theobald Smith	Princeton, N. J.
Torald Hermann Sollmann	Cleveland, Ohio
Charles Wardell Stiles	Washington, D. C.
Richard Pearson Strong	Boston
Fritz Bradley Talbot	Brookline
Ernest Edward Tyzzer	Wakefield
Frederick Herman Verhoeff	Boston
Joseph Treloar Wearn	Cleveland, Ohio
Soma Weiss	Cambridge
William Henry Welch	Baltimore, Md.
Benjamin White	Boston
Francis Henry Williams	Boston
Simeon Burt Wolbach	Sudbury
Hans Zinsser	Boston

Class III.—*The Social Arts.*—123.

Section I.—*Jurisprudence.*—36.

George Weston Anderson	Boston
Francis Noyes Balch	Jamaica Plain
Joseph Henry Beale	Cambridge
Harry Augustus Bigelow	Chicago, Ill.
Henry Wolf Biklé	Philadelphia, Pa.
Benjamin Nathan Cardozo	Washington, D. C.
John Dickinson	Philadelphia, Pa.
Fred Tarbell Field	Newton
Felix Frankfurter	Cambridge
Thomas Hovey Gage	Worcester
Theodore Francis Green	Providence, R. I.
Walter Perley Hall	Fitchburg
Learned Hand	New York, N. Y.
Charles Evans Hughes	Washington, D. C.
Nathan Isaacs	Cambridge
Eldon Revare James	Cambridge
Frederick Lawton	Boston
James Arnold Lowell	Newton
Sayre Macneil	Azusa, Cal.
Calvert Magruder	Cambridge
William DeWitt Mitchell	Washington, D. C.
Edmund Morris Morgan	Arlington
Herbert Parker	South Lancaster
George Wharton Pepper	Philadelphia, Pa.
Roscoe Pound	Watertown
Odin Roberts	Boston
Elihu Root	New York, N. Y.
Arthur Prentice Rugg	Worcester
Francis Bowes Sayre	Washington, D. C.
Austin Wakeman Scott	Cambridge
Harlan Fiske Stone	Washington, D. C.
Edward Sampson Thurston	Cambridge
William Cushing Wait	Medford
Eugene Wambaugh	Cambridge

Edmund Allen Whitman	Cambridge
George Woodward Wickersham	New York, N. Y.

Class III., Section II.—*Government, International Law, and Diplomacy.*—24.

George Hubbard Blakeslee	Worcester
Edwin Montefiore Borchard	New Haven, Conn.
William Richards Castle, Jr.	Washington, D. C.
Joseph Perkins Chamberlain	New York, N. Y.
Robert Treat Crane	New York, N. Y.
William Yandell Elliott	Belmont
Sidney Bradshaw Fay	Cambridge
William Cameron Forbes	Norwood
Joseph Clark Grew	Tokyo, Japan
Albert Bushnell Hart	Cambridge
Arthur Norman Holcombe	Cambridge
Manley Ottmer Hudson	Cambridge
Philip Carryl Jessup	New York, N. Y.
Abbott Lawrence Lowell	Boston
William MacDonald	New York, N. Y.
Charles Edward Merriam	Chicago, Ill.
Ogden Livingston Mills	New York, N. Y.
John Bassett Moore	New York, N. Y.
William Bennett Munro	Pasadena, Cal.
Westel Woodbury Willoughby	Baltimore, Md.
William Franklin Willoughby	Washington, D. C.
George Grafton Wilson	Cambridge
Quincy Wright	Chicago, Ill.
Henry Aaron Yeomans	Cambridge

Class III., Section III.—*Economics and Sociology.*—38.

Charles Jesse Bullock	Cambridge
Harold Hitchings Burbank	Cambridge
Richard Clarke Cabot	Cambridge
Thomas Nixon Carver	Cambridge
John Bates Clark	New York, N. Y.
John Candler Cobb	Milton
Arthur Harrison Cole	Cambridge

Melvin Thomas Copeland	Cambridge
William Leonard Crum	Cambridge
William James Cunningham	Cambridge
Clive Day	New Haven, Conn.
Davis Rich Dewey	Cambridge
Arthur Stone Dewing	Cambridge
Wallace Brett Donham	Boston
Irving Fisher	New Haven, Conn.
James Ford	Cambridge
Edwin Francis Gay	Cambridge
Sheldon Glueck	Cambridge
Henry Wyman Holmes	Cambridge
Walter Wallace McLaren	Williamstown
Leon Carroll Marshall	Baltimore, Md.
Edward Sagendorph Mason	Cambridge
Frederick Cecil Mills	New York, N. Y.
Wesley Clair Mitchell	New York, N. Y.
Harold Glenn Moulton	Washington, D. C.
William Fielding Ogburn	Chicago, Ill.
Warren Milton Persons	New York, N. Y.
George Byron Roorbach	Cambridge
Leo S. Rowe	Washington, D. C.
Josef Alois Schumpeter	Cambridge
Carl Snyder	New York, N. Y.
Pitirim Alexandrovich Sorokin	Winchester
Oliver Mitchell Wentworth Sprague	Cambridge
Frank William Taussig	Cambridge
William Isaac Thomas	New York, N. Y.
Donald Skeele Tucker	Cambridge
Abbott Payson Usher	Belmont
John Henry Williams	Cambridge

CLASS III., SECTION IV.—*Administration and Affairs.*—25.

Charles Francis Adams	Concord
Charles Foster Batchelder	Cambridge
George Hoyt Bigelow	Milton
Ingersoll Bowditch	Jamaica Plain

John Albert Cousens	Tufts College
Henry Sturgis Dennison	Framingham
William Lusk Webster Field	Milton
Frank Johnson Goodnow	Baltimore, Md.
Francis Russell Hart	Boston
Edward Jackson Holmes	Boston
Nathaniel Thayer Kidder	Milton
Thomas William Lamont	New York, N. Y.
James Vance May	Boston
Everett Morss	Boston
Robert Lincoln O'Brien	Dedham
Thomas Nelson Perkins	Westwood
Andrew James Peters	Jamaica Plain
Herbert Putnam	Washington, D. C.
Alfred Lawrence Ripley	Andover
Jeremiah Smith, Jr.	Cambridge
Payson Smith	Brookline
Albert Warren Stearns	Billerica
Charles Henry Taylor	Boston
Edwin Sibley Webster	Brookline
Benjamin Loring Young	Weston

Class IV.—*The Humanities.*—195.

Section I.—*Theology, Philosophy, and Psychology.*—45.

Michael Joseph Ahern	Weston
Gordon Willard Allport	Cambridge
James Rowland Angell	New Haven, Conn.
John Gilbert Beebe-Center	Swampscott
Edwin Garrigues Boring	Cambridge
Edgar Sheffield Brightman	Newton
Henry Addington Bruce	Cambridge
Leonard Carmichael	Providence, R. I.
J(ames) McKeen Cattell	Garrison, N. Y.
George Croft Cell	Reading
Walter Fenno Dearborn	Cambridge
Edmund Burke Delabarre	Providence, R. I.
Raymond Dodge	New Haven, Conn.

Curt John Ducasse	Providence, R. I.
William Henry Paine Hatch	Cambridge
William Healy	Boston
William Arthur Heidel	Middletown, Conn.
William Ernest Hocking	Cambridge
Walter Samuel Hunter	Worcester
Frederick John Foakes Jackson	Englewood, N. J.
Truman Lee Kelley	Cambridge
Albert Cornelius Knudson	Cambridge
Karl Spencer Lashley	Chicago, Ill.
William Lawrence	Boston
Clarence Irving Lewis	Cambridge
Lee Sullivan McCollester	Tufts College
William McDougall	Durham, N. C.
Edward Caldwell Moore	Cambridge
Paul Elmer More	Princeton, N. J.
Arthur Darby Nock	Cambridge
William Cardinal O'Connell	Boston
Johnson O'Connor	Boston
Charles Edwards Park	Boston
Leighton Parks	London, England
Francis Greenwood Peabody	Cambridge
Carroll Cornelius Pratt	Cambridge
James Hugh Ryan	Washington, D. C.
Henry Knox Sherrill	Boston
Willard Learoyd Sperry	Cambridge
Russell Henry Stafford	Brookline
Henry Bradford Washburn	Cambridge
John Broadus Watson	New York, N. Y.
Frederic Lyman Wells	Newton Highlands
James Haughton Woods	Cambridge
Robert Mearns Yerkes	New Haven, Conn.

CLASS IV., SECTION II.—*History, Archæology, and Anthropology.*—40.

Charles McLean Andrews	New Haven, Conn.
James Phinney Baxter, 3d	Cambridge
Carl Lotus Becker	Ithaca, N. Y.

Robert Pierpont Blake	Cambridge
Franz Boas	New York, N. Y.
William Brooks Cabot	Boston
George Henry Chase	Cambridge
Roland Burrage Dixon	Cambridge
Wilberforce Eames	New York, N. Y.
Ephraim Emerton	Cambridge
Max Farrand	San Marino, Cal.
William Scott Ferguson	Cambridge
Worthington Chauncey Ford	Cambridge
Henry Thatcher Fowler	Providence, R. I.
Evarts Boutell Greene	New York, N. Y.
Charles Homer Haskins	Cambridge
Charles Downer Hazen	New York, N. Y.
Bert Hodge Hill	Athens, Greece
Earnest Albert Hooton	Cambridge
Halford Lancaster Hoskins	Tufts College
Aleš Hrdlička	Washington, D. C.
Alfred Vincent Kidder	Washington, D. C.
Alfred Louis Kroeber	Berkeley, Cal.
Kirsopp Lake	Cambridge
George LaPiana	Cambridge
Waldo Gifford Leland	Washington, D. C.
Charles Howard McIlwain	Cambridge
Roger Bigelow Merriman	Cambridge
Samuel Eliot Morison	Boston
George Andrew Reisner	Boston
Michael Ivanovich Rostovtzeff	New Haven, Conn.
Edward Sapir	New Haven, Conn.
George Sarton	Cambridge
Arthur Meier Schlesinger	Cambridge
Herbert Joseph Spinden	Brooklyn, N. Y.
John Osborne Sumner	Boston
Charles Holt Taylor	Cambridge
Charles Cutler Torrey	New Haven, Conn.
Alfred Marston Tozzer	Cambridge
Clark Wissler	New York, N. Y.

Class IV., Section III.—*Philology.*—53.

Edward Cooke Armstrong	Princeton, N. J.
William Nickerson Bates	Philadelphia, Pa.
Campbell Bonner	Ann Arbor, Mich.
Carleton Brown	New York, N. Y.
Carl Darling Buck	Chicago, Ill.
Edward Capps	Princeton, N. J.
Walter Eugene Clark	Cambridge
Ronald Salmon Crane	Chicago, Ill.
Morris William Croll	Princeton, N. J.
Samuel Hazzard Cross	Cambridge
Franklin Edgerton	New Haven, Conn.
Frank Edgar Farley	Middletown, Conn.
Jeremiah Denis Mathias Ford	Cambridge
James Geddes, Jr.	Brookline
Charles Hall Grandgent	Cambridge
Louis Herbert Gray	New York, N. Y.
William Chase Greene	Cambridge
Charles Burton Gulick	Cambridge
Roy Kenneth Hack	Cincinnati, Ohio
Raymond Dexter Havens	Baltimore, Md.
George Lincoln Hendrickson	New Haven, Conn.
William Guild Howard	Cambridge
Eugene Xavier Louis Henry Hyvernat	Washington, D. C.
Carl Newell Jackson	Cambridge
James Richard Jewett	Cambridge
(Ralph) Hayward Keniston	Chicago, Ill.
George Lyman Kittredge	Cambridge
Hans Kurath	Providence, R. I.
Henry Roseman Lang	New Haven, Conn.
Ernest Felix Langley	Cambridge
Charles Rockwell Lanman	Cambridge
Ivan Mortimer Linforth	Berkeley, Cal.
David Gordon Lyon	Cambridge
Francis Peabody Magoun, Jr.	Cambridge
Albert Matthews	Boston
William Albert Nitze	Chicago, Ill.

George Rapall Noyes	Berkeley, Cal.
Milman Parry	Belmont
Howard Rollin Patch	Northampton
Arthur Stanley Pease	Cambridge
Edward Kennard Rand	Cambridge
Fred Norris Robinson	Cambridge
Robert Kilburn Root	Princeton, N. J.
Rudolph Schevill	Berkeley, Cal.
Horatio Elwin Smith	Providence, R. I.
Herbert Weir Smyth	Cambridge
Franklin Bache Stephenson	Washington, D. C.
William Thomson	Cambridge
George Benson Weston	Cambridge
Joshua Whatmough	Cambridge
Ernest Hatch Wilkins	Oberlin, Ohio
Harry Austryn Wolfson	Cambridge
Karl Young	New Haven, Conn.

Class IV., Section IV.—*The Fine Arts and Belles Lettres.*—57.

George Pierce Baker	New Haven, Conn.
Stephen Vincent Benét	New York, N. Y.
Frank Weston Benson	Salem
(William) Welles Bosworth	New York, N. Y.
LeBaron Russell Briggs	Cambridge
John Alden Carpenter	Chicago, Ill.
Chalmers Dancy Clifton	New York, N. Y.
Charles Collens	Newton Center
Kenneth John Conant	Cambridge
Frederick Shepherd Converse	Westwood
Charles Allerton Coolidge	Boston
Charles Townsend Copeland	Cambridge
Ralph Adams Cram	Boston
Cyrus Edwin Dallin	Arlington Heights
George Harold Edgell	Cambridge
William Emerson	Cambridge
Carl Engel	Washington, D. C.
John Erskine	New York, N. Y.
Arthur Fairbanks	Hanover, N. H.

Edward Waldo Forbes	Cambridge
Robert Frost	South Shaftsbury, Vt.
Wallace Goodrich	Boston
Robert Grant	Boston
Chester Noyes Greenough	Cambridge
Edward Burlingame Hill	Cambridge
Robert Silliman Hillyer	Cambridge
Charles Hopkinson	Manchester
Mark Antony DeWolfe Howe	Boston
Archer Milton Huntington	New York, N. Y.
Henry James	New York, N. Y.
William James	Cambridge
Leo Rich Lewis	Tufts College
John Ellerton Lodge	Washington, D. C.
Charles Martin Tornov Loeffler	Medfield
John Livingston Lowes	Cambridge
Charles Donagh Maginnis	Brookline
Paul Manship	New York, N. Y.
Daniel Gregory Mason	New York, N. Y.
Frank Jewett Mather	Princeton, N. J.
Harold Murdock	Chestnut Hill
Kenneth Ballard Murdock	Cambridge
William Allan Neilson	Northampton
Curtis Hidden Page	Gilmanton, N. H.
William Lyon Phelps	New Haven, Conn.
Anthony John Philpott	Arlington
Chandler Rathfon Post	Cambridge
Edwin Arlington Robinson	New York, N. Y.
Denman Waldo Ross	Cambridge
Paul Joseph Sachs	Cambridge
Ellery Sedgwick	Boston
Henry Dwight Sedgwick	Dedham
Henry Richardson Shepley	Brookline
David Stanley Smith	New Haven, Conn.
Edmund C. Tarbell	Boston
Charles Howard Walker	Boston
Owen Wister	Philadelphia, Pa.
Charles Henry Conrad Wright	Cambridge

FOREIGN HONORARY MEMBERS.—119.

(Number limited to one hundred and thirty.)

CLASS I.—*Mathematical and Physical Sciences.*—33.

SECTION I.—*Mathematics and Astronomy.*—9.

Arthur Stanley Eddington	Cambridge, England
Jacques Salomon Hadamard	Paris
Godfrey Harold Hardy	Cambridge, England
Ejnar Hertzsprung	Leyden
Tullio Levi-Civita	Rome
Charles Emile Picard	Paris
Willem de Sitter	Leyden
Charles Jean de la Vallée Poussin	Louvain
Hermann Weyl	Princeton, N. J.

CLASS I., SECTION II.—*Physics.*—9.

Vilhelm Frimann Koren Bjerknes	Oslo
Albert Einstein	Princeton, N. J.
James Franck	Baltimore, Md.
Abram F. Joffé	Leningrad
Sir Joseph Larmor	Cambridge, England
Friedrich Paschen	Charlottenburg
Max Planck	Berlin
Sir Ernest Rutherford	Cambridge, England
Sir Joseph John Thomson	Cambridge, England

CLASS I., SECTION III.—*Chemistry.*—9.

Johannes N. Brönsted	Copenhagen
Peter Debye	Zürich
Fritz Haber	Berlin
Jaroslav Heyrovsky	Prague
Henri Louis Le Chatelier	Paris
Archibald Byron Macallum	London, Ontario
Fritz Paneth	Killean, Tayinloan, Scotland
Sören Peter Lauritz Sörensen	Copenhagen
Heinrich Wieland	Munich

CLASS I., SECTION IV.—*Technology and Engineering.*—6.

Maurice d'Ocagne	Paris
Ludwig Prandtl	Göttingen
Emil Probst	Karlsruhe
Aurel Stodola	Zürich
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Margerie, E. de. FHM, II: 1
Mark, E. L. F, II: 3
Mark, K. L. F, I: 3
Marks, L. S. F, I: 4
Marshall, L. C. F, III: 3
Martin, E. G. F, II: 3
Mason, D. G. F, IV: 4
Mason, E. S. F, III: 3
Mather, F. J. F, IV: 4
Mather, K. F. F, II: 1
Mathews, E. B. F, II: 1
Matthews, A. F, IV: 3
May, J. V. F, III: 4
Mayo, W. J. F, II: 4
Mazon, P. FHM, IV: 3
Mead, A. D. F, II: 3
Meillet, A. FHM, IV: 3
Meinecke, F. FHM, IV: 2
Melander, A. L. F, II: 3
Mendel, L. B. F, I: 3
Merriam, C. E. F, III: 2
Merriam, J. C. F, II: 1
Merrill, E. D. F, II: 2
Merriman, R. B. F, IV: 2
Merritt, E. G. F, I: 2
Miller, D. C. F, I: 2
Miller, G. A. F, I: 1
Miller, G. S. F, II: 3

Miller, J. A. F, I: 1
Miller, W. J. F, II: 1
Millikan, R. A. F, I: 2
Mills, F. C. F, III: 3
Mills, O. L. F, III: 2
Minot, G. R. F, II: 4
Mitchell, S. A. F, I: 1
Mitchell, W. C. F, III: 3
Mitchell, W. DeW. F, III: 1
Miyabe, K. FHM, II: 2
Miyajima, M. FHM, II: 4
Molengraaff, G. A. F. FHM, II: 1
Moore, E. C. F, IV: 1
Moore, J. B. F, III: 2
More, P. E. F, IV: 1
Morgan, E. M. F, III: 1
Morgan, T. H. F, II: 3
Morison, S. E. F, IV: 2
Morley, F. F, I: 1
Morris, F. K. F, II: 1
Morse, H. W. F, I: 2
Morse, M. F, I: 1
Morss, E. F, III: 4
Moss, W. L. F, II: 4
Moulton, F. R. F, I: 1
Moulton, H. G. F, III: 3
Mueller, E. F, I: 3
Müller, F. von. FHM, II: 4
Müller, G. E. FHM, IV: 1
Mueller, J. H. F, II: 4
Mulliken, S. P. F, I: 3
Munro, W. B. F, III: 2
Munroe, C. E. F, I: 3
Murdock, H. F, IV: 4
Murdock, K. B. F, IV: 4
Murray, G. FHM, IV: 4
Neal, H. V. F, II: 3
Neilson, W. A. F, IV: 4
Newell, L. C. F, I: 3
Nichols, E. L. F, I: 2
Nitze, W. A. F, IV: 3
Nock, A. D. F, IV: 1
Norris, J. F. F, I: 3
Norton, A. E. F, I: 4
Norton, C. L. F, I: 2
Noyes, A. A. F, I: 3
Noyes, G. R. F, IV: 3
Noyes, W. A. F, I: 3
Noyes, W. A., Jr. F, I: 3
Nuttall, G. H. F. FHM, II: 3
O'Brien, R. L. F, III: 4
d'Ocagne, M. FHM, I: 4
O'Connell, W. H. F, IV: 1
O'Connor, J. F, IV: 1
Oertel, H. FHM, IV: 3
Ogburn, W. F. F, III: 3
Oldenberg, O. F, I: 2
Olmsted, F. L. F, I: 4
Osborn, H. F. F, II: 3
Osgood, R. B. F, II: 4
Osterhout, W. J. V. F, II: 2
Page, C. H. F, IV: 4
Palache, C. F, II: 1
Paneth, F. FHM, I: 3
Park, C. E. F, IV: 1
Park, C. F. F, I: 4
Parker, G. H. F, II: 3
Parker, H. F, III: 1
Parks, L. F, IV: 1
Parry, M. F, IV: 3
Paschen, F. FHM, I: 2
Patch, H. R. F, IV: 3
Peabody, F. G. F, IV: 1
Pearl, R. F, II: 3
Pearse, L. F, I: 4
Pearson, K. FHM, IV: 2
Pease, A. S. F, IV: 3
Peers, E. A. FHM, IV: 4
Peirce, G. J. F, II: 2
Pender, H. F, I: 4
Pepper, G. W. F, III: 1
Perkins, T. N. F, III: 4
Persons, W. M. F, III: 3
Peters, A. J. F, III: 4
Phelps, W. L. F, IV: 4
Phillips, H. B. F, I: 1
Phillips, J. C. F, II: 3
Philpott, A. J. F, IV: 4
Picard, C. E. FHM, I: 1
Pickard, G. W. F, I: 4

Pickering, W. H. F, I: 1
Pidal, R. M. FHM, IV: 3
Pierce, G. W. F, I: 2
Pigou, A. C. FHM, III: 3
Pilsbry, H. A. F, II: 3
Pirenne, H. FHM, IV: 2
Planck, M. FHM, I: 2
Poincaré, R. FHM, III: 4
Pollock, Sir F. FHM, III: 1
Poor, C. L. F, I: 1
Post, C. R. F, IV: 4
Pound, R. F, III: 1
Prandtl, L. FHM, I: 4
Pratt, C. C. F, IV: 1
Pratt, F. H. F, II: 3
Pratt, J. H. F, II: 4
Prescott, S. C. F, I: 3
Probst, E. FHM, I: 4
Pupin, M. I. F, I: 2
Putnam, H. F, III: 4
Rabaud, H. FHM, IV: 4
Rand, E. K. F, IV: 3
Rand, H. W. F, II: 3
Rapport, D. F, II: 3
Raymond, P. E. F, II: 1
Reading, Lord. FHM, III: 4
Redfield, A. C. F, II: 3
Redlich, J. FHM, III: 1
Regan, C. T. FHM, II: 3
Rehder, A. F, II: 2
Reisner, G. A. F, IV: 2
Rendle, A. B. FHM, II: 2
Renner, O. FHM, II: 2
Richards, R. H. F, I: 3
Richardson, R. G. D. F, I: 1
Ripley, A. L. F, III: 4
Rist, C. FHM, III: 3
Ritter, W. E. F, II: 3
Roberts, O. F, III: 1
Robinson, B. L. F, II: 2
Robinson, E. A. F, IV: 4
Robinson, F. N. F, IV: 3
Rogers, A. F. F, II: 1
Roorbach, G. B. F, III: 3
Root, E. F, III: 1
Root, R. K. F, IV: 3
Rosanoff, M. A. F, I: 3
Rosenau, M. J. F, II: 4
Ross, D. W. F, IV: 4
Rostovtzeff, M. I. F, IV: 2
Rowe, A. W. F, I: 3
Rowe, L. S. F, III: 3
Rugg, A. P. F, III: 1
Russell, G. E. F, I: 4
Russell, H. N. F, I: 1
Rutherford, Sir E. FHM, I: 2
Ruthven, A. G. F, II: 3
Rutledge, G. F, I: 1
Ryan, J. H. F, IV: 1
Sachs, P. J. F, IV: 4
Sapir, E. F, IV: 2
Sarton, G. F, IV: 2
Saunders, F. A. F, I: 2
Sauveur, A. F, I: 4
Sax, K. F, II: 2
Sayles, R. W. F, II: 1
Sayre, F. B. F, III: 1
Scatchard, G. F, I: 3
Schaller, W. T. F, II: 1
Schevill, R. F, IV: 3
Schlesinger, A. M. F, IV: 2
Schlesinger, F. F, I: 1
Schuchert, C. F, II: 1
Schumb, W. C. F, I: 3
Schumpeter, J. A. F, III: 3
Scott, A. W. F, III: 1
Scott, W. B. F, II: 1
Sedgwick, E. F, IV: 4
Sedgwick, H. D. F, IV: 4
Sellards, A. W. F, II: 4
Setchell, W. A. F, II: 2
Seward, A. C. FHM, II: 2
Shapley, H. F, I: 1
Shattuck, F. C. F, II: 4
Shaw, Sir W. N. FHM, II: 1
Shepley, H. R. F, IV: 4
Sherrill, H. K. F, IV: 1
Sherrill, M. S. F, I: 3
Sherrington, Sir C. S. FHM, II: 4
Shimer, H. W. F, II: 1

Sitter, W. de. FHM, I: 1
Slater, J. C. F, I: 2
Slipher, V. M. F, I: 1
Slocum, F. F, I: 1
Smith, D. S. F, IV: 4
Smith, H. E. F, IV: 3
Smith, H. M. F, I: 3
Smith, J., Jr. F, III: 4
Smith, P. F, III: 4
Smith, T. F, II: 4
Smyth, H. L. F, I: 4
Smyth, H. W. F, IV: 3
Snyder, C. F, III: 3
Snyder, V. F, I: 1
Sörensen, S. P. L. FHM, I: 3
Sollmann, T. H. F, II: 4
Sombart, W. FHM, III: 3
Sorokin, P. A. F, III: 3
Spemann, H. FHM, II: 3
Sperry, W. L. F, IV: 1
Spinden, H. J. F, IV: 2
Spofford, C. M. F, I: 4
Sprague, O. M. W. F, III: 3
Stafford, R. H. F, IV: 1
Stakman, E. C. F, II: 2
Stammler, R. FHM, III: 1
Stamp, Sir J. FHM, III: 4
Stearns, A. W. F, III: 4
Stebbins, J. F, I: 1
Stein, Sir A. FHM, IV: 2
Stephenson, F. B. F, IV: 3
Stetson, H. T. F, I: 1
Stevens, J. F. F, I: 4
Stieglitz, J. O. F, I: 3
Stiles, C. W. F, II: 4
Stiles, P. G. F, II: 3
Stodola, A. FHM, I: 4
Stone, H. F. F, III: 1
Stone, J. S. F, I: 2
Stone, M. H. F, I: 1
Strong, R. P. F, II: 4
Struik, D. J. F, I: 1
Sumner, J. O. F, IV: 2
Taber, H. F, I: 1
Talbot, F. B. F, II: 4
Tamarkin, J. D. F, I: 1
Tarbell, E. C. F, IV: 4
Taussig, F. W. F, III: 3
Taylor, C. H. F, III: 4
Taylor, C. H. F, IV: 2
Taylor, F. B. F, II: 1
Terry, C. S. FHM, IV: 2
Theiler, Sir A. FHM, II: 4
Thomas, F. W. FHM, IV: 3
Thomas, W. I. F, III: 3
Thompson, D'A. W. FHM, II: 3
Thompson, M. deK. F, 1: 2
Thomson, E. F, I: 2
Thomson, Sir J. J. FHM, I: 2
Thomson, W. F, IV: 3
Thurston, E. S. F, III: 1
Timonoff, V. E. FHM, I: 4
Tolman, R. C. F, I: 3
Torrey, C. C. F, IV: 2
Tozzer, A. M. F, IV: 2
Trelease, W. F, II: 2
Trevelyan, G. M. FHM, IV: 2
Tucker, D. S. F, III: 3
Tyler, H. W. F, I: 1
Tyzzer, E. E. F, II: 4
Underwood, H. W., Jr. F, I: 3
Usher, A. P. F, III: 3
Vallarta, M. S. F, I: 2
Vallée Poussin, C. J. de la. FHM, I: 1
Vaughan, T. W. F, II: 1
Veblen, O. F, I: 1
Vecchio, G. Del. FHM, III: 1
Verhoeff, F. H. F, II: 4
Vries, H. de. FHM, II: 2
Wagner, K. W. FHM, I: 4
Wait, W. C. F, III: 1
Walker, C. H. F, IV: 4
Walker, W. H. F, I: 3
Walsh, J. L. F, I: 1
Wambaugh, E. F, III: 1
Warner, E. P. F, I: 4
Warren, C. H. F, II: 1
Washburn, H. B. F, IV: 1
Watson, J. B. F, IV: 1
Wearn, J. T. F, II: 4

Weatherby, C. A. F, II: 2
Webster, D. L. F, I: 2
Webster, E. S. F, III: 4
Weiss, S. F, II: 4
Welch, W. H. F, II: 4
Wells, A. E. F, I: 4
Wells, F. L. F, IV: 1
Weston, G. B. F, IV: 3
Weston, R. S. F, I: 4
Weston, W. H. F, II: 2
Wetmore, R. H. F, II: 2
Weyl, H. FHM, I: 1
Weysse, A. W. F, II: 3
Whatmough, J. F, IV: 3
Wheeler, W. M. F, II: 3
White, B. F, II: 4
White, D. F, II: 1
Whitehead, A. N. F, I: 1
Whitlock, H. P. F, II: 1
Whitman, E. A. F, III: 1
Whitney, W. R. F, I: 3
Wickersham, G. W. F, III: 1
Widder, D. V. F, I: 1
Wieland, H. FHM, I: 3
Wilkins, E. H. F, IV: 3
Williams, F. H. F, II: 4
Williams, J. H. F, III: 3
Williams, R. S. F, I: 3
Willis, B. F, II: 1
Willoughby, W. F. F, III: 2
Willoughby, W. W. F, III: 2
Wilson, E. B. F, II: 3
Wilson, E. B. F, I: 2
Wilson, G. G. F, III: 2
Winsor, F. E. F, I: 4
Wissler, C. F, IV: 2
Wister, O. F, IV: 4
Wolbach, S. B. F, II: 4
Wolff, J. E. F, II: 1
Wolfson, H. A. F, IV: 3
Wood, R. W. F, I: 2
Woodman, A. G. F, I: 3
Woods, F. A. F, II: 3
Woods, F. S. F, I: 1
Woods, J. H. F, IV: 1
Worcester, J. R. F, I: 4
Wright, C. H. C. F, IV: 4
Wright, F. E. F, II: 1
Wright, Q. F, III: 2
Wyman, J., Jr. F, II: 3
Yeomans, H. A. F, III: 2
Yerkes, R. M. F, IV: 1
Young, B. L. F, III: 4
Young, K. F, IV: 3
Zeleny, J. F, I: 2
Zinsser, H. F, II: 4

STATUTES AND STANDING VOTES.

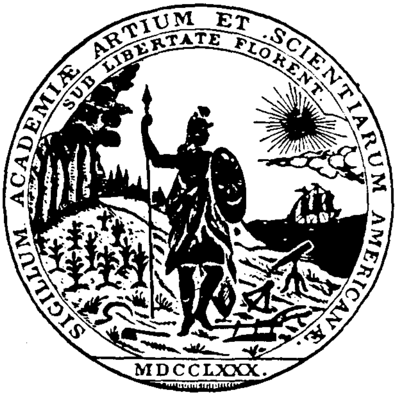
STATUTES.

Adopted November 8, 1911: *amended May* 8, 1912, *January* 8, *and May* 14, 1913, *April* 14, 1915, *April* 12, 1916, *April* 10, 1918, *May* 14, 1919, *February* 8, *April* 12, *and December* 13, 1922, *February* 14, *March* 14, *and October* 10, 1923, *March* 10, 1926, *May* 9, 1928, *April* 8, 1931, *and November* 11, 1931, *and April* 12, 1933.

CHAPTER I.

THE CORPORATE SEAL.

ARTICLE 1. The Corporate Seal of the Academy shall be as here depicted:


SIGILLUM ACADEMIÆ ARTIUM ET SCIENTIARUM AMERICANÆ.
SUB LIBERTATE FLORENT.
MDCCLXXX.

ARTICLE 2. The Recording Secretary shall have the custody of the Corporate Seal.

See Chap. v, art. 3; chap. vi, art. 2.

CHAPTER II.

FELLOWS AND FOREIGN HONORARY MEMBERS AND DUES.

ARTICLE 1. The Academy consists of Fellows, who are either citizens or residents of the United States of America, and Foreign Honorary Members. They are arranged in four Classes, according to the Arts and Sciences in which they are severally proficient, and each Class is divided into four Sections, namely:

CLASS I. *The Mathematical and Physical Sciences*

Section 1. Mathematics and Astronomy
Section 2. Physics
Section 3. Chemistry
Section 4. Technology and Engineering

CLASS II. *The Natural and Physiological Sciences*

Section 1. Geology, Mineralogy, and Physics of the Globe
Section 2. Botany
Section 3. Zoölogy and Physiology
Section 4. Medicine and Surgery

CLASS III. *The Social Arts*

Section 1. Jurisprudence
Section 2. Government, International Law, and Diplomacy
Section 3. Economics and Sociology
Section 4. Administration and Affairs

CLASS IV. *The Humanities*

Section 1. Theology, Philosophy, and Psychology
Section 2. History, Archæology, and Anthropology
Section 3. Philology
Section 4. The Fine Arts and Belles Lettres

ARTICLE 2. The number of Fellows shall not exceed Eight hundred, of whom not more than Six hundred shall be residents of Massachusetts, nor shall there be more than Two hundred and ten in any one Class.

ARTICLE 3. The number of Foreign Honorary Members shall not exceed One hundred and thirty. They shall be chosen from among citizens of foreign countries most eminent for their discoveries and attainments in any of the Classes above enumerated. There shall not be more than Thirty-five in any one Class.

ARTICLE 4. If any person, after being notified of his election as Fellow, shall neglect for six months to accept in writing, or, if a Fellow resident within fifty miles of Boston shall neglect to pay his Admission Fee, his election shall be void; and if any Fellow resident within fifty miles of Boston shall neglect to pay his Annual Dues for six months after they are due, provided his attention shall have been called to this Article of the Statutes in the meantime, he shall cease to be a Fellow; but the Council may suspend the provisions of this Article for a reasonable time.

With the previous consent of the Council, the Treasurer may dispense (*sub silentio*) with the payment of the Admission Fee or of the Annual Dues or both whenever he shall deem it advisable. In the case of officers of the Army or Navy who are out of the Commonwealth on duty, payment of the Annual Dues may be waived during such absence if continued during the whole financial year and if notification of such expected absence be sent to the Treasurer. Upon similar notification to the Treasurer, similar exemption may be accorded to Fellows subject to Annual Dues, who may temporarily remove their residence for at least two years to a place more than fifty miles from Boston.

If any person elected a Foreign Honorary Member shall neglect for six months after being notified of his election to accept in writing, his election shall be void.

See Chap. vii, art. 2.

ARTICLE 5. Every Fellow resident within fifty miles of Boston hereafter elected shall pay an Admission Fee of Ten dollars, unless previously as an Associate he has paid an Admission Fee of like amount.

Every Fellow resident within fifty miles of Boston shall, and others may, pay such Annual Dues, not exceeding Fifteen dollars, as shall be voted by the Academy at each Annual Meeting, when they shall become due; but any Fellow shall be exempt from the annual payment if, at any time after his admission, he shall pay into the treasury

Two hundred dollars in addition to his previous payments. Any Fellow shall also be exempt from Annual Dues who has paid such dues for forty years, or, having attained the age of seventy-five has paid dues for twenty-five years.

All Commutations of the Annual Dues shall be and remain permanently founded, the interest only to be used for current expenses.

Any Fellow not previously subject to Annual Dues who takes up his residence within fifty miles of Boston, shall pay to the Treasurer within three months thereafter Annual Dues for the current year, failing which his Fellowship shall cease; but the Council may suspend the provisions of this Article for a reasonable time.

Only Fellows who pay Annual Dues or have commuted them may hold office in the Academy or serve on the Standing Committees or vote at meetings.

Article 6. Fellows who pay or have commuted the Annual Dues and Foreign Honorary Members shall be entitled to receive gratis one copy of all Publications of the Academy issued after their election.

See Chap. xi, art. 2.

Article 7. Diplomas signed by the President and the Vice-President of the Class to which the member belongs, and countersigned by the Secretaries, shall be given to Foreign Honorary Members and to Fellows on request.

Article 8. If, in the opinion of a majority of the entire Council, any Fellow or Foreign Honorary Member shall have rendered himself unworthy of a place in the Academy, the Council shall recommend to the Academy the termination of his membership; and if three-fourths of the Fellows present, out of a total attendance of not less than fifty at a Stated Meeting, or at a Special Meeting called for the purpose, shall adopt this recommendation, his name shall be stricken from the Roll.

See Chap. iii; chap. vi, art. 1; chap. x, art. 1, 7; chap. xi, art. 2.

CHAPTER III.

Election of Fellows and Foreign Honorary Members.

The procedure in the election of Fellows and Foreign Honorary Members shall be as follows:

Nominations to Fellowship or Foreign Honorary Membership in any Section must be signed by Two Fellows of that Section, or by three Fellows of any Sections, and sent to the Corresponding Secretary accompanied by a statement of the qualifications of the nominee and brief biographical data.

Notice shall be sent to every Fellow not later than the fifteenth of January in each year, reminding him that all nominations must be in the hands of the Corresponding Secretary before the fifteenth of February following.

A list of the nominees, giving a brief account of each, with the names of the nominators, shall be sent to every Fellow with a request that he return the list with such confidential comments and indications of preference as he may choose to make.

All the nominations, with any comments thereon and with expressions of preference on the part of the Fellows, shall be referred to the appropriate Class Committees, which shall canvass them, and report their recommendations in writing to the Council before the Stated Meeting of the Academy in April.

Elections of Fellows and Foreign Honorary Members shall be made by the Council before the Annual Meeting in May, and announced at that meeting.

Persons nominated in any year, but not elected, may be carried over to the list of nominees for the next year at the discretion of the Council, but shall not be further continued unless renominated.

See Chap. ii; chap. vi, art. 1; chap. x, art. 1.

CHAPTER IV.

OFFICERS.

ARTICLE 1. The Officers of the Academy shall be a President (who shall be Chairman of the Council), four Vice-Presidents (one from each Class), a Corresponding Secretary (who shall be Secretary of the Council), a Recording Secretary, a Treasurer, a Librarian, and an Editor, all of whom shall be elected by ballot at the Annual Meeting, and shall hold their respective offices for one year, and until others are duly chosen and installed.

There shall be also sixteen Councillors, one from each Section of each Class. At each Annual Meeting four Councillors, one from each Class, shall be elected by ballot to serve for the full term of four years and until others are duly chosen and installed. The same Fellow shall not be eligible for two successive terms.

The Councillors, with the other officers previously named, and the Chairman of the House Committee, *ex officio*, shall constitute the Council.

See Chap. xi, art. 1.

ARTICLE 2. If any officer be unable, through death, absence, or disability, to fulfill the duties of his office, or if he shall resign, his place may be filled by the Council in its discretion for any part or the whole of the unexpired term.

ARTICLE 3. At the Stated Meeting in March, the President shall appoint a Nominating Committee of four Fellows having the right to vote, one from each Class. This Committee shall prepare a list of nominees for the several offices to be filled, and for the Standing Committees, and file it with the Recording Secretary not later than four weeks before the Annual Meeting.

See Chap. vi, art. 2.

ARTICLE 4. Independent nominations for any office, if signed by at least twenty Fellows having the right to vote, and received by the Recording Secretary not less than ten days before the Annual Meeting, shall be inserted in the call therefor, and shall be mailed to all the Fellows having the right to vote.

See Chap. vi, art. 2.

ARTICLE 5. The Recording Secretary shall prepare for use in voting at the Annual Meeting a ballot containing the names of all persons duly nominated for office.

CHAPTER V.

THE PRESIDENT.

ARTICLE 1. The President, or in his absence the senior Vice-President present (seniority to be determined by length of continuous Fellowship in the Academy), shall preside at all meetings of the Academy. In the absence of all these officers, a Chairman of the meeting shall be chosen by ballot.

ARTICLE 2. Unless otherwise ordered, all Committees which are not elected by ballot shall be appointed by the presiding officer.

ARTICLE 3. Any deed or writing to which the Corporate Seal is to be affixed, except leases of real estate, shall be executed in the name of the Academy by the President or, in the event of his death, absence, or inability, by one of the Vice-Presidents, when thereto duly authorized.

See Chap. ii, art. 7; chap. iv, art. 1, 3; chap. vi, art. 2; chap. vii, art. 1; chap. x, art. 6; chap. xi, art. 1, 2; chap. xii, art. 1.

CHAPTER VI.

THE SECRETARIES.

ARTICLE 1. The Corresponding Secretary shall conduct the correspondence of the Academy and of the Council, recording or making an entry of all letters written in its name, and preserving for the files all official papers which may be received. At each meeting of the Council he shall present the communications addressed to the Academy which have been received since the previous meeting, and at the next meeting of the Academy he shall present such as the Council may determine.

He shall notify all persons who may be elected Fellows or Foreign Honorary Members, send to each a copy of the Statutes, and on their acceptance issue the proper Diploma. He shall also notify all meetings of the Council; and in case of the death, absence, or inability of the Recording Secretary he shall notify all meetings of the Academy.

Under the direction of the Council, he shall keep a List of the Fellows and Foreign Honorary Members, arranged in their several Classes and Sections. It shall be printed annually and issued as of the first day of July.

See Chap. ii, art. 7; chap. iii; chap. iv, art. 1; chap. x, art. 6; chap. xi, art. 1; chap. xii, art. 1.

ARTICLE 2. The Recording Secretary shall have the custody of the Charter, Corporate Seal, Archives, Statute-Book, Journals, and all literary papers belonging to the Academy.

Fellows borrowing such papers or documents shall receipt for them to their custodian.

The Recording Secretary shall attend the meetings of the Academy and keep a faithful record of the proceedings with the names of the Fellows present; and after each meeting is duly opened, he shall read the record of the preceding meeting.

He shall notify the meetings of the Academy to each Fellow and by mail at least seven days beforehand, and in his discretion may also cause the meetings to be advertised; he shall apprise Officers and Committees of their election or appointment, and inform the Treasurer of appropriations of money voted by the Academy.

After all elections, he shall insert in the Records the names of the Fellows by whom the successful nominees were proposed.

He shall send the Report of the Nominating Committee in print to every Fellow having the right to vote at least three weeks before the Annual Meeting.

See Chap. iv, art. 3.

In the absence of the President and of the Vice-Presidents he shall, if present, call the meeting to order, and preside until a Chairman is chosen.

See Chap. i; chap. ii, art. 7; chap. iv, art. 3, 4, 5; chap. x, art. 6; chap. xi, art. 1, 2; chap. xii, art. 1, 3.

ARTICLE 3. The Secretaries, with the Editor, shall have authority to publish such of the records of the meetings of the Academy as may seem to them likely to promote its interests.

CHAPTER VII.

THE TREASURER AND THE TREASURY.

ARTICLE 1. The Treasurer shall collect all money due or payable to the Academy, and all gifts and bequests made to it. He shall pay all bills due by the Academy, when approved by the proper officers, except those of the Treasurer's office, which may be paid without such approval; in the name of the Academy he shall sign all leases of real estate; and, with the written consent of a member of the Committee on Finance, he shall make all transfers of stocks, bonds, and other securities belonging to the Academy, all of which shall be in his official custody.

He shall keep a faithful account of all receipts and expenditures, submit his accounts annually to the Auditing Committee, and render them at the expiration of his term of office, or whenever required to do so by the Academy or the Council.

He shall keep separate accounts of the income of the Rumford Fund, and of all other special Funds, and of the Appropriation thereof, and render them annually.

His accounts shall always be open to the inspection of the Council.

ARTICLE 2. He shall report annually to the Council at its March meeting on the expected income of the various Funds and from all other sources during the ensuing financial year. He shall also report the names of all Fellows who may be then delinquent in the payment of their Annual Dues.

ARTICLE 3. He shall give such security for the trust reposed in him as the Academy may require.

ARTICLE 4. With the approval of a majority of the Committee on Finance, he may appoint an Assistant Treasurer to perform his duties, for whose acts, as such assistant, he shall be responsible; or, with like approval and responsibility, he may employ any Trust Company doing business in Boston as his agent for the same purpose, the compensation of such Assistant Treasurer or agent to be fixed by the Committee on Finance and paid from the Funds of the Academy.

ARTICLE 5. At the Annual Meeting he shall report in print all his official doings for the preceding year, stating the amount and condition of all the property of the Academy entrusted to him, and the character of the investments.

ARTICLE 6. The Financial Year of the Academy shall begin with the first day of April.

ARTICLE 7. No person or committee shall incur any debt or liability in the name of the Academy, unless in accordance with a previous vote and appropriation therefor by the Academy or the Council, or sell or otherwise dispose of any property of the Academy, except cash or invested funds, without previous consent and approval of the Council.

See Chap. ii, art. 4, 5; chap. vi, art. 2; chap. x, art. 6; chap. xi, art. 1, 2, 3; chap. xii, art. 1.

CHAPTER VIII.

THE LIBRARIAN AND THE LIBRARY.

ARTICLE 1. The Librarian shall have charge of the printed books, keep a correct catalogue thereof, and provide for their delivery from the Library.

At the Annual Meeting, as Chairman of the Committee on the Library, he shall make a Report on its condition.

ARTICLE 2. In conjunction with the Committee on the Library he shall have authority to expend such sums as may be appropriated by the Academy for the purchase of books, periodicals, etc., and for defraying other necessary expenses connected with the Library.

ARTICLE 3. All books procured from the income of the Rumford Fund or of other special Funds shall contain a book-plate expressing the fact.

ARTICLE 4. Books taken from the Library shall be receipted for to the Librarian or his assistant.

ARTICLE 5. Books shall be returned in good order, regard being had to necessary wear with good usage. If any book shall be lost or injured, the Fellow to whom it stands charged shall replace it by a new volume or by a new set, if it belongs to a set, or pay the current price thereof to the Librarian, whereupon the remainder of the set, if any, shall be delivered to the Fellow so paying, unless such remainder be valuable by reason of association.

ARTICLE 6. All books shall be returned to the Library for examination at least one week before the Annual Meeting.

ARTICLE 7. The Librarian shall have the custody of the Publications of the Academy. With the advice and consent of the President, he may effect exchanges with other associations.

See Chap. ii, art. 6; chap. xi, art. 1, 2.

CHAPTER IX.

THE EDITOR AND THE PUBLICATIONS.

ARTICLE 1. The Editor shall have charge of the conduct through the press of the Proceedings and the Memoirs, and all correspondence relative thereto, and shall have power to fix the price at which individual numbers of the Proceedings and Memoirs are sold.

ARTICLE 2. In conjunction with the Committee of Publication, he shall have authority to expend such sums as may be appropriated by the Academy for printing the publications and for defraying other expenses therewith connected.

ARTICLE 3. All publications which are financed in whole or in part from the income of the Rumford Fund or from the income of other special funds, and all publications of work done with the aid of the Rumford Fund or other special funds, shall contain a conspicuous statement of this fact.

ARTICLE 4. Two hundred extra copies of each paper printed in the Proceedings or Memoirs shall be placed at the disposal of the author without charge.

If, on account of the number of communications offered for publication, it shall be necessary to decline for publication communications otherwise acceptable, members of the Academy shall be given preference in each of the several Classes over non-members; but whenever it shall be necessary to exercise this preference, the Editor shall inform the Council of the fact.

See Chap. iv, art. 1; chap. vi, art. 3; chap. x, art. 6; chap. xi, art. 4.

CHAPTER X.

THE COUNCIL.

ARTICLE 1. The Council shall exercise a discreet supervision over all nominations and elections to membership, and in general supervise all the affairs of the Academy not explicitly reserved to the Academy as a whole or entrusted by it or by the Statutes to standing or special committees.

It shall consider all nominations duly sent to it by any Class Committee, and act upon them in accordance with the provisions of Chapter III.

With the consent of the Fellow interested, it shall have power to make transfers between the several Sections, reporting its action to the Academy.

See Chap. iii, art. 2, 3; chap. xi, art. 1.

Article 2. Nine members shall constitute a quorum.

Article 3. It shall establish rules and regulations for the transaction of its business, and provide all printed and engraved blanks and books of record.

Article 4. It shall act upon all resignations of officers, and all resignations and forfeitures of Fellowship; and cause the Statutes to be faithfully executed.

It shall appoint all agents and subordinates not otherwise provided for by the Statutes, prescribe their duties, and fix their compensation. They shall hold their respective positions during the pleasure of the Council.

Article 5. It may appoint, for terms not exceeding one year, and prescribe the functions of, such committees of its number, or of the Fellows of the Academy, as it may deem expedient, to facilitate the administration of the affairs of the Academy or to promote its interests.

Article 6. At its March meeting it shall receive reports from the President, the Secretaries, the Treasurer, and the Standing Committees, on the appropriations severally needed for the ensuing financial year. At the same meeting the Treasurer shall report on the expected income of the various Funds and from all other sources during the same year.

A report from the Council shall be submitted to the Academy, for action, at the March meeting, recommending the appropriation which in the opinion of the Council should be made.

On the recommendation of the Council, special appropriations may be made at any Stated Meeting of the Academy, or at a Special Meeting called for the purpose.

See Chap. xi, art. 3.

Article 7. After the death of a Fellow or Foreign Honorary Member, it shall appoint a member of the Academy to provide a biographical notice for publication in the Proceedings.

Article 8. It shall report at every meeting of the Academy such business as it may deem advisable to present.

See Chap. ii, art. 4, 5, 8; chap. iv, art. 1, 2; chap. vi, art. 1; chap. vii, art. 1; chap. xii, art. 1, 4.

CHAPTER XI.

Standing Committees.

Article 1. The Class Committee of each Class shall consist of the Vice-President, who shall be chairman, and the four Councillors of the Class, together with such other officer or officers annually elected as may belong to the Class. It shall consider nominations to Fellowship in its own Class, and report in writing to the Council such as may receive at a Class Committee Meeting a majority of the votes cast, provided at least three shall have been in the affirmative.

See Chap. iii, art. 2.

Article 2. At the Annual Meeting the following Standing Committees shall be elected by ballot to serve for the ensuing year:

(i) *The Committee on Finance,* to consist of four Fellows, who, through the Treasurer, shall have full control and management of the funds and trusts of the Academy, with the power of investing the funds and changing the investments thereof in their discretion.

See Chap. iv, art. 3; chap. vii, art. 1, 4; chap. x, art. 6.

(ii) *The Rumford Committee,* to consist of seven Fellows, who shall report to the Academy on all applications and claims for the Rumford Premium. It alone shall authorize the purchase of books, publications and apparatus at the charge of the income from the Rumford Fund, and generally shall see to the proper execution of the trust.

See Chap. iv, art. 3; chap. x, art. 6.

(iii) *The Cyrus Moors Warren Committee,* to consist of seven Fellows, who shall consider all applications for appropriations from the income of the Cyrus Moors Warren Fund, and generally shall see to the proper execution of the trust.

See Chap. iv, art. 3; chap. x, art. 6.

(iv) *The Committee of Publication,* to consist of the Editor, *ex officio,* as Chairman, and four other Fellows, one from each Class, to whom all communications submitted to the Academy for publication shall be referred, and to whom the printing of the Proceedings and the Memoirs shall be entrusted.

It shall fix the price at which volumes of the publications shall be sold; but Fellows may be supplied at half price with volumes which they are not entitled to receive gratis.

It shall determine when the pressure of material offered for publication makes it necessary to give preference to members of the Academy as compared with non-members, or to give priority to certain members as compared with others, and to what extent this preference or priority shall be applied in each of the four Classes, to the end that a proper balance of the facilities of publication with respect to subject matter and authors may be maintained.

See Chap. iv, art. 3; chap. vi, art. 1, 3; chap. ix; chap. x, art. 6.

(v) *The Committee on the Library*, to consist of the Librarian, *ex officio*, as Chairman, and four other Fellows, one from each Class, who shall examine the Library and make an annual report on its condition and management.

See Chap. iv, art. 3; chap. viii, art. 1, 2; chap. x, art. 6.

(vi) *The House Committee*, to consist of four Fellows, who shall have charge of all expenses connected with the House, including the general expenses of the Academy not specifically assigned to the care of other Committees or Officers.

See Chap. iv, art. 1, 3; chap. x, art. 6.

(vii) *The Committee on Meetings*, to consist of the President, the Recording Secretary, and four other Fellows, who shall have charge of plans for meetings of the Academy.

See Chap. iv, art. 3; chap. x, art. 6.

(viii) *The Auditing Committee*, to consist of two Fellows, who shall audit the accounts of the Treasurer, with power to employ an expert and to approve his bill.

See Chap. iv, art. 3; chap. vii, art. 1; chap. x, art. 6.

(ix) *The Committee on Biographical Notices*, to consist of six Fellows, two to be elected each year, six in 1933, one of them to be a Secretary of the Academy, to see that biographical notices of the Fellows are provided.

See Chap. x, art. 7.

ARTICLE 3. The Standing Committees shall report annually to the Council in March on the appropriations severally needed for the ensuing financial year; and all bills incurred on account of these Committees, within the limits of the several appropriations made by the Academy, shall be approved by their respective Chairmen.

In the absence of the Chairman of any Committee, bills may be approved by any member of the Committee whom he shall designate for the purpose.

See Chap. vii. art, 1, 7; chap. x, art. 6.

CHAPTER XII.

MEETINGS, COMMUNICATIONS, AND AMENDMENTS.

ARTICLE 1. There shall be annually eight Stated Meetings of the Academy, namely, on the second Wednesday of October, November, December, January, February, March, April, and May. Only at these meetings, or at adjournments thereof regularly notified, or at Special Meetings called for the purpose, shall appropriations of money be made or amendments of the Statutes or Standing Votes be effected.

The Stated Meeting in May shall be the Annual Meeting of the Corporation.

Special Meetings shall be called by either of the Secretaries at the request of the President, of a Vice-President, of the Council, or of ten Fellows having the right to vote; and notifications thereof shall state the purpose for which the meeting is called.

A meeting for receiving and discussing literary or scientific communications may be held on the fourth Wednesday of each month, excepting July, August, and September; but no business shall be transacted at said meetings.

ARTICLE 2. Twenty-five Fellows having the right to vote shall constitute a quorum for the transaction of business at Stated or Special Meetings. Eighteen Fellows shall be sufficient to constitute a meeting for literary or scientific communications and discussions.

ARTICLE 3. Upon the request of the presiding officer or the Recording Secretary, any motion or resolution offered at any meeting shall be submitted in writing.

ARTICLE 4. No report of any paper presented at a meeting of the Academy shall be published by any Fellow without the consent of the author; and no report shall in any case be published by any Fellow in a newspaper as an account of the proceedings of the Academy without the previous consent and approval of the Council. The Council, in its discretion, by a duly recorded vote, may delegate its authority in this regard to one or more of its members.

ARTICLE 5. No Fellow shall introduce a guest at any meeting of the Academy until after the business has been transacted, and especially until after the result of the balloting upon nominations has been declared.

ARTICLE 6. The Academy shall not express its judgment on literary or scientific memoirs or performances submitted to it, or included in its Publications.

ARTICLE 7. All proposed Amendments of the Statutes shall be referred to a committee, and on its report, at a subsequent Stated Meeting or at a Special Meeting called for the purpose, two-thirds of the ballot cast, and not less than twenty-five, must be affirmative to effect enactment.

ARTICLE 8. Standing Votes may be passed, amended, or rescinded at a Stated Meeting, or at a Special Meeting called for the purpose, by a vote of two-thirds of the members present. They may be suspended by a unanimous vote.

See Chap. ii, art. 5, 8; chap. iii; chap. iv, art. 3, 4, 5; chap. v, art. 1; chap. vi, art. 1, 2; chap. x, art. 8.

STANDING VOTES.

1. Communications of which notice has been given to either of the Secretaries shall take precedence of those not so notified.

2. Fellows may take from the Library six volumes at any one time, and may retain them for three months, and no longer. Upon special application, and for adequate reasons assigned, the Librarian may permit a larger number of volumes, not exceeding twelve, to be drawn from the Library for a limited period.

3. Works published in numbers, when unbound, shall not be taken from the Hall of the Academy without the leave of the Librarian.

4. The Council, under such rules respecting nominations as it may prescribe, may elect as Associates of the Academy a limited number of men of mark in affairs or of distinguished service in the community.

Associates shall be entitled to the same privileges as Fellows, but shall not have the right to vote.

The admission fee and annual dues of Associates shall be the same as those of Fellows residing within fifty miles of Boston.

5. Communications offered for publication in the Proceedings or Memoirs of the Academy shall not be accepted for publication before the author shall have informed the Committee on Meetings of his readiness, either himself or through some agent, to use such time as the Committee may assign him at such meeting as may be convenient both to him and to the Committee, for the purpose of presenting to the Academy a general statement of the nature and significance of the results contained in his communication.

RUMFORD PREMIUM.

In conformity with the terms of the gift of Sir Benjamin Thompson, Count Rumford, of a certain Fund to the American Academy of Arts and Sciences, and with a decree of the Supreme Judicial Court of Massachusetts for carrying into effect the general charitable intent and purpose of Count Rumford, as expressed in his letter of gift, the Academy is empowered to make from the income of the Rumford Fund, as it now exists, at any Annual Meeting, an award of a gold and a silver medal, being together of the intrinsic value of three hundred dollars, as a Premium to the author of any important discovery or useful improvement in light or heat, which shall have been made and published by printing, or in any way made known to the public, in any part of the continent of America, or any of the American islands; preference always being given to such discoveries as, in the opinion of the Academy, shall tend most to promote the good of mankind; and, if the Academy sees fit, to add to such medals, as a further Premium for such discovery and improvement, a sum of money not exceeding three hundred dollars.

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